ON THE LATTICE OF $((\delta, \gamma))$ - FUZZY IDEALS OF A LATTICE

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Abstract

In this paper we prove that for a lattice $X$, the family of all $((\delta, \gamma))$-fuzzy ideals are also lattices. We further claim that the lattice of all $((\delta, \gamma))$-fuzzy ideals is in fact a sublattice of the lattice of all $((\delta, \gamma))$-fuzzy sublattices.

Key words and phrases: $((\delta, \gamma))$-fuzzy sets, $((\delta, \gamma))$-fuzzy sublattices, $((\delta, \gamma))$-fuzzy subnear-ring, fuzzy two-sided $N$-subgroup.

1. Introduction

The notions of fuzzy ideals were introduced by S-Abou-Zaid in 1991 [8,1]. The notion of fuzzy subgroup was introduced by A. Rosenfeld [5] in his pioneering paper. Subsequently the definition of fuzzy subgroup was generalized by Negoita and Ralescu [7]. Fuzzy ideals of a ring were first introduced by Liu[13]. T. Ali and A.K. Ray [2] studied the concepts of fuzzy sublattices and fuzzy ideals of a lattice. The notions of fuzzy subnear-ring, fuzzy ideal and fuzzy $R$-subgroup of a near-ring were introduced by Salah Abou-Zahid [8] and it has been studied by several authors [11,12, 3, 4] and also we introduce the notion of a $((\delta, \gamma))$-fuzzy ideal of a near-ring and we prove a correspondence theorem between the families of $((\delta, \gamma))$-fuzzy ideals of two homomorphic lattices. This is an extension of the result of M. J. Rani [10] and T. Manikantan [9].

2. Preliminaries

In this section We recall some definitions and results that will be needed in the sequel. The interval $[0,1]$ is a lattice and this entity $(([0,1],[\leq])$ is denoted by $I$.

Definition 2.1[10] Let $\mu, \nu \in I^A$. If $\mu(x) \leq \nu(x) \forall x \in X$, then we say that $\mu$ is contained in $\nu$ and we write $\mu \leq \nu$. Clearly the inclusion relation $\leq$ is a partial ordering on $I^A$.

Definition 2.2[10] Let $\mu, \nu \in I^A$. Define $\mu \cup \nu$ and $\mu \cap \nu$ as follows.

$\forall x \in A, (\mu \cup \nu)(x) = \mu(x) \vee \nu(x)$ and $(\mu \cap \nu)(x) = \mu(x) \wedge \nu(x)$

Then $(\mu \cup \nu)$ and $(\mu \cap \nu)$ are respectively the lub and glb and they are called the union and intersection of $\mu$ and $\nu$ respectively. It is also known that under the natural ordering, $I^A$ is a complete lattice for any nonempty set $A$. Its largest and smallest element are $1_A$ (where $1_A(x) = 1 \forall x \in A$) and $0_A$ (where $0_A(x) = 0 \forall x \in A$).
Definition 2.3[10] A fuzzy subset $\mu$ of $X$ is said to be a fuzzy sublattice of $X$ if $\forall x, y \in X$, 
(i) $\mu(x \lor y) \geq \mu(x) \land \mu(y)$, 
(ii) $\mu(x \land y) \geq \mu(x) \land \mu(y)$.

Definition 2.4[10] Let $\mu \in I^X$, then $\mu$ is called a fuzzy ideal of $X$ if $\forall x, y \in X$, 
(1) $\mu(x \lor y) \geq \mu(x) \land \mu(y)$, 
(2) $\mu(x \land y) \geq \mu(x) \land \mu(y)$.

If $I_2$ holds, then $\mu(x \land y) \geq \mu(x) \land \mu(y)$. Thus by $I_1$ and $I_2$, $\mu \in FL(X)$, (i.e) a fuzzy ideal of $X$ is fuzzy sublattice of $X$.

Definition 2.5[8] A fuzzy subset $A$ of $N$ is called a fuzzy subnear-ring of $N$ if $\forall x, y \in N$, 
(i) $A(x - y) \geq \min\{A(x), A(y)\}$, 
(ii) $A(xy) \geq \min\{A(x), A(y)\}$.

Definition 2.6[5] A fuzzy subset $A$ of a group $(G, +)$ is said to be a fuzzy subgroup of $G$ if $\forall x, y \in G$, 
(i) $A(x + y) \geq \min\{A(x), A(y)\}$, 
(ii) $A(-x) = A(x)$, or equivalently $A(x - y) \geq \min\{A(x), A(y)\}$.

If $A$ is a fuzzy subgroup of a group $G$, then $A(0) \geq A(x \forall x \in G.)$

Definition 2.7[8] A fuzzy subset $A$ of $N$ is said to be a fuzzy two-sided $N$-subgroup of $N$ if 
(i) $A$ is a fuzzy subgroup of $(N, +)$, 
(ii) $A(xy) \geq A(x) \forall x, y \in N$, 
(iii) $A(xy) \geq A(y) \forall x, y \in N$.

If $A$ satisfies (i),(ii) then $A$ is called a fuzzy right $N$-subgroup of $N$. If $A$ satisfies (i) and (iii), then $A$ is called a fuzzy left $N$-subgroup of $N$.

Definition 2.8[8] A fuzzy subset $A$ of $N$ is said to be a fuzzy ideal of $N$ if 
(i) $A$ is a fuzzy subnear-ring of $N$, 
(ii) $A(y + x - y) = A(x) \forall x, y \in N$, 
(iii) $A(xy) \geq A(y) \forall x, y \in N$. 
(iv) $A(a(b + i) - ab) \geq A(i) \forall a, b, i \in N$.

A fuzzy subset with (i),(ii) and (iii) is called a fuzzy right ideal of $N$ whereas a fuzzy subset with (i),(ii) and (iv) is called a fuzzy left ideal of $N$.

3. $(\delta, \gamma)$-Fuzzy ideals of a lattice
Based on the notion of $(\lambda, \mu)$-fuzzy ideals introduced by B. You [6]. In this section we introduce $(\delta, \gamma)$-fuzzy ideals of lattice. In the following discussion, we always assume that $0 \leq \delta < \gamma \leq 1$.

Definition 3.1 A $(\delta, \gamma)$-fuzzy subset $\beta$ of $X$ is said to be a $(\delta, \gamma)$-fuzzy sublattice of $X$ if $\forall x, y \in X$, 
(i) $\beta(x \lor y) \lor \delta \geq \beta(x) \land \beta(y) \land \gamma$, 
(ii) $\beta(x \land y) \lor \delta \geq \beta(x) \land \beta(y) \land \gamma$. 
Definition 3.2 Let $\beta, \nu \in I^{A}_{(\delta, \gamma)}$. Define $\beta \cup \nu$ and $\beta \cap \nu$ as follows.

\[ \forall x \in A, ((\beta \cup \nu(x)) \vee \delta = (\beta \vee \nu(x)) \wedge \gamma \text{ and } ((\beta \cap \nu(x)) \vee \delta = (\beta \wedge \nu(x)) \wedge \gamma} \]

Then $(\beta \cup \nu)$ and $(\beta \cap \nu)$ are respectively the lub and glb and they are called the union and intersection of $\beta$ and $\nu$ respectively. It is also known that under the natural ordering, $I^{A}_{(\delta, \gamma)}$ is a complete lattice for any nonempty set $A$. Its largest and smallest element are $1_A$ (where $1_A(x) = 1 \forall x \in A$) and $0_A$ (where $0_A(x) = 0 \forall x \in A$).

Definition 3.3 Let $\beta, \nu \in I^{A}_{(\delta, \gamma)}$. If $\beta(x) \leq \nu(x), \forall x \in X$, then we say that $\beta$ is contained in $\nu$ and we write $\beta \leq \nu$. Clearly the inclusion relation $\leq$ is a partial ordering on $I^{A}_{(\delta, \gamma)}$.

Example 3.4 If $X$ is any lattice and $t \in I$, then $\beta(x) \vee \delta = t \wedge \gamma, \forall x \in X$ is a $(\delta, \gamma)$-fuzzy sublattice of $X$.

Example 3.5 If $X$ is any subset of $N$ with usual ordering and $\beta \in I^{x}_{(\delta, \gamma)}$ is given by

$\beta(x) \vee \delta = 1/x \wedge \gamma$ then $\beta$ is a $(\delta, \gamma)$-fuzzy sublattice of $X$.

Notation 3.6 $FL_{(\delta, \gamma)}(X)$ denotes the set of all $(\delta, \gamma)$-fuzzy sublattice of $X$.

Result 3.7 If $\beta \in FL_{(\delta, \gamma)}(X)$, then the set $\beta^* = \{x \in X, \beta(x) \vee \delta > 0 \wedge \gamma\}$ is a $(\delta, \gamma)$-fuzzy sublattice of $X$.

Proof. Omitted.

Let $\beta \in I^X$, then $\beta$ is called a $(\delta, \gamma)$-fuzzy ideal of $X$ if $\forall x, y \in X$,

(i). $\beta(x \vee y) \vee \delta \geq \beta(x) \wedge \beta(y) \wedge \gamma$,

(ii). $\beta(x \wedge y) \vee \delta \geq \beta(x) \wedge \beta(y) \wedge \gamma$.

If $I_2$ holds, then $\beta(x \wedge y) \vee \delta \geq \beta(x) \wedge \beta(y) \wedge \gamma$. Thus by $I_1$ and $I_2, \beta \in FL_{(\delta, \gamma)}(X)$, (i.e) a $(\delta, \gamma)$-fuzzy ideal of $X$ is $(\delta, \gamma)$-fuzzy sublattice of $X$.

Notation 3.9 The set of all $(\delta, \gamma)$-fuzzy ideals of $X$ is denoted by $FL_{(\delta, \gamma)}(X)$. Let $\beta \in I^x_{(\delta, \gamma)}$ satisfies $I_2$ if and only if $\beta(x \wedge y) \vee \delta \geq \beta(x) \wedge \gamma, \forall x, y \in X$. Since from $I_2, \beta(x \wedge y) \vee \delta \geq \beta(x) \wedge \gamma$ and conversely if $\beta(x \wedge y) \vee \delta \geq \beta(x) \wedge \gamma, \forall x \in X$, then $\beta(x \wedge y) \vee \delta = \beta(y \wedge x) \vee \delta \geq \beta(y) \wedge \gamma, \forall x, y \in X$. Thus $I_2$ is equivalent to $(I_1)$. $\beta(x \wedge y) \vee \delta \geq \beta(x) \wedge \gamma, \forall x \in X$. Hence a $(\delta, \gamma)$-fuzzy sublattice $\beta$ of $X$ is $(\delta, \gamma)$-fuzzy ideal of $X$ if and only if $\beta(x \wedge y) \vee \delta \geq \beta(x) \wedge \gamma, \forall x \in X$.

Example 3.10 Consider the lattice $X = \{a, b, c, d\}$ where $a > c > d, a > b > d$ and $b, c$ are non comparable. Where $\delta = 0.1$ and $\gamma = 0.9$. Define $\beta \in I^x_{(\delta, \gamma)}$ as follows.

$\beta(a) = 0.1, \beta(b) = 0.2, \beta(c) = 0.4, \beta(d) = 0.5$ then $\beta$ is a $(\delta, \gamma)$-fuzzy ideal of $X$.

Example 3.11 If $X$ and $Y$ with a smallest element $0$ be lattice and $f$ a lattice homomorphism of $X$ onto $Y$, then $ker\ f = \{x \in X | f(x) = 0\}$ is an ideal of $X$ and $\chi_{ker\ f}$ is a fuzzy ideal of $X$ (from $\chi$) and also $\chi_{ker\ f}$ is a $(\delta, \gamma)$-fuzzy ideal of $X$.

Theorem 3.12 Let $A \subseteq X$. Then $A$ is an $(\delta, \gamma)$-fuzzy ideal of $X$ if and only if $\chi_A$ is a $(\delta, \gamma)$-fuzzy ideal of $X$. Proof. Suppose $A$ is an $(\delta, \gamma)$-fuzzy ideal of $X$. Therefore $A$ is a $(\delta, \gamma)$-fuzzy sublattice of $X$. 
Proof. Suppose \( x, y \in X \) be elements of \( A \). Therefore both \( x \land y \) and \( x \lor y \in A \). Then

\[
\chi_A(x) \lor \delta = 1 \land \gamma, \quad \chi_A(y) \lor \delta = 1 \land \gamma, \quad \chi_A(x \lor y) \lor \delta = 1 \land \gamma \quad \text{and} \quad \chi_A(x \lor y) \lor \delta = 1 \land \gamma.
\]

Then

\[
\chi_A(x \lor y) \lor \delta \geq \chi_A(x) \land \chi_A(y) \lor \gamma = 1 \land \gamma
\]

Therefore \( \chi_A \) is a \((\delta, \gamma)\)-fuzzy ideal of \( X \).

Suppose \( x \in A \) and \( y \in A \). Then \( \chi_A(x) \lor \delta = 1 \land \gamma \) and \( \chi_A(y) \lor \delta = 0 \land \gamma \).

But, since \( A \) is an ideal, \((x \land y) \lor \delta \in A \land \gamma \).

Now since \( \chi_A(y) \lor \delta = 0 \land \gamma \) and \( \chi_A(x \lor y) \lor \delta \geq \chi_A(x) \land \chi_A(y) \lor \gamma = 1 \land \gamma \) and \( \chi_A(x \land y) \lor \delta \geq \chi_A(x) \lor \chi_A(y) \lor \gamma = 1 \land \gamma \).

Thus \( \chi_A \) is a \((\delta, \gamma)\)-fuzzy ideal of \( X \).

Conversely, suppose that \( \chi_A \) is a \((\delta, \gamma)\)-fuzzy ideal of \( X \).

Let \( x, y \in A \) then \( \chi_A(x) \lor \delta = 1 \land \gamma, \chi_A(y) \lor \delta = 1 \land \gamma \).

Since \( \chi_A(x \lor y) \lor \delta \geq \chi_A(x) \land \chi_A(y) \lor \gamma = 1 \land \gamma \), \( \chi_A(x \lor y) \lor \delta = 1 \land \gamma \).

Therefore \((x \lor y) \lor \delta \in A \land \gamma \).

Similarly \( \chi_A(x \lor y) \lor \delta = 1 \land \gamma \).

Therefore \( x \land y \in A \).

Therefore \( A \) is a \((\delta, \gamma)\)-fuzzy sub lattice of \( X \).

Let \( x \in A \) and \( y \in X \). Therefore \( \chi_A(x) \lor \delta = 1 \land \gamma \). Then, since

\[
\chi_A(x \lor y) \lor \delta \geq \chi_A(x) \lor \chi_A(y) \lor \gamma = 1 \land \gamma, \quad x, y \in A.
\]

Therefore \( A \) is an \((\delta, \gamma)\)-fuzzy ideal of \( X \).

Theorem 3.13 \( \mu \in FL_{(\delta, \gamma)}(X) \) is a \((\delta, \gamma)\)-fuzzy ideal of \( X \), when \( \beta(x \lor y) \lor \delta \geq a \land \gamma \) holds if and only if \( \beta(x) \lor \delta \geq a \land \gamma \) and \( \beta(y) \lor \delta \geq a \land \gamma \), \( \forall a \in I \).

Proof. Suppose that \( \beta(x \lor y) \lor \delta \geq a \land \gamma \) holds if and only if \( \beta(x) \lor \delta \geq a \land \gamma \) and \( \beta(y) \lor \delta \geq a \land \gamma \), \( \forall a \in I \). Let \( x, y \in X \). Let \( \beta(x) \land \beta(y) \lor \delta = a \land \beta \). Then \( \beta(x) \lor \delta \geq a \land \gamma \) and \( \beta(y) \lor \delta \geq a \land \gamma \). Therefore by assumption \( \beta(x \lor y) \lor \delta \geq a \land \gamma \).

That is \( \beta(x \lor y) \lor \delta \geq (\beta(x) \land \beta(y)) \lor \delta \). Thus \( (I_1) \).

Let \( \beta(x) = b \). Then \( x \in B_b \). Now since \( X \) is a lattice, \( x \lor \delta = (x \lor (x \land y)) \lor \delta \)

Therefore \( (x \lor (x \land y)) \lor \delta \in B_b \). That is \( \beta((x \lor (x \land y)) \lor \delta) \geq b \land \gamma \)

Therefore \( \beta(x \lor y) \lor \delta \geq b \land \gamma \) by assumption that is \( \beta(x \lor y) \lor \delta \geq \beta(x) \land \gamma \). Thus \( I_3 \).

Hence \( \beta \in FL_{(\delta, \gamma)}(X) \).

Theorem 3.14 Let \( \beta \in FL_{(\delta, \gamma)}(X) \). Then

(i) If \( X \) has a smallest element \( 0 \), then \( \beta(0) \lor \delta \geq \beta(x) \land \gamma \forall x \).

(ii) If \( X \) has a greatest element \( 1 \), then \( \beta(1) \lor \delta \geq \beta(x) \land \gamma \forall x \in X \).

(iii) \( \beta \) is an \((\delta, \gamma)\)-fuzzy ideal of \( X \).

Proof. (i) \( \beta(0) \lor \delta = \beta(0 \land x) \lor \delta = \beta(x \land 0) \lor \delta \geq \beta(x) \land \gamma \) by \( I_3 \).

(ii) \( \beta(1) \lor \delta = \beta(1 \land x) \lor \delta = \beta(x \land 1) \lor \delta \geq \beta(x) \land \gamma \) by \( I_3 \).

Therefore \( \beta(x) \leq (X) \)

(iii) By definition of \((\delta, \gamma)\)-fuzzy sublattice of \( X \).. \( \beta^* = \{x | \beta(x) \lor \delta > 0 \land \beta \} \).

Let \( x, y \in \beta^* \). Then both \( \beta(x) \lor \delta > 0 \land \gamma \) and \( \beta(y) \lor \delta > 0 \land \gamma \)

Therefore \( (\beta(x) \lor \beta(y)) \lor \delta > 0 \land \gamma \) and \( (\beta(x) \land \beta(y)) \lor \delta > 0 \land \gamma \), since \( (\beta(x) \lor \beta(y)) \lor \delta = \max\{\beta(x), \beta(y), \delta\} \) and \( (\beta(x) \land \beta(y)) \lor \delta = \min\{\beta(x), \beta(y), \gamma\} \) in \( I \). Now


\((x \lor y) \lor \delta \geq (\beta(x) \land \beta(y)) \lor \delta > 0\), since \(\beta \in FI_{(\delta, \gamma)}(X)\)

Therefore \(x \lor y \in \beta^*\)

Also \((\beta(x \land y)) \lor \delta \geq (\beta(x) \lor \beta(y)) \land \gamma > 0\)

Therefore \(x \land y \in \beta^*\). Suppose \(z \notin \beta^*\).

Therefore \(\beta(Z) = 0\), \(x \land z \in \beta^*\). Thus \(\beta^*\) is an ideal of \(X\).

References


