ON THE LATTICE OF (δ,γ)- FUZZY IDEALS OF A LATTICE

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Abstract
In this paper we prove that for a lattice X, the family of all (δ,γ)-fuzzy ideals are also lattices.We further claim that the lattice of all (δ,γ)-fuzzy ideals is infact a sublattice of the lattice of all (δ,γ)-fuzzy sublattices.

Key words and phrases: (δ,γ)-fuzzy sets, (δ,γ)-fuzzy sublattices, (δ,γ)-fuzzy subnear-ring, fuzzy two-sided N-subgroup.

1. Introduction

The notions of fuzzy ideals were introduced by S-Abou-Zaid in 1991 [8,1]. The notion of fuzzy subgroup was introduced by A. Rosenfeld [5] in his pioneering paper. Subsequently the definition of fuzzy subgroup was generalized by Negoita and Ralescu [7]. Fuzzy ideals of a ring were first introduced by Liu[13]. T. Ali and A.K. Ray [2] studied the concepts of fuzzy sublattices and fuzzy ideals of a lattice. The notions of fuzzy subnear-ring, fuzzy ideal and fuzzy R-subgroup of a near-ring were introduced by Salah Abou-Zahid [8] and it has been studied by several authors [11,12, 3, 4] and also ,we introduce the notion of a (δ,γ)-fuzzy ideal of a near-ring and we prove a correspondence theorem between the families of (δ,γ)-fuzzy ideals of two homomorphic lattices. This is an extension of the result of M. J. Rani [10] and T. Manikantan [9].

2. Preliminaries

In this section We recall some definitions and results that will be needed in the sequel. The interval $[0,1]$ is a lattice and this entity $([0,1],\leq)$ is denoted by I.

Definition 2.1[10] Let $\mu,\nu\in I^A$. If $\mu(x)\leq\nu(x),\forall\, x\in X$, then we say that μ is contained in ν and we write $\mu\leq\nu$. Clearly the inclusion relation $\leq$ is a partial ordering on I^A.

Definition 2.2[10] Let $\mu,\nu\in I^A$. Difine $\mu\cup\nu$ and $\mu\cap\nu$ as follows.

$$\forall\, x\in A, (\mu\cup\nu)(x)=\mu(x)\vee\nu(x) \text{ and } (\mu\cap\nu)(x)=\mu(x)\wedge\nu(x)$$

Then $(\mu\cup\nu)$ and $(\mu\cap\nu)$ are respectively the lub and glb and they are called the union and intersection of μ and ν respectively. It is also known that under the natural ordering, I^A is a complete lattice for any nonempty set A. Its largest and smallest element are 1_A (where $1_A(x)=1\,\forall\, x\in A$) and 0_A (where $0_A(x)=0\,\forall\, x\in A$)

Definition 2.3[10] A fuzzy subset μ of X is said to be a fuzzy sublattice of X if $\forall\, x, y \in X$,

(i) $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$,

(ii) $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$.

Definition 2.4[10] Let $\mu \in I^X$, then μ is called a fuzzy ideal of X if $\forall\, x, y \in X$,

(I_1). $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$,

(I_2). $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$.

If I_2 holds, then $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$. Thus by I_1 and $I_2, \mu \in FL(X)$, (i.e) a fuzzy ideal of X is fuzzy sublattice of X.

Definition 2.5[8] A fuzzy sub set A of N is called a fuzzy subnear-ring of N if $\forall\, x, y \in N$,

(i) $A(x - y) \geq min\{A(x), A(y)\}$,

(ii) $A(xy) \geq min\{A(x), A(y)\}$.

Definition 2.6[5] A fuzzy sub set A of a group $(G, +)$ is said to be a fuzzy subgroup of G if $\forall\, x, y \in G$,

(i) $A(x + y) \geq min\{A(x), A(y)\}$,

(ii) $A(-x) = A(x)$, or equivalently $A(x - y) \geq min\{A(x), A(y)\}$.

If A is a fuzzy subgroupof a group G, then $A(0) \geq A(x \forall\, x \in G.)$

Definition 2.7[8] A fuzzy sub set A of N is said to be a fuzzy two-sided N-subgroup of N if

(i) A is a fuzzy subgroup of $(N, +)$,

(ii) $A(xy) \geq A(x)\ \forall\, x, y \in N$,

(iii) $A(xy) \geq A(y)\ \forall\, x, y \in N$.

If A satisfies (i),(ii) then A is called a fuzzy right N-subgroup of N. If A satisfies (i) and (iii), then A is called a fuzzy left N-subgroup of N.

Definition 2.8 [8] A fuzzy sub set A of N is said to be a fuzzy ideal of N if

(i) A is a fuzzy subnear-ring of N,

(ii) $A(y + x - y) = A(x)\ \forall\, x, y \in N$,

(iii) $A(xy) \geq A(y)\ \forall\, x, y \in N$.

(iv) $A(a(b+i) - ab) \geq A(i)\ \forall\, a, b, i \in N$.

A fuzzy subset with (i),(ii) and (iii) is called a fuzzy right ideal of N whereas a fuzzy subset with (i),(ii) and (iv) is called a fuzzy left ideal of N.

3. (δ, γ)-Fuzzy ideals of a lattice

Based on the notion of (λ, μ)-fuzzy ideals introduced by B. You [6]. In this section we introduce (δ, γ)-fuzzy ideals of lattice. In the following discussion, we always assume that $0 \leq \delta < \gamma \leq 1$.

Definition 3.1 A (δ, γ)-fuzzy subset β of X is said to be a (δ, γ)-fuzzy sublattice of X if $\forall\, x, y \in X$,

(i) $\beta(x \vee y) \vee \delta \geq \beta(x) \wedge \beta(y) \wedge \gamma$,

(ii) $\beta(x \wedge y) \vee \delta \geq \beta(x) \wedge \beta(y) \wedge \gamma$.

Definition 3.2 Let $\beta, \nu \in I^{A}_{(\delta,\gamma)}$. Difine $\beta \cup \nu$ and $\beta \cap \nu$ as follows.

$\forall x \in A, ((\beta \cup \nu)(x)) \vee \delta = (\beta(x) \vee \nu(x)) \wedge \gamma$ and $((\beta \cap \nu)(x)) \vee \delta = (\beta(x) \wedge \nu(x)) \wedge \gamma$

Then $(\beta \cup \nu)$ and $(\beta \cap \nu)$ are respectively the lub and glb and they are called the union and intersection of β and ν respectively. It is also known that under the natural ordering, $I^{A}_{(\delta,\gamma)}$ is a complete lattice for any nonempty set A. Its largest and smallest element are 1_A (where $1_A(x) = 1 \, \forall x \in A$) and 0_A (where $0_A(x) = 0 \, \forall x \in A$)

Definition 3.3 Let $\beta, \nu \in I^{A}_{(\delta,\gamma)}$. If $\beta(x) \le \nu(x), \forall x \in X$, then we say that β is contained in ν and we write $\beta \le \nu$. Clearly the inclusion relation $\le$ is a partial ordering on $I^{A}_{(\delta,\gamma)}$.

Example 3.4 If X is any lattice and $t \in I$, then $\beta(x) \vee \delta = t \wedge \gamma, \forall x \in X$ is a (δ,γ)-fuzzy sublattice of X.

Example 3.5 If X is any subset of N with usual ordering and $\beta \in I^{X}_{(\delta,\gamma)}$ is given by $\beta(x) \vee \delta = 1/x \wedge \gamma$ then β is a (δ,γ)-fuzzy sublattice of X.

Notation 3.6 $FL_{(\delta,\gamma)}(X)$ denotes the set of all (δ,γ)-fuzzy sublattice of X.

Result 3.7 If $\beta \in FL_{(\delta,\gamma)}(X)$, then the set $\beta^{*} = \{x \in X, \beta(x) \vee \delta > 0 \wedge \gamma\}$ is a (δ,γ)-fuzzy sublattice of X.

Proof. Omitted.

Let $\beta \in I^{X}$, then β is called a (δ,γ)-fuzzy ideal of X if $\forall x, y \in X$,

(I_1). $\beta(x \vee y) \vee \delta \ge \beta(x) \wedge \beta(y) \wedge \gamma$,

(I_2). $\beta(x \wedge y) \vee \delta \ge \beta(x) \wedge \beta(y) \wedge \gamma$.

If I_2 holds, then $\beta(x \wedge y) \vee \delta \ge \beta(x) \wedge \beta(y) \wedge \gamma$. Thus by I_1 and $I_2, \beta \in FL_{(\delta,\gamma)}(X)$, (i.e) a (δ,γ)-fuzzy ideal of X is (δ,γ)-fuzzy sublattice of X.

Notation 3.9 The set of all (δ,γ)-fuzzy ideals of X is denoted by $FI_{(\delta,\gamma)}(X)$. Let $\beta \in I^{X}_{(\delta,\gamma)}$ satisfies I_2 if and only if $\beta(x \wedge y) \vee \delta \ge \beta(x) \wedge \gamma, \forall x, y \in X$. Since from $I_2, \beta(x \wedge y) \vee \delta \ge \beta(x) \wedge \gamma$ and conversely if $\beta(x \wedge y) \vee \delta \ge \beta(x) \wedge \gamma, \forall x \in X$, then $\beta(x \wedge y) \vee \delta = \beta(y \wedge x) \vee \delta \ge \beta(y) \wedge \gamma, \forall x, y \in X$. Thus I_2 is equivalent to (I_3). $\beta(x \wedge y) \vee \delta \ge \beta(x) \wedge \gamma, \forall x \in X$. Hence a (δ,γ)-fuzzy sublattice β of X is (δ,γ)-fuzzy ideal of X if and only if $\beta(x \wedge y) \vee \delta \ge \beta(x) \wedge \gamma, \forall x \in X$.

Example 3.10 Consider the lattice $X = \{a, b, c, d\}$ where $a > c > d, a > b > d$ and b, c are non comparable.Where $\delta = 0.1$ and $\gamma = 0.9$. Define $\beta \in I^{X}_{(\delta,\gamma)}$ as follows.
$\beta(a) = 0.1, \beta(b) = 0.2, \beta(c) = 0.4, \beta(d) = 0.5$ then β is a (δ,γ)-fuzzy ideal of X.

Example 3.11 If X and Y with a smallest element 0 be lattice and f a lattice homomorphism of X onto Y, then $ker\ f = \{x \in X \mid f(x) = 0\}$ is an ideal of X and $\chi_{ker\ f}$ is a fuzzy ideal of X (from) and also $\chi_{ker\ f}$ is a (δ,γ)-fuzzy ideal of X.

Theorem 3.12 Let $A \subseteq X$. Then A is an (δ,γ)-fuzzy ideal of X if and only if χ_A is a (δ,γ)-fuzzy ideal of X. **Proof.** Suppose A is an (δ,γ)-fuzzy ideal of X .Therefore A is a (δ,γ)-fuzzy sublattice of X.

Proof. Suppose $x, y \in X$ be elements of A. Therefore both $x \wedge y$ and $x \vee y \in A$. Then $\chi_A(x) \vee \delta = 1 \wedge \gamma$, $\chi_A(y) \vee \delta = 1 \wedge \gamma$, $\chi_A(x \wedge y) \vee \delta = 1 \wedge \gamma$ and $\chi_A(x \vee y) \vee \delta = 1 \wedge \gamma$.

Then $\chi_A(x \vee y) \vee \delta \geq \chi_A(x) \wedge \chi_A(y) \wedge \gamma = 1 \wedge \gamma$ and $\chi_A(x \wedge y) \vee \delta \geq \chi_A(x) \vee \chi_A(y) \wedge \gamma = 1 \wedge \gamma$

Therefore χ_A is (δ, γ)-fuzzy ideal of X.

Suppose $x \in A$ and $y \notin A$. Then $\chi_A(x) \vee \delta = 1 \wedge \gamma, \chi_A(y) \vee \delta = 0 \wedge \gamma$.

But, since A is an ideal, $(x \wedge y) \vee \delta \in A \wedge \gamma$.

Now since $\chi_A(y) \vee \delta = 0 \wedge \gamma, \chi_A(x \vee y) \vee \delta \geq \chi_A(x) \wedge \chi_A(y) \wedge \gamma = 1 \wedge \gamma$ and $\chi_A(x \wedge y) \vee \delta \geq \chi_A(x) \vee \chi_A(y) \wedge \gamma = 1 \wedge \gamma$.

Thus χ_A is a (δ, γ)-fuzzy ideal of X.

Conversely, suppose that χ_A is a (δ, γ)-fuzzy ideal of X.

Let $x, y \in A$ then $\chi_A(x) \vee \delta = 1 \wedge \gamma$, $\chi_A(y) \vee \delta = 1 \wedge \gamma$.

Since $\chi_A(x \vee y) \vee \delta \geq \chi_A(x) \wedge \chi_A(y) \wedge \gamma = 1 \wedge \gamma$, $\chi_A(x \vee y) \vee \delta = 1$.

Therefore $(x \wedge y) \vee \delta \in A \wedge \gamma$.

Similarly $\chi_A(x \vee y) \vee \delta = 1 \wedge \gamma$.

Therefore $x \wedge y \in A$

Therefore A is a (δ, γ)-fuzzy sub lattice of X.

Let $x \in A$ and $y \in X$. Therefore $\chi_A(x) \vee \delta = 1 \wedge \gamma$. Then, since $\chi_A(x \wedge y) \vee \delta \geq \chi_A(x) \vee \chi_A(y) \wedge \gamma = 1 \wedge \gamma$, $x \wedge y \in A$.

Therefore A is an (δ, γ)-fuzzy ideal of X.

Theorem 3.13 $\mu \in FL_{(\delta,\gamma)}(X)$ is a (δ, γ)-fuzzy ideal of X, when $\beta(x \vee y) \vee \delta \geq a \wedge \gamma$ holds if and only if $\beta(x) \vee \delta \geq a \wedge \gamma$ and $\beta(y) \vee \delta \geq a \wedge \gamma, \forall\, a \in I$.

Proof. Suppose that $\beta(x \vee y) \vee \delta \geq a \wedge \gamma$ holds if and only if $\beta(x) \vee \delta \geq a \wedge \gamma$ and $\beta(y) \vee \delta \geq a \wedge \gamma, \forall a \in I$. Let $x, y \in X$. Let $(\beta(x) \wedge \beta(y)) \vee \delta = a \wedge \beta$. Then $\beta(x) \vee \delta \geq a \wedge \gamma$ and $\beta(y) \vee \delta \geq a \wedge \gamma$.Therefore by assumption $\beta(x \vee y) \vee \delta \geq a \wedge \gamma$.

That is $\beta(x \vee y) \vee \delta \geq (\beta(x) \wedge \beta(y)) \vee \delta$. Thus (I_1).

Let $\beta(x) = b$. Then $x \in \beta_b$. Now since X is a lattice, $x \vee \delta = (x \vee (x \wedge y)) \vee \delta$

Therefore $(x \vee (x \wedge y)) \vee \delta \in \beta_b$ That is $\beta((x \vee (x \wedge y)) \vee \delta) \geq b \wedge \gamma$

Therefore $\beta(x \vee y) \vee \delta \geq b \wedge \gamma$ by assumption that is $\beta(x \vee y) \vee \delta \geq \beta(x) \wedge \gamma$. Thus I_3. Hence $\beta \in FL_{(\delta,\gamma)}(X)$.

Theorem 3.14 Let $\beta \in FL_{(\delta,\gamma)}(X)$. Then

(i) If X has a smallest Element 0, then $\beta(0) \vee \delta \geq \beta(x) \wedge \gamma\ \forall x$.

(ii) If X has a greatest Element 1, then $\beta(1) \vee \delta \geq \beta(x) \wedge \gamma\ \forall x \in X$.

(iii) β is an (δ, γ)-fuzzy ideal of X.

Proof. (i) $\beta(0) \vee \delta = \beta(0 \wedge x) \vee \delta = \beta(x \wedge 0) \vee \delta \geq \beta(x) \wedge \gamma$ by I_3.

(ii) $\beta(1) \vee \delta = \beta(1 \wedge x) \vee \delta = \beta(x \wedge 1) \vee \delta \geq \beta(x) \wedge \gamma$ by I_3.

Therefore $\beta(x) \leq (X)$

(iii) By definition of (δ, γ)-fuzzy sublattice of X., $\beta^* = \{x \mid \beta(x) \vee \delta > 0 \wedge \beta\}$.

Let $x, y \in \beta^*$. Then both $\beta(x) \vee \delta > 0 \wedge \gamma$ and $\beta(y) \vee \delta > 0 \wedge \gamma$

Therefore $(\beta(x) \vee \beta(y)) \vee \delta > 0 \wedge \gamma$ and $(\beta(x) \wedge \beta(y)) \vee \delta > 0 \wedge \gamma$, since $(\beta(x) \vee \beta(y)) \vee \delta = max\{\beta(x), \beta(y), \delta\}$ and $(\beta(x) \wedge \beta(y)) \vee \delta = min\{\beta(x), \beta(y), \gamma\}$ in I. Now

$(x \vee y) \vee \delta \geq (\beta(x) \wedge \beta(y)) \vee \delta > 0$, since $\beta \in FI_{(\delta,\gamma)}(X)$

Therefore $x \vee y \in \beta^*$

Also $(\beta(x \wedge y)) \vee \delta \geq (\beta(x) \vee \beta(y)) \wedge \gamma > 0$

Therefore $x \wedge y \in \beta^*$. Suppose $z \notin \beta^*$

Therefore $\beta(Z) = 0.$ $x \wedge z \in \beta^*.$ Thus β^* is an ideal of X.

References

[1] S. Abou-Zaid, on Fuzzy ideals and Fuzzy quotient ring, Fuzzy sets and systems 59 (1993), 205-210.

[2] Tazid Ali, On direct product of fuzzy subrings The Journal of Fuzzy Mathematics, Vol.17,No.2,2009,481-485.

[3] A.L. Narayanan, Contribution to the algebraic structures in fuzzy theory, Ph.D. Thesis, Annamalai university, India,(2001).

[4] A.L. Narayanan, Fuzzy ideals on strongly regular near-rings, The journal of the Indian Math. Soc: Vol.69. Nos.1-4,(2002),193-199.

[5] A. Rosenfeld, Fuzzy groups, J. Math. Anal. And Appl., 35 (1971),512-517.

[6] Bingxue Yao (λ, μ)-fuzzy subrings and (λ, μ)--fuzzy ideals, The Journal of Fuzzy Mathematics Vol. 15, No. 4, (2007), 981-987.

[7] C. V. Negoita and D. A. Ralescu, Applications of fuzzy sets to system analysis, John wiley and sons, New York, Toronto, (1975).

[8] Salah Abou-Zaid, On fuzzy subnear-rings and ideals, Fuzzy Sets and Systems, 44(1991), 139-146.

[9] T.Manikantan, Fuzzy Bi-ideals of Near-rings, The Journal of Fuzzy Mathematics, Vol.17, No.3, (2009), 659-671.

[10] M. J. Rani, On The Lattice of Fuzzy Ideals of A Lattice, The Journal of Fuzzy Mathematics, Vol.16, No.4,(2008),981-989.

[11] Kyung Ho Kim and Young Bae Jun, On fuzzy R-subgroups of near-rings, J. Fuzzy Mathematics, II(3)(2003), 567-580.

[12] Kyung Ho Kim and Young Bae Jun, Normal fuzzy R-subgroups in near-rings, Fuzzy Sets and Systems,121(2001), 341-345.

[13] Wang-jin-Liu, Fuzzy invariant subgroups and fuzzy ideals, Fuzzy Sets and Systems, 8 (1982), 133-139.