M-open Sets in Bitopological Spaces

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Abstract

The aim of this paper is to introduce and investigate the concept of \( \tau_i \tau_j \cdot M \)-closed sets which are introduced in a bitopological space in analogy with \( M \)-closed sets in topological spaces. Also \( M \)-closure and \( M \)-interior operators in bitopological spaces are introduced. In addition, several properties of these notions and connections to several other known ones are provided.

Keywords and phrases: \( \tau_i \tau_j \cdot M \)-closed set, \( \tau_i \tau_j \cdot Mcl(A) \), \( \tau_i \tau_j \cdot Mint(A) \).


1 Introduction and Preliminaries

Levine in 1963 initiated a new types of open set called semiopen set [8]. A subset \( A \) of a space \((X, \tau)\) is called regular open (resp., regular closed) [10] if \( A = \text{int}(\text{cl}(A)) \) (resp., \( A = \text{cl}(\text{int}(A)) \)). The delta interior [3] of a subset \( A \) of \((X, \tau)\) is the union of all regular open sets of \( X \) contained in \( A \) and is denoted by \( \delta \text{int}(A) \). A subset \( A \) of a space \((X, \tau)\) is called \( \delta \)-open [9] if \( A = \delta \text{int}(A) \). The complement of \( \delta \)-open set is called \( \delta \)-closed. Alternatively, a set \( A \) of \((X, \tau)\) is called \( \delta \)-closed [3] if \( A = \delta \text{cl}(A) \), where \( \delta \text{cl}(A) = \{ x \in X : A \cap \text{int}(\text{cl}(U)) \neq \phi, U \in \tau \} \). A subset \( A \) of a space \( X \) is called \( \theta \)-open [1] if \( A = \theta \text{int}(A) \), where \( \theta \text{int}(A) = \{ U : U \subseteq A, \ U \in \tau \} \), and a subset \( A \) is called \( \theta \)-semiopen [2] (resp., \( \delta \)-preopen [9], \( e \)-open [4] and \( M \)-open [5]) if \( A \subseteq \text{cl}(\theta \text{int}(A)) \) (resp., \( A \subseteq \text{cl}(\delta \text{int}(A)) \)) and \( A \subseteq \text{cl}(\theta \text{int}(A)) \cup \text{int}(\delta \text{cl}(A)) \). Clearly \( A \) is \( \delta \)-interior, \( \delta \)-interior, \( \delta \)-interior and \( \delta \)-closure operations, respectively. The notion of bitopological spaces (in short, Bts’s) was first introduced by kelly [6].

Through out this paper, Let \((X, \tau_1, \tau_2)\) or simply \( X \) be a Bts and \( i, j \in \{1,2\} \). A subset \( S \) of a Bts \( X \) is said to be \( \tau_{1,2} \)-open [7] if \( S = A \cup B \) where \( A \in \tau_1 \) and \( B \in \tau_2 \). A subset \( S \) of \( X \) is said to be \( \tau_{1,2} \)-closed if the complement of \( S \) is \( \tau_{1,2} \)-open. and \( \tau_{1,2} \)-clopen if \( S \) is both \( \tau_{1,2} \)-open and \( \tau_{1,2} \)-closed. For a subset \( A \) of \( X \), the interior (resp., closure) of \( A \) with respect to \( \tau_i \) will be denoted by \( \text{int}_i(A) \) (resp., \( \text{cl}_i(A) \)) for \( i = 1,2 \). In this paper, we introduce and investigate the concept of \( \tau_i \tau_j \cdot M \)-closed sets which are introduced in a bitopological spaces in analogy with \( M \)closed sets in topological spaces. Also introduce \( M \)closure and \( M \)interior operators in bitopological spaces. In addition, several properties of these notions and connections to several other known ones are provided.

2 \( M \)-open sets and their properties in bitopological spaces

Definition 2.1 Let \((X, \tau_1, \tau_2)\) be a Bts. A subset \( A \) of \( X \) is called \( \tau_i \tau_j \cdot M \)-open (briefly, \( \tau_i \tau_j \cdot M \)-o) if \( A \subseteq \text{cl}_i(\theta \text{int}_i(A)) \cup \text{int}_i(\delta \text{cl}_i(A)) \) and \( A \) is a \( \tau_i \tau_j \cdot M \)-closed (in short, \( \tau_i \tau_j \cdot M \)-c) if \( X \setminus A \) is \( \tau_i \tau_j \cdot M \)-o. A is pairwise \( M \)-open if it is both \( \tau_i \tau_j \cdot M \)-o and \( \tau_j \tau_i \cdot M \)-o. Clearly \( A \) is \( \tau_i \tau_j \cdot M \)-c if and only if \( \text{int}_j(\theta \text{cl}_j(A)) \cap \text{cl}_j(\delta \text{int}_j(A)) \subseteq A \). We denote the family of all \((i,j) \cdot M \)-o (resp., \((i,j) \cdot M \)-o) sets in a Bts \((X, \tau_1, \tau_2)\) by \( D_{MC}(\tau_i, \tau_j) \) (resp., \( D_{MO}(\tau_i, \tau_j) \)).
Definition 2.2 Let \((X,\tau_1,\tau_2)\) be a Bts. A subset \(A\) of \(X\) is called \(\tau_i\theta\)-semiopen (briefly, \(\tau_i\theta\)-so) if \(A \subseteq cl_i(\theta t_i(A))\), \(\tau_i\delta\)-preopen (briefly, \(\tau_i\delta\)-po) if \(A \subseteq int_i(\delta cl_i(A))\), \(\tau_i\text{-}e\)-open if \(A \subseteq cl_i(\delta int_i(A)) \cup int_i(\delta cl_i(A))\).

Proposition 2.1 The following implications hold:

1. \(\tau_i\text{-}o \Rightarrow \tau_i\text{-}so \Rightarrow \tau_i\text{-}M-o \Rightarrow \tau_i\text{-}e-o\).
2. \(\tau_i\text{-}o \Rightarrow \tau_i\text{-}\delta-po \Rightarrow \tau_i\text{-}M-o\).

The converse of these implications need not be true as shown by the following examples,

Example 2.1 In Bts's \((X,\tau_1,\tau_2)\) and \((X,\tau_3,\tau_4)\), \(X = \{a,b,c,d\}\), \(\tau_1 = \{\phi,X,\{a\},\{b\},\{a,b\},\{a,b,c\}\}\), \(\tau_2 = \{\phi,X,\{a\},\{b\},\{a,b\},\{d\},\{a,d\},\{b,c\}\}\) and \(\tau_3 = \{\phi,X,\{a\},\{b\},\{a,b\},\{a,c\},\{a,b,d\}\}\). Then the set \(\{b,c\}\) is a \(\tau_1\text{-}o\) set that is not a \(\tau_1\text{-}\theta\)-set; \(\{b,c\}\) is a \(\tau_1\tau_2\text{-}so\) set that is not a \(\tau_1\text{-}\theta\)-set; \(\{a,d\}\) is a \(\tau_1\tau_2\text{-}\delta\)-po set that is not a \(\tau_1\text{-}o\) set; \(\{a,b\}\) is a \(\tau_1\tau_2\text{-}M\)-o set that is not a \(\tau_1\tau_2\text{-}\theta\)-so set; \(\{a,c\}\) is a \(\tau_1\tau_2\text{-}e\)-o set that is not a \(\tau_1\tau_2\text{-}M\)-o set; \(\{a,b,d\}\) is a \(\tau_3\tau_4\text{-}M\)-o set that is not a \(\tau_3\tau_4\text{-}\delta\)-po set.

Remark 2.1 \(\tau_1\tau_2\text{-}o\) sets and \(\tau_1\tau_2\text{-}M\)-o sets are independent of each other as seen from this following example.

Example 2.2 In Example 2.1, the subsets \(\{b,d\}\) is \(\tau_1\tau_2\text{-}o\) but not \(\tau_1\tau_2\text{-}M\)-o set and the subsets \(\{a,b\}\) is \(M\)-o but not \(\tau_1\tau_2\text{-}o\) set.

Remark 2.2 According the Definitions 2, 2 and Proposition 2, the following diagram holds for a subset \(A\) of a space \(X\):

![Diagram](https://via.placeholder.com/150)

Note: \(A \rightarrow B\) denotes \(A\) implies \(B\), but not conversely.

Theorem 2.1 In Bts \((X,\tau_1,\tau_2)\), (1) Arbitrary union of \(\tau_1\tau_2\text{-}M\)-o sets are \(\tau_1\tau_2\text{-}M\)-o. (2) The intersection of an \(\tau_1\tau_2\text{-}M\)-o set with an \(\tau_1\text{-}o\) set is an \(\tau_1\tau_2\text{-}M\)-o set. (3) The intersection of arbitrary \(\tau_1\tau_2\text{-}M\)-o sets is \(\tau_1\tau_2\text{-}M\)-o.

Proof. (1) Let \(\{A_i,i \in I\}\) be a family of \(\tau_1\tau_2\text{-}M\)-o sets. Then \(A_i \subseteq cl_i(\theta t_i(A_i)) \cup int_i(\delta cl_i(A_i))\), hence \(U_i A_i \subseteq U_i \{cl_i(\theta t_i(A_i)) \cup int_i(\delta cl_i(A_i))\} \subseteq cl_i(\theta t_i(U_i A_i)) \cup int_i(\delta cl_i(U_i A_i))\), for all \(i \in I\). Thus \(U_i A_i\) is \(\tau_1\tau_2\text{-}M\)-o.

(2) and (3) are obvious.

Remark 2.3 The intersection of any two \(\tau_1\tau_2\text{-}M\)-o sets is not \(\tau_1\tau_2\text{-}M\)-o set, in Example 2(`)@, the sets \(A = \{a,b,c\}\) and \(B = \{a,c,d\}\) are \(\tau_1\tau_2\text{-}M\)-o sets but \(A \cap B = \{a,c\}\) is not \(\tau_1\tau_2\text{-}M\)-o set.

Remark 2.4 The family \(D_{MC}(\tau_1,\tau_2)\) is generally not equal to the family \(D_{MC}(\tau_2,\tau_1)\) as seen from the following example.
Example 2.3 In Example 2.1 the family $D_{MC}(\tau_3, \tau_4) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{d\}, \{c, d\}, \{b, c\}, \{a, d\}, \{a, c\}, \{a, b\}, \{d, c\}, \{a, d\}, \{b, c\}, \{a, c\}, \{b, d\}\} \cap \{\phi, X, \{a\}, \{b\}, \{c\}, \{d\}, \{a, c\}, \{b, d\}\}$. Therefore $D_{MC}(\tau_1, \tau_2) \neq D_{MC}(\tau_2, \tau_1)$.

Theorem 2.2 In a Bts $(X, \tau_1, \tau_2)$, $\tau_1 \subseteq \tau_2$ and $M$-open $(X, \tau_1) \subseteq M$-open $(X, \tau_2)$ then $D_{MC}(\tau_2, \tau_1) \subseteq D_{MC}(\tau_1, \tau_2)$.

Proof. Let $A \in D_{MC}(\tau_2, \tau_1)$ that is $A$ is an $M$-c set. To prove that $A \in D_{MC}(\tau_1, \tau_2)$. Let $G \in M$-open $(X, \tau_1)$ be such that $A \subseteq G$. Since $M$-open $(X, \tau_1) \subseteq M$-open $(X, \tau_2)$, we have $G \in M$-open $(X, \tau_2)$. As $A$ is a $(\tau_2, \tau_1)$-M-c set, we have $\tau_1 \delta pcl(A) \subseteq G$. Since $\tau_1 \subseteq \tau_2$, we have $\tau_2 \delta pcl(A) \subseteq \tau_1 \delta pcl(A)$ and it follows that $\tau_2 \delta pcl(A) \subseteq G$. Hence $A$ is a $(\tau_1, \tau_2)$-M-c. That is $A \in D_{MC}(\tau_1, \tau_2)$. Therefore $D_{MC}(\tau_2, \tau_1) \subseteq D_{MC}(\tau_1, \tau_2)$.

Theorem 2.3 Let $A$ and $B$ be subsets of $(X, \tau_1, \tau_2)$ such that $A \subseteq B$. If $A$ is an $\tau_{12}$-M-o set in $(X, \tau_1, \tau_2)$, then $A$ is an $\tau_{12}$-M-o set in $(B, \tau_1 \setminus B, \tau_2 \setminus B)$.

Proof. If $A$ is an $\tau_{12}$-M-o set in $X$,

$$A \subseteq int_1(cl_1(A)) \cup cl_1(int_1(A))$$

$$A = int_1(B) \cup cl_1(int_1(A)) \cap B \cup (cl_1(int_1(A)) \cap B)$$

Hence $A$ is an $\tau_{12}$-M-open set in $(B, \tau_1 \setminus B, \tau_2 \setminus B)$.

The converse of the Theorem 2.3 need not be true as shown by the following example, even when $A \in \tau_i$.

Example 2.4 In Example 2.1, $A = \{d\} \in \tau_1 \setminus B$ where $B = \{a, b, d\}$. Hence $A$ is an $\tau_{12}$-M-o set in $(B, \tau_1 \setminus B, \tau_2 \setminus B)$, but not an $\tau_{12}$-M-o set in $(X, \tau_1, \tau_2)$.

Definition 2.3 Let $A$ be a subset of $(X, \tau_1, \tau_2)$. Then

1. The intersection of all $\tau_{1j}$-M-c sets containing $A$ is called the $\tau_{1j}$-M closure of $A$, denoted by $\tau_{1j}cli(A)$, i.e., $\tau_{1j}cli(A) = \bigcap \{U: A \subseteq U, U \in D_{MC}(\tau_{1j}, \tau_j)\}$.
2. The union of all $\tau_{1j}$-M-o sets contained in $A$ is called the $\tau_{1j}$-M interior of $A$, denoted by $\tau_{1j}ini(A)$, i.e., $\tau_{1j}ini(A) = \bigcup \{U: U \subseteq A, U \in D_{MO}(\tau_{1j}, \tau_j)\}$.

Theorem 2.4 Let $A$ and $B$ be subsets of $(X, \tau_1, \tau_2)$ and $x \in X$. Then,

1. $A$ is $\tau_{1j}$-M-c if and only if $\tau_{1j}cli(A) = A$.
2. $A$ is $\tau_{1j}$-M-o if and only if $\tau_{1j}ini(A) = A$.
3. $x \in \tau_{1j}cli(A)$ if and only if for every $\tau_{1j}$-M-o set $U$ containing $x$, $U \cap A \neq \phi$.
4. $x \in \tau_{1j}ini(A)$ if and only if there exists an $\tau_{1j}$-M-o set $U$ such that $x \in U \subseteq A$.
5. If $A \subseteq B$, then $\tau_{1j}ini(A) \subseteq \tau_{1j}ini(B)$ and $\tau_{1j}cli(A) \subseteq \tau_{1j}cli(B)$.

Theorem 2.5 Let $\{A_\alpha: \alpha \in \Delta\}$ be a family of subsets of $X$. Then

1. $\tau_{1j}cli(\bigcap \{A_\alpha: \alpha \in \Delta\}) \subseteq \phi \cap \{\tau_{1j}cli(A_\alpha): \alpha \in \Delta\}$.
2. $\bigcup \{\tau_{1j}cli(A_\alpha: \alpha \in \Delta\}) \subseteq \tau_{1j}cli(\bigcup \{A_\alpha: \alpha \in \Delta\})$.

Theorem 2.6 The following are equivalent for a subset $A$ of $X$:

1. $A$ is $\tau_{1j}$-M-o.
2. $A = \tau_{1j}-\delta int(A) \cup \tau_{1j}-\delta sint(A)$.
3. $A \subseteq \tau_{1j}-\delta pcl(\tau_{1j}-\delta int(A))$.

Proof. (1) $\Rightarrow$ (2): Let $A$ be an $\tau_{1j}$-M-o set. Then $A \subseteq cl_1(\theta int_1(A)) \cup int_1(\delta cl_1(A))$ and $\tau_{1j}-\delta int(A) \cup \tau_{1j}-\delta sint(A) = (A \cap cl_1(\theta int_1(A))) \cup (A \cap int_1(\delta cl_1(A))) = A \cap (cl_1(\theta int_1(A)) \cup int_1(\delta cl_1(A))) = A$. Hence (2) holds.

(2) $\Rightarrow$ (3): $A = \tau_{1j}-\delta int(A) \cup \tau_{1j}-\delta sint(A) = \tau_{1j}-\delta int(A) \cup (A \cap cl_1(\theta int_1(A))) \subseteq$
\[ \tau_i \tau_j - \text{dpint}(A) \cup cl_j(\theta \text{int}_i(A)). \] Now since \( \tau_i \tau_j - \text{dpint}(A) \subseteq \tau_i \tau_j - \text{dpcl}(\tau_i \tau_j - \text{dpint}(A)) \), \( cl_j(B) \subseteq \tau_i \tau_j - \text{dpcl}(B) \) and \( \text{int}_i(B) \subseteq \tau_i \tau_j - \text{dpcl}(B) \) for every subset \( B \subseteq X \), then \( A \subseteq \tau_i \tau_j - \text{dpcl}(\tau_i \tau_j - \text{dpint}(A)) \). Thus (3) holds.

(3) \Rightarrow (1): We have \( A \subseteq \tau_i \tau_j - \text{dpcl}(\tau_i \tau_j - \text{dpint}(A)) = \tau_i \tau_j - \text{dpint}(A) \cup cl_j(\theta \text{int}_i(A)) \subseteq cl_j(\theta \text{int}_i(A)) \cup \text{int}_i(\delta \text{cl}_j(A)). \) Thus (1) holds.

**Corollary 2.1** The following are equivalent for a subset \( A \) of \( X \):
1. \( A \) is \( \tau_i \tau_j - \text{M-c} \).
2. \( A = \tau_i \tau_j - \text{dpcl}(A) \cup \tau_i \tau_j - \text{scl}(A) \).
3. \( A \subseteq \tau_i \tau_j - \text{dpint}(\tau_i \tau_j - \text{dpcl}(A)) \).

**Corollary 2.2** The following hold:
1. Every \( \tau_i \tau_j - \text{M-o} \)-set is a disjoint union of an \( \tau_i \tau_j - \text{\delta-po} \)-set and an \( \tau_i \tau_j - \theta \)-so set.
2. If \( A \) is an \( \tau_i \tau_j - \text{M-o} \)-set and \( \theta \text{int}_i(A) = \phi \), then \( A \) is an \( \tau_i \tau_j - \text{\delta-po} \)-set.

**Proof.** 1. follows from part (2) of Corollary 3(*) and the fact that, 
\[ \tau_i \tau_j - \text{dpint}(A) \cap \tau_i \tau_j - \text{\deltaint}(A) = \tau_i \tau_j - \text{dpint}(A) \setminus (A \cap \text{cl}_j(\theta \text{int}_i(A))) \]
\[ = \tau_i \tau_j - \text{dpint}(A) \setminus (\text{cl}_j(\theta \text{int}_i(A))) \],
which is \( \tau_i \tau_j - \text{\delta-po} \).
2. Obvious.

**Theorem 2.7** For a subset \( A \) of \( X \):
1. \( \tau_i \tau_j - \text{Mcl}(A) = \tau_i \tau_j - \text{scl}(A) \cap \tau_i \tau_j - \text{dpcl}(A) \).
2. \( \tau_i \tau_j - \text{Mint}(A) = \tau_i \tau_j - \text{\deltaint}(A) \cup \tau_i \tau_j - \text{dpint}(A) \).

**Proof.** We only prove part (1), as the proof of (2) is similar. Clearly, \( \tau_i \tau_j - \text{Mcl}(A) \subseteq \tau_i \tau_j - \text{scl}(A) \cap \tau_i \tau_j - \text{dpcl}(A) \). Moreover, as \( \tau_i \tau_j - \text{Mcl}(A) \) is \( \tau_i \tau_j - \text{\M-c} \), \( \tau_i \tau_j - \text{Mcl}(A) \supseteq \text{int}_i(\theta \text{cl}_j(\tau_i \tau_j - \text{Mcl}(A))) \cap \text{clj}(\delta \text{int}(\tau_i \tau_j - \text{Mcl}(A))) \) \supseteq \text{int}_i(\theta \text{cl}_j(A)) \cap \text{clj}(\delta \text{int}(A)). \) Thus \( \tau_i \tau_j - \text{Mcl}(A) \supseteq A \cup \text{int}(\theta \text{cl}_j(A)) = \tau_i \tau_j - \text{scl}(A) \cap \tau_i \tau_j - \text{dcl}(A) \).

**Corollary 2.3** For a subset \( A \) of \( X \):
1. \( \tau_i \tau_j - \text{Mcl}(\theta \text{int}_i(A)) = \theta \text{int}_i(\tau_i \tau_j - \text{Mcl}(A)) = \theta \text{int}_i(\text{cl}_j(\theta \text{int}_i(A))) \).
2. \( \tau_i \tau_j - \text{Mint}(\delta \text{cl}_j(A)) = \delta \text{cl}_j(\tau_i \tau_j - \text{Mint}(A)) = \delta \text{cl}_j(\text{int}_i(\delta \text{cl}_j(A))) \).
3. \( \tau_i \tau_j - \text{Mcl}(\tau_i \tau_j - \text{\deltaint}(A)) = \tau_i \tau_j - \text{scl}(\tau_i \tau_j - \text{\deltaint}(A)) \).
4. \( \tau_i \tau_j - \text{Mint}(\tau_i \tau_j - \text{\deltaint}(A)) = \tau_i \tau_j - \text{\deltaint}(\tau_i \tau_j - \text{\deltaint}(A)) \).
5. \( \tau_i \tau_j - \text{dint}(\tau_i \tau_j - \text{Mcl}(A)) = \tau_i \tau_j - \text{\deltaint}(\tau_i \tau_j - \text{\deltaint}(A)) \).
6. \( \tau_i \tau_j - \text{dcl}(\tau_i \tau_j - \text{Mcl}(A)) = \tau_i \tau_j - \text{\deltaint}(\tau_i \tau_j - \text{\deltaint}(A)) \).
7. \( \tau_i \tau_j - \text{dint}(\tau_i \tau_j - \text{Mcl}(A)) = \tau_i \tau_j - \text{Mcl}(\tau_i \tau_j - \text{dint}(A)) = \tau_i \tau_j - \text{dint}(\tau_i \tau_j - \text{dcl}(A)) \).
8. \( \tau_i \tau_j - \text{dcl}(\tau_i \tau_j - \text{Mint}(A)) = \tau_i \tau_j - \text{Mint}(\tau_i \tau_j - \text{dcl}(A)) = \tau_i \tau_j - \text{dcl}(\tau_i \tau_j - \text{dint}(A)) \).
9. \( \tau_i \tau_j - \text{dint}(\tau_i \tau_j - \text{Mcl}(A)) = \tau_i \tau_j - \text{Mcl}(\tau_i \tau_j - \text{dint}(A)) = \tau_i \tau_j - \text{dint}(\tau_i \tau_j - \text{dcl}(A)) \).
10. \( \tau_i \tau_j - \text{dcl}(\tau_i \tau_j - \text{Mint}(A)) = \tau_i \tau_j - \text{Mint}(\tau_i \tau_j - \text{dcl}(A)) = \tau_i \tau_j - \text{dcl}(\tau_i \tau_j - \text{dint}(A)) \).
11. \( \tau_i \tau_j - \text{Mint}(\tau_i \tau_j - \text{Mcl}(A)) = \tau_i \tau_j - \text{Mcl}(\tau_i \tau_j - \text{Mint}(A)) \).

**Theorem 2.8.** If \( A \) and \( B \) be subsets of \( X \). Then
1. \( \tau_i \tau_j - \text{Mcl}(X) = X \) and \( \tau_i \tau_j - \text{Mcl}(\phi) = \phi \).
2. \( A \subseteq \tau_i \tau_j - \text{Mcl}(A) \).
3. If \( B \) is any \( \tau_i \tau_j - \text{\M-c} \)-set containing \( A \), then \( \tau_i \tau_j - \text{Mcl}(A) \subseteq B \).

**Proof.** Follows from Definition 2(*)@
Theorem 2.9. Let A and B be subsets of X and i, j ∈ {1, 2} be fixed integers. If A ⊆ B, then \( \tau_{ij} \cdot \text{Mcl}(A) \subseteq \tau_{ij} \cdot \text{Mcl}(B) \).

Proof. Let A ⊆ B. By Definition 2(8) \( \tau_{ij} \cdot \text{Mcl}(B) = \bigcap \{F : B \subseteq F \in D_{MC}(\tau_{ij})\} \). If B ⊆ F ∈ D_{MC}(\tau_{ij}), since A ⊆ B, A ⊆ B ⊆ F ∈ D_{MC}(\tau_{ij})}, we have \( \tau_{ij} \cdot \text{Mcl}(A) \subseteq F \). Therefore \( \tau_{ij} \cdot \text{Mcl}(A) \subseteq \bigcap \{F : B \subseteq F \in D_{MC}(\tau_{ij})\} = \tau_{ij} \cdot \text{Mcl}(B) \). That is \( \tau_{ij} \cdot \text{Mcl}(A) \subseteq \tau_{ij} \cdot \text{Mcl}(B) \).

Theorem 2.10. Let A be a subset of X. If \( \tau_1 \subseteq \tau_2 \) and M-o \( (X, \tau_1) \subseteq M-o \( (X, \tau_2) \), then \( (\tau_1, \tau_2) \cdot \text{Mcl}(A) \subseteq (\tau_2, \tau_1) \cdot \text{Mcl}(A) \).

Proof. By Definition 2.3, \( (\tau_1, \tau_2) \cdot \text{Mcl}(A) = \bigcap \{F : A \subseteq F \in D_{MC}(\tau_1, \tau_2)\} \). Since \( \tau_1 \subseteq \tau_2 \) and M-open \( (X, \tau_1) \subseteq M \cdot \text{open} \( (X, \tau_2) \) in \( (X, \tau_1, \tau_2) \) then \( D_{MC}(\tau_2, \tau_1) \subseteq D_{MC}(\tau_1, \tau_2) \). Therefore \( \bigcap \{F : A \subseteq F \in D_{MC}(\tau_1, \tau_2)\} \subseteq \bigcap \{F : A \subseteq F \in D_{MC}(\tau_2, \tau_1)\} = (\tau_2, \tau_1) \cdot \text{Mcl}(A) \). Hence \( (\tau_1, \tau_2) \cdot \text{Mcl}(A) \subseteq (\tau_2, \tau_1) \cdot \text{Mcl}(A) \).

Theorem 2.11. Let A be a subset of X and i, j ∈ {1, 2} be fixed integers, then \( \tau_{ij} \cdot \text{Mcl}(A) \subseteq \tau_{i} \cdot \text{cl}(A) \).

Proof. By Definition 2.3, it follows that \( \tau_{ij} \cdot \text{Mcl}(A) \subseteq \tau_{i} \cdot \text{cl}(A) \). Now to prove that \( \tau_{ij} \cdot \text{Mcl}(A) \subseteq \tau_{i} \cdot \text{cl}(A) \). By Definition of closure, \( \tau_{i} \cdot \text{cl}(A) = \{F \subseteq X : A \subseteq F \) and \( F \) is \( \tau_{i} \cdot \text{c} \). If \( A \subseteq F \) and \( F \) is \( \tau_{i} \cdot \text{c} \), then \( \tau_{ij} \cdot \text{Mcl}(A) \subseteq \{F \subseteq X : A \subseteq F \) and \( F \) is \( \tau_{i} \cdot \text{c} \). Therefore \( \tau_{ij} \cdot \text{Mcl}(A) \subseteq \bigcap \{F \subseteq X : A \subseteq F \) and \( F \) is \( \tau_{i} \cdot \text{c} \} = \tau_{i} \cdot \text{cl}(A) \).

Example 2.5. Let \( X = \{a, b, c, d\} \), \( \tau_1 = \{\phi, X, \{a\}, \{b\}, \{c\}, \{d\}\} \) and \( \tau_2 = \{\phi, X, \{a\}, \{b, d\}\} \). Then \( \tau_2 \cdot \text{c} \) sets are \( \{X, \phi, \{a, c\}, \{a, c, d\}\} \) and \( (12) \cdot \text{c} \) sets are \( \{\phi, X, \{a\}, \{a, c\}, \{a, c, d\}\} \). Take \( A = \{b\} \). Then \( \tau_2 \cdot \text{cl}(A) = X \) and \( (12) \cdot \text{Mcl}(A) = \{b, c, d\}. \) Now \( A \subseteq (12) \cdot \text{Mcl}(A) \), but \( A \not\subseteq (12) \cdot \text{Mcl}(A) \).

Theorem 2.12. Let A be a subset of X and i, j ∈ {1, 2} be fixed integers. If A is \( \tau_{ij} \cdot \text{M} \), then \( \tau_{ij} \cdot \text{Mcl}(A) = A \).

Proof. Let A be a \( \tau_{ij} \cdot \text{M} \) subset of X. We know that \( A \subseteq \tau_{ij} \cdot \text{Mcl}(A) \). Also \( A \subseteq A \) and \( A \) is \( \tau_{ij} \cdot \text{M} \). By Theorem 11(8)(iii), \( \tau_{ij} \cdot \text{Mcl}(A) \subseteq A \). Hence \( \tau_{ij} \cdot \text{Mcl}(A) = A \).

Theorem 2.13. The operator \( \tau_{ij} \cdot \text{Mcl} \) in Definition 2, (i) is the Kuratowski closure operator on X.

Proof. (i) \( \tau_{ij} \cdot \text{Mcl}(\phi) = \phi \), by Theorem 11(8)(i).

(ii) \( E \subseteq \tau_{ij} \cdot \text{Mcl}(E) \) for any subset E in X by Theorem 11(8)(i).

(iii) Suppose E and F are two subsets of X. It follows from Theorem 3(8)(ii), that \( \tau_{ij} \cdot \text{Mcl}(E \cup F) \subseteq \tau_{ij} \cdot \text{Mcl}(E) \cup \tau_{ij} \cdot \text{Mcl}(F) \) and that \( \tau_{ij} \cdot \text{Mcl}(E) \subseteq \tau_{ij} \cdot \text{Mcl}(E \cup F) \). Hence we have \( \tau_{ij} \cdot \text{Mcl}(E) \subseteq \tau_{ij} \cdot \text{Mcl}(E \cup F) \) and \( \tau_{ij} \cdot \text{Mcl}(F) \subseteq \tau_{ij} \cdot \text{Mcl}(E \cup F) \). Now if \( x \not\in \tau_{ij} \cdot \text{Mcl}(E) \cup \tau_{ij} \cdot \text{Mcl}(F) \), then \( x \not\in (i, j) \cdot \text{Mcl}(E) \) and \( x \not\in (i, j) \cdot \text{Mcl}(F) \), it follows that there exist \( A, B \in D_{MC}(\tau_{ij}) \) such that \( E \subseteq A, x \not\in A \) and \( F \subseteq B, x \not\in B \). Hence \( E \cup F \subseteq A \cup B, x \not\in A \cup B \). Since \( A \cup B \) is \( \tau_{ij} \cdot \text{M} \) and \( A, B \in D_{MC}(\tau_{ij}) \), then \( A \cup B \in D_{MC}(\tau_{ij}) \) so \( x \not\in \tau_{ij} \cdot \text{Mcl}(E \cup F) \). Then we have \( \tau_{ij} \cdot \text{Mcl}(E \cup F) \subseteq \tau_{ij} \cdot \text{Mcl}(E) \cup \tau_{ij} \cdot \text{Mcl}(F) \). From the above discussions we have \( \tau_{ij} \cdot \text{Mcl}(E \cup F) = \tau_{ij} \cdot \text{Mcl}(E) \cup \tau_{ij} \cdot \text{Mcl}(F) \).

(iv) Let E be any subset of X. By the definition of \( \tau_{ij} \cdot \text{Mcl}, \tau_{ij} \cdot \text{Mcl}(E) = \bigcap \{A : E \subseteq A \in D_{MC}(\tau_{ij})\} \). If \( A \subseteq E \in D_{MC}(\tau_{ij}) \), then \( \tau_{ij} \cdot \text{Mcl}(E) \subseteq A \). Since A is a \( \tau_{ij} \cdot \text{M} \) set containing \( \tau_{ij} \cdot \text{Mcl}(E) \), by Theorem 11(8)(iii), \( \tau_{ij} \cdot \text{Mcl}(\tau_{ij} \cdot \text{Mcl}(E)) \subseteq A \). Hence \( \tau_{ij} \cdot \text{Mcl}(E) \subseteq \tau_{ij} \cdot \text{Mcl}(\tau_{ij} \cdot \text{Mcl}(E)) \) is true by Theorem 11(8)(iii). Then we have \( \tau_{ij} \cdot \text{Mcl}(E) = \tau_{ij} \cdot \text{Mcl}(\tau_{ij} \cdot \text{Mcl}(E)) \). Hence \( \tau_{ij} \cdot \text{Mcl} \) is a Kuratowski closure operator on X.

From this theorem \( \tau_{ij} \cdot \text{Mcl} \) defines the new topology on X.

Definition 2.4. Let i, j ∈ {1, 2} be two fixed integers. Let \( \tau_{ij} \cdot \text{M} \) be topology on X. 
generated by \((\tau_i, \tau_j) - M\ cl\) in the usual manner. That is \(\tau_M - (\tau_i, \tau_j) = \{E \subseteq X: (\tau_i, \tau_j) - M\ cl(E^c) = E^c\}.

**Theorem 2.14.** Let \(X\) be a Bts and \(i, j \in \{1, 2\}\) be two fixed integers, then \(\tau_i \subseteq \tau_M(\tau_i, \tau_j)\).

**Proof.** Let \(G \in \tau_i\), it follows that \(G^c\) is \(\tau_i\)-c. By Proposition 2 (ii), \(G^c\) is \((\tau_i, \tau_j) - M\)-c. Therefore \((\tau_i, \tau_j) - M\ cl(G^c) = G^c\), by Theorem 3(')@. That is \(G \in \tau_M(\tau_i, \tau_j)\) and hence \(\tau_i \subseteq \tau_M(\tau_i, \tau_j)\).

**Remark 2.5.** Containment relation in the above Theorem 3(')@ may be proper as seen from the following Example.

**Example 2.6** Let \(X = \{a, b, c, d\}, \tau_1 = \{\phi, X, \{a\}, \{b, c\}, \{a, b, c\}\} \) and \(\tau_2 = \{\phi, X, \{b\}, \{b, d\}\}\). Then \((\tau_1, \tau_2) - M\ c\ sets\ are\ \{\phi, X, \{a\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{c, d\}, \{b, c, d\}, \{a, b, c, d\}\} \) and \(\tau_M(\tau_1, \tau_2) = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{b, d\}, \{b, c, d\}, \{a, b, d\}\}\). Clearly \(\tau_2 \subseteq \tau_M(\tau_1, \tau_2)\) but \(\tau_2 \neq \tau_M(\tau_1, \tau_2)\).

**Theorem 2.15** Let \((X, \tau_1, \tau_2)\) be a Bts and \(i, j \in \{1, 2\}\) be two fixed integers. If a subset \(E\) of \(X\) is \(\tau_i\tau_j\-M\ c\) then \(E\) is \(\tau_M(\tau_i, \tau_j)\).

**Proof.** Let a subset \(E\) of \(X\) be \(\tau_i\tau_j\-M\ c\). By Theorem 3('@) \(\tau_i\tau_j\-M\ cl(E) = E\). That is \(\tau_i\tau_j\-M\ cl\{E^c\} = (E^c)^c\), it follows that \(E^c \in \tau_M(\tau_i, \tau_j)\). Therefore \(E\) is \(\tau_M(\tau_i, \tau_j)\).

**Example 2.7** For \((X, \tau_1, \tau_2)\) of Example 2.6, the subset \(A = \{b, c\}\) is \(\tau_M(\tau_1, \tau_2)\), but not \(\tau_1\tau_2\-M\ c\).

**Theorem 2.16.** If \(\tau_1 \subseteq \tau_2\) and \(M\-open\ \(X, \tau_1) \subseteq M\-open\ \(X, \tau_2)\) in \(X\), then \(\tau_M(\tau_2, \tau_1) \subseteq \tau_M(\tau_1, \tau_2)\).

**Proof.** Let \(G \in \tau_M(\tau_2, \tau_1)\), then \((\tau_2, \tau_1) - M\ cl(G^c) = G^c\). To prove that \(G \in \tau_M(\tau_1, \tau_2)\). That is to prove \((\tau_1, \tau_2) - M\ cl(G^c) = G^c\). Now \((\tau_1, \tau_2) - M\ cl(G^c) = \cap \{F \subseteq X: G^c \subseteq F \in D_M(\tau_1, \tau_2)\}\). Since \(\tau_1 \subseteq \tau_2\) and \(M\-open\ \(X, \tau_1) \subseteq M\-open\ \(X, \tau_2)\), by Therom 2('@) \(D_M(\tau_2, \tau_1) \subseteq D_M(\tau_1, \tau_2)\). Thus \(\cap \{F \subseteq X: G^c \subseteq F \in D_M(1,2)\} \subseteq \cap \{F \subseteq X: G^c \subseteq F \in D_M(2,1)\}\). That is \((\tau_1, \tau_2) - M\ cl(G^c) \subseteq (\tau_2, \tau_1) - M\ cl(G^c)\), and so \((\tau_1, \tau_2) - M\ cl(G^c) \subseteq G^c\). Conversely \(G^c \subseteq (\tau_1, \tau_2) - M\ cl(G^c)\) is true by the Theorem 11('@) (ii). Then we have \((\tau_1, \tau_2) - M\ cl(G^c) = G^c\). That is \(G \in \tau_M(\tau_1, \tau_2)\) and hence \(\tau_M(\tau_2, \tau_1) \subseteq \tau_M(\tau_1, \tau_2)\).

**References**


