Fuzzy $e$-closed and Generalized Fuzzy $e$-closed Sets in Double Fuzzy Topological Spaces

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Abstract

The purpose of this paper is to introduce and study a new class of fuzzy sets called $(r,s)$-generalized fuzzy $e$-closed sets in double fuzzy topological spaces. Furthermore, the relationship between the new concepts are introduced and established with some interesting examples.

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1. Introduction

A progressive development of fuzzy sets [9] has been made to discover the fuzzy analogues of the crisp sets theory. On the other hand, the ideal of intuitionistic fuzzy sets was first introduced by Atanassov [2]. Later on, Coker [3] presented the notion of intuitionistic fuzzy topology. Samanta and Mondal [7], introduced and characterized the intuitionistic gradation of openness of fuzzy sets which is a generalization of smooth topology and the topology of intuitionistic fuzzy sets. The name "intuitionistic" is disconnected in mathematics and applications. Gracia and Rodabaugh [5] concluded that they word under the name "double". In 2008, Erdal Ekici [4] introduced $e$-open sets in general topology. In 2014, Seenivasan et. al [8] introduce fuzzy $e$-open sets in fuzzy topological spaces. As a generalization of the results in References [4, 8], we introduce and study $(r,s)$-fuzzy $e$-closed sets and $(r,s)$-generalized fuzzy $e$-closed sets in double fuzzy topological spaces. Also the relationship between $(r,s)$-fuzzy $e$-closed (resp. $(r,s)$-generalized fuzzy $e$-closed) sets and and some other sets are introduced and established with some interesting couter examples.

2. Preliminaries

Throughout this paper, $X$ will be a non-empty set, $I = [0,1], I_0 = (0,1]$ and $I_1 = [0,1)$. A fuzzy set $A$ is quasi-coincident with a fuzzy set $B$ (denoted by, $A_{qB}$) iff there exists $x \in X$ such that $A(x) + B(x) > 1$ and they are not quasi-coincident otherwise (denoted by, $A_{q}B$). The family of all fuzzy sets on $X$ is denoted by $I^X$. By $0$ and $1$, we denote the smallest and the greatest fuzzy sets on $X$. For a fuzzy set $A \in I^X$, $1 - A$ denotes its complement. All other notations are standard notations of fuzzy set theory.

Now, we recall the following definitions which are useful in the sequel.

Definition 2.1 [7] A double fuzzy topology $(T,T^\bar{a})$ on $X$ is a pair of maps $T,T^\bar{a} : I^X \rightarrow I$, which satisfies the following properties:

(i) $T(A) \leq 1 - T^\bar{a}(A)$ for each $A \in I^X$.

(ii) $T(A_1 \land A_2) \geq T(A_1) \land T(A_2)$ and $T^\bar{a}(A_1 \land A_2) \leq T^\bar{a}(A_1) \lor T^\bar{a}(A_2)$, for each $A_1, A_2 \in I^X$. 


The triplet $(X, T, T^\#)$ is called a double fuzzy topological space (briefly, dfts). A fuzzy set $A$ is called an $(r, s)$-fuzzy open (briefly, $(r, s)$-fo) if $T(A) \geq r$ and $T^\#(A) \leq s$. A fuzzy set $A$ is called an $(r, s)$-fuzzy closed (briefly, $(r, s)$-fc) set iff $1 - A$ is an $(r, s)$-fo set.

**Theorem 2.1** [6] Let $(X, T, T^\#)$ be a dfts. Then double fuzzy closure operator and double fuzzy interior operator of $A \in I^X$ are defined by

$$C_{T, T^\#}(A, r, s) = \big\{ B \in I^X \mid A \subseteq B, T(1 - B) \geq r, T^\#(1 - B) \leq s \big\}.$$ $$I_{T, T^\#}(A, r, s) = \big\{ B \in I^X \mid A \supseteq B, T(B) \geq r, T^\#(B) \leq s \big\}.$$ 

Where $r \in I_0$ and $s \in I_1$ such that $r + s \leq 1$.

**Definition 2.2** [1] Let $(X, T, T^\#)$ be a dfts. For each $A \in I^X$, $r \in I_0$ and $s \in I_1$, a fuzzy set $A$ is called an $(r, s)$-generalized fuzzy closed (briefly, $(r, s)$-gfc) if $C_{T, T^\#}(A, r, s) \subseteq B$, $A \subseteq B$, $T(B) \geq r$ and $T^\#(B) \leq s$. A is called an $(r, s)$-generalized fuzzy open (briefly, $(r, s)$-gfo) iff $1 - A$ is $(r, s)$-gfc set.

3. $(r, s)$ fuzzy $e$-closed and $(r, s)$-generalized fuzzy $e$-closed sets

**Definition 3.1** Let $(X, T, T^\#)$ be a dfts. Then for each $A \in I^X$, $r \in I_0$ and $s \in I_1$, a fuzzy set $A$ is called an $(r, s)$-fuzzy regular open (briefly, $(r, s)$-fro) if $A = I_{T, T^\#}(C_{T, T^\#}(A, r, s), r, s)$ and $(r, s)$-fuzzy regular closed (briefly, $(r, s)$-frc) if $A = C_{T, T^\#}(I_{T, T^\#}(A, r, s), r, s)$.

**Definition 3.2** Let $(X, T, T^\#)$ be a dfts. Then for each $A \in I^X$, $r \in I_0$ and $s \in I_1$, we define operators $\delta C_{T, T^\#}$ and $\delta I_{T, T^\#}: I^X \times I_0 \times I_1 \to I^X$ as follows:

$$\delta C_{T, T^\#}(A, r, s) = \big\{ B \in I^X \mid A \subseteq B, T(B) \geq r, T^\#(B) \leq s \big\}.$$ $$\delta I_{T, T^\#}(A, r, s) = \big\{ B \in I^X \mid A \supseteq B, T(B) \geq r, T^\#(B) \leq s \big\}.$$ 

**Definition 3.3** Let $(X, T, T^\#)$ be a dfts. Then for each $A \in I^X$, $r \in I_0$ and $s \in I_1$, a fuzzy set $A$ is called an

(i) $(r, s)$-fuzzy $\delta$ semiopen (briefly, $(r, s)$-f $\delta$ so) if $A \leq \delta C_{T, T^\#}(A, r, s)$ and $(r, s)$-fuzzy $\delta$ semi closed (briefly, $(r, s)$-f $\delta$ sc) if $A \geq \delta I_{T, T^\#}(A, r, s)$.

(ii) $(r, s)$-fuzzy $\delta$ pre open (briefly, $(r, s)$-f $\delta$ po) if $A \leq I_{T, T^\#}(\delta C_{T, T^\#}(A, r, s), r, s)$ and $(r, s)$-fuzzy $\delta$ pre closed (briefly, $(r, s)$-f $\delta$ pc) if $A \geq C_{T, T^\#}(\delta I_{T, T^\#}(A, r, s), r, s)$.

(iii) $(r, s)$-fuzzy $\beta$ open (briefly, $(r, s)$-f $\beta$ o) if $A \leq C_{T, T^\#}(I_{T, T^\#}(A, r, s), r, s) r, s$ and $(r, s)$-fuzzy $\beta$ closed (briefly, $(r, s)$-f $\beta$ c) if $A \geq I_{T, T^\#}(C_{T, T^\#}(A, r, s), r, s)$.

(iv) $(r, s)$-fuzzy $e$-open (briefly, $(r, s)$-feo) if $A \leq C_{T, T^\#}(\delta I_{T, T^\#}(A, r, s), r, s) \lor I_{T, T^\#}(\delta C_{T, T^\#}(A, r, s), r, s)$ and $(r, s)$-fuzzy $e$-closed (briefly, $(r, s)$-feco) if $A \geq I_{T, T^\#}(\delta C_{T, T^\#}(A, r, s), r, s) \land C_{T, T^\#}(\delta I_{T, T^\#}(A, r, s), r, s)$.

**Definition 3.4** Let $(X, T, T^\#)$ be a dfts. Then for each $A \in I^X$, $r \in I_0$ and $s \in I_1$, we define operators $eC_{T, T^\#}$ (resp. $\delta SC_{T, T^\#}$, $\delta PC_{T, T^\#}$ and $\beta C_{T, T^\#}$) and

$$e I_{T, T^\#}(\delta S I_{T, T^\#}, \delta P I_{T, T^\#} and \beta I_{T, T^\#}) : I^X \times I_0 \to I^X$$ $$e C_{T, T^\#}(\delta S C_{T, T^\#}, \delta P C_{T, T^\#} and \beta C_{T, T^\#})(A, r, s) = \big\{ B \in I^X : A \subseteq B, B \text{ is } (r, s) - e \text{ closed (resp. } f \delta sc, f \delta pc \text{ and } f \beta c) \big\}$$. 
\[ e_{I,J}^{T} (r,s) = (A,r,s) = \] 
\[ \{ B \in I^X : B \leq A, B \text{ is } (r,s) \text{-feo} \} \].

**Definition 3.5** Let \((X,T,T')\) be a dfts, \(A \in I^X\), \(r \in I_0\) and \(s \in I_1\), \(A\) is called an \((r,s)\)-fuzzy \(e\)-\(Q\) -neighborhood of \(x \in P(X)\) if there exists an \((r,s)\)-feo set \(B \in I^X\) such that \(x \in B\) and \(B \leq A\).

The family of all \((r,s)\)-fuzzy \(e\)-\(Q\)-neighborhood of \(x_i\) denoted by \(e\)-\(Q(x_i,r,s)\).

**Theorem 3.1** Let \((X,T,T')\) be a dfts. Then for each \(A \in I^X\), \(r \in I_0\) and \(s \in I_1\), the operator \(e_{I,J}^{T} (r,s)\) satisfies the following statements:

(i) \(e_{I,J}^{T} (Q,r,s) = \emptyset, \; e_{I,J}^{T} (1,r,s) = 1\).
(ii) \(A \leq e_{I,J}^{T} (A,r,s) \).
(iii) If \(A \leq B\), then \(e_{I,J}^{T} (A,r,s) \leq e_{I,J}^{T} (B,r,s)\).
(iv) If \(A\) is an \((r,s)\)-fc set, then \(A = e_{I,J}^{T} (A,r,s)\).
(v) If \(A\) is an \((r,s)\)-feo set, then \(BqA\) if \(BqC_{I,J}^{T} (A,r,s)\).
(vi) \(e_{I,J}^{T} (e_{I,J}^{T} (A,r,s),r,s) = e_{I,J}^{T} (A,r,s)\).
(vii) \(e_{I,J}^{T} (A,r,s) \lor e_{I,J}^{T} (B,r,s) \leq e_{I,J}^{T} (A \lor B,r,s)\).
(viii) \(e_{I,J}^{T} (A,r,s) \land e_{I,J}^{T} (B,r,s) \geq e_{I,J}^{T} (A \land B,r,s)\).

**Proof.** (i), (ii), (iii) and (iv) are proved easily.

(v) Let \(BqA\) and \(B\) is an \((r,s)\)-feo set, then \(A \leq 1 - B\). But we have, \(BqA\) if \(BqC_{I,J}^{T} (A,r,s)\) and \(e_{I,J}^{T} (A,r,s) \leq e_{I,J}^{T} (1 - B,r,s) = 1 - B\), so \(BqC_{I,J}^{T} (A,r,s)\), which is contradiction. Then \(BqA\) if \(BqC_{I,J}^{T} (A,r,s)\).

(vi) Let \(x_i\) be a fuzzy point such that \(x_i \in e_{I,J}^{T} (A,r,s)\). Then there is an \((r,s)\)-fuzzy \(e\)-\(Q\)-neighborhood \(B\) of \(x_i\) such that \(BqA\). But by (v), we have an \((r,s)\)-fuzzy \(e\)-\(Q\)-neighborhood \(B\) of \(x_i\) such that \(BqC_{I,J}^{T} (A,r,s)\). Also, \(x_i \in e_{I,J}^{T} (e_{I,J}^{T} (A,r,s),r,s)\). Then \(e_{I,J}^{T} (e_{I,J}^{T} (A,r,s),r,s) \leq e_{I,J}^{T} (A,r,s)\). But we have, \(e_{I,J}^{T} (e_{I,J}^{T} (A,r,s),r,s) \geq e_{I,J}^{T} (A,r,s)\). Therefore \(e_{I,J}^{T} (e_{I,J}^{T} (A,r,s),r,s) = e_{I,J}^{T} (A,r,s)\).

(vii) and (viii) are obvious.

Similarly the other operators (i.e) \(\delta SC_{I,J}^{T}, \delta PC_{I,J}^{T}\) and \(\beta C_{I,J}^{T}\) satisfies the above conditions.

**Theorem 3.2** Let \((X,T,T')\) be a dfts. Then for each \(A \in I^X\), \(r \in I_0\) and \(s \in I_1\), the operator \(e_{I,J}^{T} (r,s)\) satisfies the following statements:

(i) \(e_{I,J}^{T} (1 - A,r,s) = 1 - e_{I,J}^{T} (A,r,s)\), \(e_{I,J}^{T} (1 - A,r,s) = 1 - e_{I,J}^{T} (A,r,s)\).
(ii) \(e_{I,J}^{T} (0,r,s) = 0, \; e_{I,J}^{T} (1,r,s) = 1\).
(iii) \(e_{I,J}^{T} (A,r,s) \leq A\).
(iv) If \(A\) is an \((r,s)\)-fc set, then \(A = e_{I,J}^{T} (A,r,s)\).
(v) If \(A \leq B\), then \(e_{I,J}^{T} (A,r,s) \leq e_{I,J}^{T} (B,r,s)\).
(vi) \(e_{I,J}^{T} (e_{I,J}^{T} (A,r,s),r,s) = e_{I,J}^{T} (A,r,s)\).
(vii) \(e_{I,J}^{T} (A \lor B,r,s) \geq e_{I,J}^{T} (A,r,s) \lor e_{I,J}^{T} (B,r,s)\).
(viii) \( el_{T,T}^\delta (A \vee B, r, s) \leq el_{T,T}^\delta (A, r, s) \land el_{T,T}^\delta (B, r, s) \).

**Proof.** It is similar to Theorem 3.1. Similarly the other operators (i.e) \( \delta SI_{T,T}^\delta \), \( \delta PI_{T,T}^\delta \) and \( \beta I_{T,T}^\delta \) satisfies the above conditions.

**Definition 3.6** Let \((X,T,T^\delta)\) be a dfts. Then for each \( A \in I^X \), \( r \in I_0 \) and \( s \in I_1 \): a fuzzy set \( A \) is called

(i) \((r,s)\)-generalized fuzzy \( \delta \) semiopen (briefly, \((r,s)\)-gf \( \delta \) sc) if \( B \leq \delta SI_{T,T}^\delta (A, r, s) \) whenever \( B \leq A \) and \( T(1-B) \geq r, T^\delta (1-B) \leq s \).

(ii) \((r,s)\)-generalized fuzzy \( \delta \) preopen (briefly, \((r,s)\)-gf \( \delta \) p o) if \( B \leq \delta PI_{T,T}^\delta (A, r, s) \) whenever \( B \leq A \) and \( T(1-B) \geq r, T^\delta (1-B) \leq s \).

(iii) \((r,s)\)-generalized fuzzy \( \beta \) -open (briefly, \((r,s)\)-gf \( \beta \) o) if \( B \leq \beta I_{T,T}^\delta (A, r, s) \) whenever \( B \leq A \) and \( T(1-B) \geq r, T^\delta (1-B) \leq s \).

(iv) \((r,s)\)-generalized fuzzy \( e \) -open (briefly, \((r,s)\)-gf \( e \) o) if \( B \leq el_{T,T}^\delta (A, r, s) \) whenever \( B \leq A \) and \( T(1-B) \geq r, T^\delta (1-B) \leq s \).

(v) \((r,s)\)-generalized fuzzy \( \delta \) semiclosed (briefly, \((r,s)\)-gf \( \delta \) s c) if \( \delta SC_{T,T}^\delta (A, r, s) \leq B \) whenever \( A \leq B \) and \( T(B) \geq r, T^\delta (B) \leq s \).

(vi) \((r,s)\)-generalized fuzzy \( \delta \) preclosed (briefly, \((r,s)\)-gf \( \delta \) p c) if \( \delta PC_{T,T}^\delta (A, r, s) \leq B \) whenever \( A \leq B \) and \( T(B) \geq r, T^\delta (B) \leq s \).

(vii) \((r,s)\)-generalized fuzzy \( \beta \) -closed (briefly, \((r,s)\)-gf \( \beta \) c) if \( \beta C_{T,T}^\delta (A, r, s) \leq B \) whenever \( A \leq B \) and \( T(B) \geq r, T^\delta (B) \leq s \).

(viii) \((r,s)\)-generalized fuzzy \( e \) -closed (briefly, \((r,s)\)-gf \( e \) c) if \( eC_{T,T}^\delta (A, r, s) \leq B \) whenever \( A \leq B \) and \( T(B) \geq r, T^\delta (B) \leq s \).

**Example 3.1** Let \( X = \{x,y\} \). Defined \( B, C, D \) and \( E \) by \( B(x) = 0.3, B(y) = 0.4; C(x) = 0.4, C(y) = 0.5; D(x) = 0.8, D(y) = 0.8; E(x) = 0.4, E(y) = 0.6; F(x) = 0.4, F(y) = 0.4 \).

\[
T(A) = \begin{cases} 
1, & \text{if } A \in [0,1], \\
1/2, & \text{if } A \in \{B,C\}, \\
0, & \text{otherwise.} 
\end{cases} 
T^\delta (A) = \begin{cases} 
1, & \text{if } A \in [0,1], \\
1/2, & \text{if } A \in \{B,C\}, \\
1, & \text{otherwise.} 
\end{cases}
\]

(i) \( C \) is an \( \left(\frac{1}{2},\frac{1}{2}\right)\)-f \( \delta \) sc (resp. \( \left(\frac{1}{2},\frac{1}{2}\right)\)-gf \( \delta \) sc) but not an \( \left(\frac{1}{2},\frac{1}{2}\right)\)-fc (resp. \( \left(\frac{1}{2},\frac{1}{2}\right)\)-gfc).

(ii) \( C \) is an \( \left(\frac{1}{2},\frac{1}{2}\right)\)-f ec (resp. \( \left(\frac{1}{2},\frac{1}{2}\right)\)-gfe c) but not an \( \left(\frac{1}{2},\frac{1}{2}\right)\)-f pc (resp. \( \left(\frac{1}{2},\frac{1}{2}\right)\)-gf pc).

(iii) \( D \) is an \( \left(\frac{1}{2},\frac{1}{2}\right)\)-f pc, \( \left(\frac{1}{2},\frac{1}{2}\right)\)-e c but not an \( \left(\frac{1}{2},\frac{1}{2}\right)\)-fc, \( \left(\frac{1}{2},\frac{1}{2}\right)\)-\( \delta \) sc.

(iv) \( E \) is an \( \left(\frac{1}{2},\frac{1}{2}\right)\)-f \( \beta \) c but not an \( \left(\frac{1}{2},\frac{1}{2}\right)\)-f ec.

(v) \( F \) is an \( \left(\frac{1}{2},\frac{1}{2}\right)\)-gf \( e \) c, \( \left(\frac{1}{2},\frac{1}{2}\right)\)-g \( \delta \) pc but not an \( \left(\frac{1}{2},\frac{1}{2}\right)\)-gf \( \delta \) sc, \( \left(\frac{1}{2},\frac{1}{2}\right)\)-gfc.
**Example 3.2** Let \( X = \{ x, y \} \). Defined \( G, H, I, J, K \) and \( L \) by \( G(x) = 0.1, G(y) = 0.3; H(x) = 0.3, H(y) = 0.2; I(x) = 0.1, I(y) = 0.2; J(x) = 0.3, J(y) = 0.3; K(x) = 0.7, K(y) = 0.6; L(x) = 0.4, L(y) = 0.3).\n
\[
T(A) = \begin{cases} 
1, & \text{if } A \in \{0,1\}, \\
\frac{1}{2}, & \text{if } A \in \{G, H, I, J, K\}, \\
0, & \text{otherwise}. 
\end{cases}
\]

\[
T^a(A) = \begin{cases} 
\frac{1}{2}, & \text{if } A \in \{G, H, I, J, K\}, \\
1, & \text{otherwise}. 
\end{cases}
\]

(i) \( L \) is an \((\frac{1}{2}, \frac{1}{2})\)-gf \( \beta \) c but not an \((\frac{1}{2}, \frac{1}{2})\)-gf e sc.

**Theorem 3.3** Let \( (X, T, T^a) \) be a dfts, \( A \in I^X \), is \((r, s)\)-gfeo set \( r \in I_0 \) and \( s \in I_1 \), if and only if \( B \leq eI_{T,T^a}(A, r, s) \) whenever \( B \leq A \), \( T(1-B) \geq r \) and \( T^a(1-B) \leq s \).

**Proof.** Suppose that \( A \) is an \((r, s)\)-gfeo set in \( I^X \), and let \( T(1-B) \geq r \) and \( T^a(1-B) \leq s \) such that \( B \leq A \). By the definition, \( 1-A \) is an \((r, s)\)-gfec set in \( I^X \). So, \( eC_{T,T^a}(1-A, r, s) \leq 1-B \). Also, \( 1-eI_{T,T^a}(A, r, s) \leq 1-B \). And then, \( B \leq eI_{T,T^a}(A, r, s) \). Conversely, let \( B \leq A, T(1-B) \geq r \) and \( T^a(1-B) \leq s, r \in I_0 \) and \( s \in I_1 \), such that \( B \leq eI_{T,T^a}(A, r, s) \). Now \( 1-eI_{T,T^a}(A, r, s) \leq 1-B \). Thus \( eC_{T,T^a}(1-A, r, s) \leq 1-B \). That is, \( 1-A \) is an \((r, s)\)-gfec set, then \( A \) is an \((r, s)\)-gfeo set.

**Theorem 3.4** Let \( (X, T, T^a) \) be a dfts, \( A \in I^X \), is \((r, s)\)-gfeo set \( r \in I_0 \) and \( s \in I_1 \), if \( A \) is an \((r, s)\)-gfec set, then

(i) \( eC_{T,T^a}(A, r, s) - A \) does not contain any non-zero \((r, s)\)-fc sets.

(ii) \( A \) is an \((r, s)\)-fc iff \( eC_{T,T^a}(A, r, s) - A \) is \((r, s)\)-fc.

(iii) \( B \) is \((r, s)\)-gfec set for each set \( B \in I^X \) such that \( A \leq B \leq eC_{T,T^a}(A, r, s) \).

(iv) For each \((r, s)\)-fo set \( B \in I^X \) such that \( B \leq A \), \( B \) is an \((r, s)\)-gfec relative to \( A \) if and only if \( B \) is an \((r, s)\)-gfec in \( I^X \).

(v) For each \((r, s)\)-fo set \( B \in I^X \) such that \( eC_{T,T^a}(A, r, s) - B \) iff \( \bar{A}qB \).
Proof. (i) Suppose that $T(1-B) \geq r$ and $T^A(1-B) \leq s$, $r \in I_0$ and $s \in I_1$, such that $B \leq eC_{T,T^A}(A,r,s) - A$ whenever $A \in I^X$ is an $(r,s)$-gfec set. Since $1-B$ is an $(r,s)$-fo set,

$$A \leq (1-B) \Rightarrow eC_{T,T^A}(A,r,s) \leq (1-B)$$

$$\Rightarrow B \leq (1-eC_{T,T^A}(A,r,s))$$

$$\Rightarrow B \leq (1-eC_{T,T^A}(A,r,s)) \land (eC_{T,T^A}(A,r,s) - A)$$

$$= 0$$

and hence $B = 0$ which is a contradiction. Then $eC_{T,T^A}(A,r,s) - A$ does not contain any non-zero $(r,s)$-fc sets.

(ii) Let $A$ be an $(r,s)$-gfec set. So, for each $r \in I_0$ and $s \in I_1$ if $A$ is an $(r,s)$-fec set then, $eC_{T,T^A}(A,r,s) - A = 0$ which is an $(r,s)$-fc set.

Conversely, suppose that $eC_{T,T^A}(A,r,s) - A$ is an $(r,s)$-fc set. Then by (i), $eC_{T,T^A}(A,r,s) - A$ does not contain any non-zero an $(r,s)$-fc set. But $eC_{T,T^A}(A,r,s) - A$ is an $(r,s)$-fc set, then $eC_{T,T^A}(A,r,s) - A = 0 \Rightarrow A = eC_{T,T^A}(A,r,s)$. So, $A$ is an $(r,s)$-fec set.

(iii) Suppose that $T(C) \geq r$ and $T^A(C) \leq s$ where $r \in I_0$ and $s \in I_1$ such that $B \leq C$ and let $A$ be an $(r,s)$-gfec set such that $A \leq C$. Then $eC_{T,T^A}(A,r,s) \leq C$. So, $eC_{T,T^A}(A,r,s) = eC_{T,T^A}(B,r,s)$. Therefore $eC_{T,T^A}(B,r,s) \leq C$. So, $B$ is an $(r,s)$-gfec set.

(iv) Let $A$ be an $(r,s)$-gfec and $T(A) \geq r$ and $T^A(A) \leq s$, where $r \in I_0$ and $s \in I_1$. Then $eC_{T,T^A}(A,r,s) \leq A$. But, $B \leq A$ so, $eC_{T,T^A}(B,r,s) \leq eC_{T,T^A}(A,r,s)$. Also, since $B$ is an $(r,s)$-gfec relative to $A$, then $A \land eC_{T,T^A}(A,r,s) = eC_{T,T^A}(B,r,s)$ so $eC_{T,T^A}(A,r,s) = eC_{T,T^A}(B,r,s) \leq A$.

Now, if $B$ is an $(r,s)$-gfec relative to $A$ and $T(C) \geq r$ and $T^A(C) \leq s$ where $r \in I_0$ and $s \in I_1$ such that $B \leq C$, then for each an $(r,s)$-fo set $C \land A = B \land A \leq C \land A$. Hence $B$ is an $(r,s)$-gfec relative to $A$. $eC_{T,T^A}(B,r,s) = eC_{T,T^A}(A,r,s) \leq (C \land A) \leq C$. Therefore, $B$ is an $(r,s)$-gfec in $I^X$.

Conversely, let $B$ be an $(r,s)$-gfec set in $I^X$ and $T(C) \geq r$ and $T^A(C) \leq s$ whenever $C \leq A$ such that $B \leq A$, $r \in I_0$ and $s \in I_1$. Then for each an $(r,s)$-fo set $D \in I^X$, $C = D \land A$. But we have, $B$ is an $(r,s)$-gfec set in $I^X$ such that $B \leq D$.

$eC_{T,T^A}(B,r,s) \leq D \Rightarrow eC_{T,T^A}(B,r,s) = eC_{T,T^A}(B,r,s) \land A \leq D \land A = C$. That is, $B$ is an $(r,s)$-gfec relative to $A$.

(v) Suppose $B$ is an $(r,s)$-fo and $\bar{A}qB$, $r \in I_0$ and $s \in I_1$. Then $A \leq (1-B)$. Since $(1-B)$ is an $(r,s)$-fo set of $I^X$ and $A$ is an $(r,s)$-gfec set, then $eC_{T,T^A}(A,r,s)\bar{q}B$.

Conversely, let $B$ be an $(r,s)$-fbc set of $I^X$ such that $A \leq B$, $r \in I_0$ and $s \in I_1$. Then $\bar{A}q(1-B)$. But $eC_{T,T^A}(A,r,s)\bar{q}(1-B) \Rightarrow eC_{T,T^A}(A,r,s) \leq B$. Hence $A$ is an $(r,s)$-gfec.

Proposition 3.1 Let $(X,T,T^A)$ be a dfts, $A \in I^X$, $r \in I_0$ and $s \in I_1$.

(i) If $A$ is an $(r,s)$-gfec and an $(r,s)$-fo set, then $A$ is an $(r,s)$-fs set.

(ii) If $A$ is an $(r,s)$-fo and an $(r,s)$-gfec, then $A \land B$ is an $(r,s)$-gfec set whenever $B \leq eC_{T,T^A}(A,r,s)$.

Proof. (i) Suppose $A$ is an $(r,s)$-gfec and an $(r,s)$-fo set such that $A \leq B$, $r \in I_0$ and $s \in I_1$. Then
But we have, \( A \leq eC_{T,r,s} (A, r, s) \). Then, \( A = eC_{T,r,s} (A, r, s) \). Therefore, \( A \) is an \((r, s)\)-fec set.

(ii) Suppose that \( A \) is an \((r, s)\)-fo and an \((r, s)\)-gfec set, \( r \in I_0 \) and \( s \in I_1 \). Then
\[
\begin{align*}
eC_{T,r,s} (A, r, s) & \leq A \Rightarrow A \text{ is an } (r, s)\text{-fec set} \\
& \Rightarrow A \wedge B \text{ is an } (r, s)\text{-fec}
\end{align*}
\]

\(A \wedge B\) is an \((r, s)\)-gfec.

References