Fuzzy $M$-continuity Mappings in $\hat{S}$ostak’s Fuzzy Topological Spaces

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Abstract

We introduce and investigate some new classes of mappings called fuzzy $M$-continuous, fuzzy $\theta$-continuous and fuzzy $\theta$-semicontinuous to the fuzzy topological spaces in $\hat{S}$ostak’s sense. Also, some of their fundamental properties are studied. Moreover, we investigate the relationships between fuzzy continuous, fuzzy $\theta$-semicontinuous, fuzzy $\theta$-continuous, fuzzy $\delta$-semicontinuous, fuzzy $\delta$-precontinuous, fuzzy $\alpha$-continuous, fuzzy $M$-continuous, fuzzy $e$-continuous and fuzzy $e^*$-continuous mappings.

Keywords and phrases: fuzzy continuous, fuzzy $\theta$-semicontinuous, fuzzy $\theta$-continuous, fuzzy $\delta$-semicontinuous, fuzzy $\delta$-precontinuous, fuzzy $\alpha$-continuous, fuzzy $M$-continuous, fuzzy $e$-continuous and fuzzy $e^*$-continuous mappings.


Introduction

$\hat{S}$ostak [30] introduced the fuzzy topology as an extension of Chang’s fuzzy topology[4]. It has been developed in many directions [12,13,27]. Weaker forms of fuzzy continuity between fuzzy topological spaces have been considered by many authors [2, 3, 5, 9, 11, 20, 23] using the concepts of fuzzy semi-open sets[2], fuzzy regular open sets[2], fuzzy preopen sets, fuzzy strongly semiopen sets [3], fuzzy $\gamma$-open sets[11], fuzzy $\delta$-semiopen sets[1], fuzzy $\delta$-preopen sets[1], fuzzy semi $\delta$-preopen sets[34] and fuzzy $e$-open sets[29] Ganguly and Saha [10] introduced the notions of fuzzy $\delta$-cluster points in Chang’s [4] fuzzy topological spaces. Kim and Park [14] introduced $r$-$\delta$-cluster points and $\delta$-closure operators in $\hat{S}$ostak's fuzzy topological spaces.

It is a good extension of the notions of Ganguly and Saha[10]. Park et al. [14] introduced the fuzzy semi-preopen. In 2008, the initiations of $e$-open sets, $e^*$-open sets and $\alpha$-open sets in topological spaces are due to Erdal Ekici[[7],[8]]. Sobana et al. [33] defined $t$-fuzzy $e$-open sets in $\hat{S}$ostak's fuzzy topological space. Vadivel et al. [36] introduced $t$-fuzzy $e^*$-open sets in $\hat{S}$ostak's fuzzy topological space. In 1968, Velicko studied $\theta$-open sets [35] and $\delta$-open sets for the purpose of investigating the characterizations of $H$-closed topological spaces. semi-open set [17] were initiated by Levine in 1963. In 1993, Raychaudhuri and Mukherjee defined $\delta$-preopen sets[26]. In 1997, $\delta$-semiopen sets was obtained by Park [19] and $\theta$-semi-open sets were obtained by Caldas in 2008[6]. Shafei introduced fuzzy $\theta$-closed [31] and fuzzy $\theta$-open sets in 2006. Maghrabi et al.[21] introduced the notion of $M$-open sets in topological spaces in 2011.
In this paper fuzzy $M$-continuous, fuzzy $\theta$-continuous and fuzzy $\theta$-semicontinuous to the fuzzy topological spaces in Šostak's sense are introduced and some of their fundamental properties are studied. Moreover, we investigate the relationships between fuzzy continuous, fuzzy $\theta$-semicontinuous, fuzzy $\theta$-continuous, fuzzy $\delta$-semicontinuous, fuzzy $\delta$-precontinuous, fuzzy $a$-continuous, fuzzy $M$-continuous, fuzzy $e$-continuous and fuzzy $e'$-continuous mappings.

1. Preliminaries

Throughout this article, we denote nonempty sets by $X, Y$ etc., $I = [0, 1]$ and $I_0 = (0, 1]$. For $\alpha \in I$, $\tilde{\alpha}(x) = \alpha, \forall x \in X$. A fuzzy point $x_i$ for $t \in I_0$ is an element of $I^X$ such that $x_i(y) = \begin{cases} t & \text{if } y \text{ is equal to } x \\ 0 & \text{if } y \text{ is not equal to } x. \end{cases}$

Let $Pr(X)$ denote the set of all fuzzy points in $X$. A fuzzy point $x_i \in \mu$ iff $t < \mu(x)$. $\mu \in I^X$ is quasi-coincident with $\nu$, denoted by $\mu \trianglerighteq \nu$, if $\exists x \in X$ such that $\mu(x) + \nu(x) > 1$.

If $\mu$ is not quasi-coincident with $\nu$, we denoted $\mu \triangleright \nu$. If $A$ is a subset of $X$, we define the characteristic function $\chi_A$ on $X$ by $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$

All notations and definitions will be standard in the fuzzy set theory.

**Lemma 1.1** [30] Consider $X$ be a nonempty set and $\mu, \nu \in I^X$. Then

(i) $\mu \trianglerighteq \nu$ iff there exists $x_i \in \mu$ such that $x_i \triangleright \nu$.

(ii) $\mu \trianglerighteq \nu$, then $\mu \land \nu \neq 0$.

(iii) $\mu \triangleright \nu$ iff $\mu \leq \tilde{1} - \nu$.

(iv) $\mu \triangleright \nu$ iff $x_i \in \mu$ implies $x_i \in \nu$ iff $x_i \triangleright \nu$ implies $x_i \triangleright \nu$.

(v) $x_i \triangleright \nu$ iff there exists $i_0 \in \mu$ such that $x_i \triangleright \nu$.

**Definition 1.1** [30] A function $\tau : I^X \to I$ is called a fuzzy topology on $X$ if it satisfies the following conditions:

1. $\tau(\tilde{0}) = \tau(\tilde{1}) = 1$,
2. $\tau(\bigvee_{i \in I} \nu_i) \geq \bigwedge_{i \in I} \tau(\nu_i)$, for any $\{\nu_i\}_{i \in I} \subset I^X$,
3. $\tau(\nu_1 \land \nu_2) \geq \tau(\nu_1) \land \tau(\nu_2)$, for any $\nu_1, \nu_2 \in I^X$.

The pair $(X, \tau)$ is called a fuzzy topological space or Šostak's fuzzy topological space or smooth topological space (for short, fts, sfts, sts).

**Remark 1.1** [25] Let $(X, \tau)$ be a sfts. Then, for every $t \in I_0$, $\tau_r = \{\nu \in I^X : \tau(\nu) \geq t\}$ is a Change's fuzzy topology on $X$.

**Theorem 1.1** [27] Let $(X, \tau)$ be a sfts. Then for each $\mu \in I^X, t \in I_0$, we define an operator $C_t : I^X \times I_0 \to I^X$ as follows:

$C_t(\mu, t) = \bigwedge\{\nu \in I^X : \mu \leq \nu, \tau(\tilde{1} - \nu) \geq t\}$.

For $\mu, \nu \in I^X$ and $t, s \in I_0$, the operator $C_t$ satisfies the following conditions:
(1) $C_r(0,t) = 0$,
(2) $\mu \leq C_r(\mu, t)$,
(3) $C_r(\mu, t) \lor C_r(\nu, t) = C_r(\mu \lor \nu, t)$,
(4) $C_r(\mu, t) \leq C_r(\mu, s)$ if $t \leq s$,
(5) $C_r(C_r(\mu, t), t) = C_r(\mu, t)$.

**Theorem 1.2** [27] Let $(X, \tau)$ be a sfts. Then for each $t \in I, \mu \in I^X$ we define an operator $I_r : I^X \times I \rightarrow I^X$ as follows:

$I_r(\mu, t) = \bigvee \{\nu \in I^X : \mu \geq \nu, \tau(\nu) \geq t\}$.

For $\mu, \nu \in I^X$ and $s, t \in I$, the operator $I_r$ satisfies the following conditions:

(i) $I_r(\bar{1}, t) = \bar{1}$,
(ii) $\mu \geq I_r(\mu, t)$,
(iii) $I_r(\mu, t) \land I_r(\nu, t) = I_r(\mu \land \nu, t)$,
(iv) $I_r(\mu, t) \leq I_r(\mu, s)$ if $s \leq t$,
(v) $I_r(I_r(\mu, t), t) = I_r(\mu, t)$,
(vi) $I_r(1 - \mu, t) = 1 - C_r(\mu, t)$ and $C_r(1 - \mu, t) = 1 - I_r(\mu, t)$.

**Definition 1.2** [15] Let $(X, \tau)$ be a sfts. Then for each $\nu \in I^X, x \in P(X)$ and $t \in nI_0$, $\nu$ is called

(i) $t$-open $Q_{\tau}$-neighbourhood of $x$, if $x, q, \nu$ with $\tau(\nu) \geq t$.
(ii) $t$-open $R_{\tau}$-neighbourhood of $x$, if $x, q, \nu$ with $\nu = I_r(C_r(\mu, t), t)$.

We denote $Q_{\tau}(x, t) = \{\nu \in I^X : x, q, \tau(\nu) \geq t\}$, $R_{\tau}(x, t) = \{\nu \in I^X : x, q, \nu = I_r(C_r(\mu, t), t)\}$.

**Definition 1.3** [15] Let $(X, \tau)$ be a sfts. Then for each $\mu \in I^X, x \in P(X)$ and $t \in nI_0$, $x$ is called

(i) $t$-$\tau$ cluster point of $\mu$ if for every $\nu \in Q_{\tau}(x, t)$, we have $v \mu$.
(ii) $t$-cluster point of $\mu$ if for every $\nu \in R_{\tau}(x, t)$, we have $v \mu$.
(iii) An $\delta$-closure operator is a mapping $D_{\delta} : I^X \times I \rightarrow I^X$ defined as follows: $\delta C_r(\mu, t)$ or $D_r(\mu, t) = \bigvee \{x \in P(X) : x, i\}$ is a $\delta$-cluster point of $\mu$.

**Definition 1.4** [17] Let $(X, \tau)$ be a sfts. For $\mu, \nu \in I^X$ and $t \in I_0$, $\mu$ is called an

(i) $t$-fuzzy $\delta$-semiopen (resp. $t$-fuzzy $\delta$-semiclosed) set if $\mu \leq C_r(\delta I_r(\mu, t), t)$,
(ii) $t$-fuzzy $\delta$-preopen (resp. $t$-fuzzy $\delta$-preclosed) set if $\mu \leq C_r(\delta I_r(\mu, t), t)$,
(iii) $t$-fuzzy $\mu$-open (resp. $t$-fuzzy $\mu$-closed) set if $\mu \leq I_r(C_r(\delta I_r(\mu, t), t), t)$. 
(resp. \( C_r(I_r(\delta C_r(\mu, t), t), t) \leq \mu \)).

(iv) \( t \)-fuzzy \( e \)-open (resp. \( t \)-fuzzy \( e \)-closed) set if \( \mu \leq C_r(\delta I_r(\mu, t), t) \vee I_r(\delta C_r(\mu, t), t) \)
(resp. \( C_r(\delta I_r(\mu, t), t) \wedge I_r(\delta C_r(\mu, t), t) \leq \mu \)).

(v) \( t \)-fuzzy \( e \)-open (resp. \( t \)-fuzzy \( e \)-closed) set if \( \mu \leq C_r(I_r(\delta C_r(\mu, t), t), t) \)
(resp. \( I_r(C_r(\delta I_r(\mu, t), t), t) \leq \mu \)).

(v) \( t \)-fuzzy semiopen (resp. \( t \)-fuzzy semi-closed) set if \( \mu \leq C_r(I_r(\delta C_r(\mu, t), t), t) \)
(resp. \( I_r(C_r(\delta I_r(\mu, t), t), t) \leq \mu \)).

**Definition 1.5** \(^{[37]}\)

(i) \( t \)-fuzzy \( \theta \)-interior (resp. \( t \)-fuzzy \( \theta \)-semi-interior and \( t \)-fuzzy \( \theta \)-pre-interior) of a subset \( \mu \) in a sfts \((X, \tau)\), \( \forall t \in I_0 \), denoted by \( \theta I_r(\mu, t) \) (resp. \( \theta sI_r(\mu, t) \) and \( \theta pI_r(\mu, t) \)) defined as \( \theta I_r(\mu, t) = \{ (v) : \mu \geq v, \tau(1-v) \leq t \}; \theta sI_r(\mu, t) = \{ sI_r(\mu) : \mu \geq v, v \text{ is r-fsc} \}; \theta pI_r(\mu, t) = \{ pI_r(\mu) : \mu \geq v, v \text{ is r-fsc} \}. \)

(ii) \( t \)-fuzzy \( \theta \)-closure (resp. \( t \)-fuzzy \( \theta \)-semi-closure and \( t \)-fuzzy \( \theta \)-pre-closure) of a subset \( \mu \) in a sfts \((X, \tau)\), \( \forall t \in I_0 \), denoted by \( \theta C_r(\mu, t) \) (resp. \( \theta sC_r(\mu, t) \) and \( \theta pC_r(\mu, t) \)) defined as \( \theta C_r(\mu, t) = \{ C_r(\nu) : \mu \leq v, \tau(v) \geq t \}; \theta sC_r(\mu, t) = \{ sC_r(\nu) : \mu \leq v, v \text{ is r-fsc} \}; \theta pC_r(\mu, t) = \{ pC_r(\nu) : \mu \leq v, v \text{ is r-fsc} \}. \)

**Definition 1.6** \(^{[37]}\) Let \((X, \tau)\) be a sfts. For \( \mu, \nu \in I^X \) and \( t \in I_0 \), \( \mu \) is called an

(i) \( t \)-fuzzy \( \theta \)-open (resp. \( t \)-fuzzy \( \theta \)-closed) set if \( \mu = \theta I_r(\mu, t) \) (resp. \( \mu = \theta C_r(\mu, t) \)).

(ii) \( t \)-fuzzy \( \theta \)-semiopen (resp. \( t \)-fuzzy \( \theta \)-semiclosed) set if \( \mu \leq C_r(\theta I_r(\mu, t), t) \) (resp. \( I_r(\theta C_r(\mu, t), t) \leq \mu \)).

(iii) \( t \)-fuzzy \( \theta \)-preopen (resp. \( t \)-fuzzy \( \theta \)-preclosed) set if \( \mu \leq C_r(\theta C_r(\mu, t), t) \) (resp. \( C_r(\theta I_r(\mu, t), t) \leq \mu \)).

The family of all \( t \)-fuzzy \( \theta \)-open (resp. \( t \)-fuzzy \( \theta \)-closed), \( t \)-fuzzy \( \theta \)-semiopen, (resp. \( t \)-fuzzy \( \theta \)-semiclosed), \( t \)-fuzzy \( \theta \)-preopen (resp. \( t \)-fuzzy \( \theta \)-preclosed) sets will be denoted by \( t \)-f\( \theta \)o (resp. \( t \)-f\( \theta \)c), \( t \)-f\( \theta \)s (resp. \( t \)-f\( \theta \)sc), \( t \)-f\( \theta \)p (resp. \( t \)-f\( \theta \)pc) sets.

**Lemma 1.2** \(^{[37]}\) Let \( \mu, \nu \in I^X \) and \( t \in I_0 \) in a sfts \((X, \tau)\), then

(i) \( \mu \) is \( t \)-fuzzy \( \theta \)-open iff \( \mu = \theta I_r(\mu, t) \).

(ii) If \( \mu \leq \nu \), then \( \theta I_r(\mu, t) \leq \nu \).

(iii) \( \theta I_r(\theta I_r(\mu, t)) \leq \theta I_r(\mu, t) \).

(iv) For any subset \( \mu \) of \( X \), \( \mu \leq C_r(\mu, t) \leq \delta C_r(\mu, t) \leq \theta C_r(\mu, t) \) (resp. \( \delta I_r(\mu, t) \leq \delta I_r(\mu, t) \leq I_r(\mu, t) \leq \mu \)).

(v) \( \bar{\delta} - \theta I_r(\mu, t) = \theta C_r(\bar{\delta} - \mu, t) \).

(vi) \( \bar{\delta} - \theta C_r(\mu, t) = \theta I_r(\bar{\delta} - \mu, t) \).

(vii) \( \theta I_r(\mu, t) \vee \theta I_r(\nu, t) = \theta I_r(\mu, t) \wedge \theta I_r(\nu, t) \) and \( \theta I_r(\mu, t) \vee \theta I_r(\nu, t) < \theta I_r(\mu \vee \nu, t) \).
(viii) $\theta C_r(\mu \land v, t) = \theta C_r(\mu, t) \land \theta C_r(v, t)$ and $\theta C_r(\mu \lor v, t) = \theta C_r(\mu, t) \lor \theta C_r(v, t)$.

**Proposition 1.1** [37] Let $\mu \in I^X$ and $t \in I_0$ in a sfts $(X, \tau)$, then

(i) $\theta sC_r(\mu, t) = \mu \lor I_r(\theta C_r(\mu, t))$ and $\theta sI_r(\mu, t) = \mu \land C_r(\theta I_r(\mu, t), t)$.

(ii) $\delta pC_r(\mu, t) = \mu \lor C_r(\delta I_r(\mu, t))$ and $\delta pI_r(\mu, t) = \mu \land I_r(\delta C_r(\mu, t), t)$.

(iii) $I_r(\delta I_r(\mu, t)) = \delta C_r(\bar{1} - \mu, t)$ and $I_r(\bar{1} - \mu, t) = \bar{1} - \delta C_r(\mu, t)$

**Lemma 1.3** [37] Let $v \in I^X$ and $t \in I_0$ in a sfts $(X, \tau)$, then

(i) $\delta pC_r(\mu, v, t) = v \lor C_r(\delta I_r(v, t), t)$ and $\delta pI_r(\mu, v, t) = v \land I_r(\delta C_r(v, t), t)$.

(ii) $\delta pC_r(\mu, v, t) = \delta pI_r(\mu, v, t) = \delta pI_r(\delta pI_r(\mu, v, t), t) \lor C_r(\delta I_r(v, t), t)$.

(iii) $\delta pI_r(\delta pC_r(\mu, v, t), t) = \delta pI_r(\mu, v, t) \land I_r(\delta C_r(v, t), t)$.

(iv) $\delta sI_r(\mu, v, t) = v \lor C_r(\delta I_r(v, t), t)$ and $\delta sC_r(\mu, v, t) = v \land I_r(\delta C_r(v, t), t)$.

**Lemma 1.4** [37] Let $\nu \in I^X$ and $t \in I_0$ in a sfts $(X, \tau)$, then

(i) $C_r(\delta I_r(v, t), t) = \delta C_r(\delta I_r(v, t), t)$.

(ii) $I_r(\delta C_r(v, t), t) = \delta I_r(\delta C_r(v, t), t)$.

**Definition 1.7** [37] A subset $\mu$ in a sfts $(X, \tau)$ is called a $t$-fuzzy locally closed set $\forall t \in I_0$, if $\mu = v \land \alpha$, where $\tau(v) \geq t$, $\alpha$ is $t$-fuzzy closed in $X$.

**Definition 1.8** [37] Let $(X, \tau)$ be a sfts. For $\mu \in I^X$ and $t \in I_0$, then A sfts $(X, \tau)$ is $t$-fuzzy extremely disconnected (briefly, $t$-FED) if the $t$-fuzzy closure of every $t$-fuzzy open set of $X$ is $t$-fuzzy open.

**Definition 1.9** [37] A subset $\mu \in I^X$ and $t \in I_0$ in a sfts $(X, \tau)$ is called an $t$-fuzzy open

(i) $M$-open set if $\mu \leq C_r(\theta I_r(\mu, t), t) \land I_r(\delta C_r(\mu, t), t)$.

(ii) $M$-closed set if $\mu \geq I_r(\theta C_r(\mu, t), t) \land C_r(\delta I_r(\mu, t), t)$.

**Definition 1.10** [37] $t$-fuzzy $M$-interior (resp. $t$-fuzzy $M$-closure) of $\mu$ in a sfts $(X, \tau)$, denote by $MI_r(\mu, t)$ (resp. $MC_r(\mu, t)$) defined as $MI_r(\mu, t) = \{v \in I^X : \mu \geq v, v \in \theta M \}$, $MC_r(\mu, t) = \{v \in I^X : \mu \leq v, v \in \delta M \}$.

**Proposition 1.2** [37] Let $(X, \tau)$ be a fuzzy topological space. $\lambda \in I^X$ and $r \in I_0$, then

(i) Every $\theta$-semiopen set is $\delta$-preopen.

(ii) Every $\theta$-semiopen set is $\delta$-open.

**Proposition 1.3** [37] If $\lambda$ is an $t$-fuzzy $M$-open subset of a sfts $(X, \tau)$ and $\theta I_r(\lambda, t) = 0$, then $\lambda$ is $t$-fuzzy $\delta$-preopen.
**Theorem 1.3** [37] Let \((X, \tau)\) be a fts. Let \(\lambda \in I^X\) and \(\tau \in I_o\).

(i) \(\lambda\) is \(t-fM\) o iff \(\lambda = MI_{\tau}(\lambda, \tau)\).

(ii) \(\lambda\) is \(t-fM\) c iff \(\lambda = MC_{\tau}(\lambda, \tau)\).

**Theorem 1.4** [37] Let \((X, \tau)\) be a fts. For \(\lambda \in I^X\) and \(\tau \in I_o\) we have

(i) \(MI_{\tau}(1-\lambda, \tau) = 1 - (MC_{\tau}(\lambda, \tau))\).

(ii) \(MC_{\tau}(1-\lambda, \tau) = 1 - (MI_{\tau}(\lambda, \tau))\).

**Theorem 1.5** [37] Let \((X, \tau)\) be a fts. Let \(\lambda \in I^X\) and \(\tau \in I_o\), the following statements hold:

(i) \(MC_{\tau}(0, \tau) = 0\) and \(MI_{\tau}(1, \tau) = 1\).

(ii) \(I_{\tau}(\lambda, \tau) \leq MI_{\tau}(\lambda, \tau) \leq \lambda \leq MC_{\tau}(\lambda, \tau) \leq C_{\tau}(\lambda, \tau)\).

(iii) \(\lambda \leq \mu \Rightarrow MI_{\tau}(\lambda, \tau) \leq MI_{\tau}(\mu, \tau)\) and \(MC_{\tau}(\lambda, \tau) \leq MC_{\tau}(\mu, \tau)\).

(iv) \(MC_{\tau}(MC_{\tau}(\lambda, \tau), \tau) = MC_{\tau}(\lambda, \tau)\) and \(MI_{\tau}(MC_{\tau}(\lambda, \tau), \tau) = MI_{\tau}(\lambda, \tau)\).

(v) \(MC_{\tau}(\lambda, \tau) \vee MC_{\tau}(\mu, \tau) < MC_{\tau}(\lambda \vee \mu, \tau)\) and \(MI_{\tau}(\lambda, \tau) \vee MI_{\tau}(\mu, \tau) < MI_{\tau}(\lambda \vee \mu, \tau)\).

(vi) \(MC_{\tau}(\lambda \wedge \mu, \tau) < MC_{\tau}(\lambda, \tau) \wedge MC_{\tau}(\mu, \tau)\) and \(MI_{\tau}(\lambda \wedge \mu, \tau) < MI_{\tau}(\lambda, \tau) \wedge MI_{\tau}(\mu, \tau)\).

**Theorem 1.6** [37] Let \((X, \tau)\) be a fts. For \(\lambda, \mu \in I^X\) and \(\tau \in I_o\),

(i) \(\lambda\) is \(t-fM\) o iff \(1-\lambda\) is \(t-fM\) c.

(ii) If \(\tau(\lambda) \geq \tau\), then \(\lambda\) is \(t-fM\) o set.

(iii) \(I_{\tau}(\lambda, \tau)\) is an \(t-fM\) o set.

(iv) \(C_{\tau}(\lambda, \tau)\) is an \(t-fM\) c set.

**Definition 1.11** Let \((X, \tau_1)\) and \((X, \tau_2)\) be fts's and \(f : (X, \tau_1) \rightarrow (Y, \tau_2)\) a mapping.

(i) \(f\) is called fuzzy continuous (briefly, f-cts) [25] if \(\tau_2(\mu) \leq \tau_1(f(\mu))\) for each \(\mu \in I^Y\).

(ii) \(f\) is called fuzzy semicontinuous (briefly, fs-cts) [25] if \(f^{-1}(\mu)\) is r-fso for each \(\mu \in I^X, \tau \in I_o\) with \(\tau_2(\mu) \geq \tau\).

(iii) \(f\) is called fuzzy precontinuous (briefly, fp-cts) [25] if \(f^{-1}(\mu)\) is r-fpo for each \(\mu \in I^X, \tau \in I_o\) with \(\tau_2(\mu) \geq \tau\).

**Definition 1.12** Let \((X, \tau_1)\) and \((X, \tau_2)\) be fts's and \(f : (X, \tau_1) \rightarrow (Y, \tau_2)\) a mapping.

(i) \(f\) is called fuzzy \(\delta\) -seminctinuous (briefly, f\(\delta\) s-cts) [33] if \(f^{-1}(\mu)\) is r-f\(\delta\)s\(\delta\) o for each \(\mu \in I^X, \tau \in I_o\) with \(\tau_2(\mu) \geq \tau\).

(ii) \(f\) is called fuzzy \(\delta\) -precontinuous (briefly, f\(\delta\) p-cts) [33] if \(f^{-1}(\mu)\) is r-f\(\delta\)p (resp. r-fs\(\delta\)p) for each \(\mu \in I^X, \tau \in I_o\) with \(\tau_2(\mu) \geq \tau\). \(f\) is called fuzzy \(\alpha\) -continuous (or) fuzzy semi \(\delta\) -precontinuous Error! Reference source not found. if \(f^{-1}(\mu)\) is r-fao for each \(\mu \in I^X, \tau \in I_o\) with \(\tau_2(\mu) \geq \tau\).

(iii) \(f\) is called fuzzy \(e\) -continuous (briefly, fe-cts) [33] if \(f^{-1}(\mu)\) is r-fe for each \(\mu \in I^X, \tau \in I_o\) with \(\tau_2(\mu) \geq \tau\).

(iv) \(f\) is called fuzzy \(e^*\) -continuous (briefly, fe\(^*\)-cts) [33] if \(f^{-1}(\mu)\) is r-fe\(^*\) o for each
\[ \mu \in I^X, \tau \in I_0 \text{ with } \tau_2(\mu) \geq \tau. \]

2. Fuzzy M-continuous Mappings

Definition 2.1 Let \((X, \tau_1)\) and \((X, \tau_2)\) be sfts's and \(f : (X, \tau_1) \rightarrow (Y, \tau_2)\) be a mapping. Then \(f\) is called

\(\text{(i)}\) fuzzy \(M\)-continuous (briefly, \(fM\)-cts) if \(f^{-1}(\mu)\) is \(\tau_2\)-fo for each \(\mu \in I^X, \tau \in I_0\) with \(\tau_2(\mu) \geq \tau\).

\(\text{(ii)}\) fuzzy \(\theta\)-continuous (briefly, \(f\theta\)-cts) if \(f^{-1}(\mu)\) is \(\tau_2\)-fo for each \(\mu \in I^X, \tau \in I_0\) with \(\tau_2(\mu) \geq \tau\).

\(\text{(iii)}\) fuzzy \(\theta\)-semicontinuous (briefly, \(f\theta\)s-cts) if \(f^{-1}(\mu)\) is \(\tau_2\)-fo for each \(\mu \in I^X, \tau \in I_0\) with \(\tau_2(\mu) \geq \tau\).

Remark 2.1 The following implications are true for \(\tau \in I_0\):

From the above definitions, it is clear that every \(f \delta \) p-cts map is \(fM\) -cts map and every fuzzy \(\theta s\)-cts map is \(fM\) -cts map. Also, it is clear that every \(fM\) -cts map is \(f e\)-cts map and \(f e^*\) -cts map. Also, every \(f \theta\)-cts map, \(f \delta\) -cts map, \(f a\) -cts map is \(fM\) -cts map. The converses need not be true in general, it is shown in the succeeding examples.

Example 2.1 Consider the identity mapping \(f : (X, \tau) \rightarrow (Y, \eta)\), where \(X = Y = \{x, y, z\}\), \(\lambda\) and \(\mu\) defined as follows
\[ \lambda(x) = 0.4, \quad \lambda(y) = 0.5, \quad \lambda(z) = 0.2, \quad \mu(x) = 0.5, \quad \mu(y) = 0.4, \quad \mu(z) = 0.7. \]
Then \(\tau, \eta : I^X \rightarrow I\) defined as
\[ \tau(\lambda) = \{1, 0\} \text{ if } \lambda = 0, \text{ otherwise}, \eta(\mu) = \{1, 0\} \text{ if } \mu = 0, \text{ otherwise}, \]
are fuzzy topologies on \(X\) and \(Y\). Take \(\tau = \frac{1}{2}\) and \(\eta = \frac{1}{2}\). For any \(\frac{1}{2}\)-fuzzy open set \(\mu\) in \((Y, \eta)\), \(f^{-1}(\mu) = \mu \) is \(\frac{1}{2}\)-fo in \((X, \tau)\). Then \(f\) is \(f e^*\) -cts, but \(f\) is not \(fM\) -cts, since \(f^{-1}(\mu)\) is not \(\frac{1}{2}\) -fo in \((X, \tau)\).

Example 2.2 Let \(\lambda\) and \(\mu\) be fuzzy subsets of \(X = Y = \{x, y, z\}\) defined as follows
\[ \lambda(x) = 0.5, \quad \lambda(y) = 0.3, \quad \lambda(z) = 0.2; \]
\[ \mu(x) = 0.5, \quad \mu(y) = 0.4, \quad \mu(z) = 0.4. \]
Then \(\tau, \eta : I^X \rightarrow I\) defined as
\[ \tau(\lambda) = \{1, \]
\[ \eta(\mu) = \begin{cases} 1, & \text{if } \mu = 0, \\ \frac{1}{2}, & \text{otherwise} \end{cases} \]

are fuzzy topologies on \( X \) and \( Y \). Consider the identity mapping \( f : (X, \tau) \to (Y, \eta) \). Take \( \tau = \frac{1}{2} \). For any \( \frac{1}{2} \)-fo set \( \mu \) in \((Y, \eta)\), \( f^{-1}(\mu) = \mu \) is \( \frac{1}{2} \)-feo set in \((X, \tau)\). Then \( f \) is fe-cts, but \( f \) is not f \( M \)-cts, since \( f^{-1}(\mu) \) is not \( \frac{1}{2} \)-f \( M \) o in \((X, \tau)\).

**Example 2.3** Let \( \lambda \) and \( \mu \) be fuzzy subsets of \( X = Y = \{x, y, z\} \) defined as follows

\[
\lambda(x) = 0.1, \quad \lambda(y) = 0.1, \quad \lambda(z) = 0.1; \\
\mu(x) = 0.9, \quad \mu(y) = 0.9, \quad \mu(z) = 0.9.
\]

Then \( \tau, \eta : I^X \to I \) defined as

\[
\tau(\lambda) = \begin{cases} 1, \\ \frac{1}{2}, & \text{if } \lambda = 0, \\ 0, & \text{otherwise} \end{cases}
\]

are fuzzy topologies on \( X \) and \( Y \). Consider the identity mapping \( f : (X, \tau) \to (Y, \eta) \). Take \( \tau = \frac{1}{2} \). For any \( \frac{1}{2} \)-fuzzy open set \( \mu \) in \((Y, \eta)\), \( f^{-1}(\mu) = \mu \) is \( \frac{1}{2} \)-f \( M \) o set in \((X, \tau)\). Then \( f \) is f \( M \)-cts, but \( f \) is not f \( \delta \) p-cts, f \( \delta \) -cts and f \( a \) -cts, since \( f^{-1}(\mu) \) is not \( \frac{1}{2} \)-f \( \delta \) po, \( \frac{1}{2} \)-f \( \delta \) o and \( \frac{1}{2} \)-f \( a \) o sets.

**Example 2.4** Let \( \lambda \) and \( \mu \) be fuzzy subsets of \( X = Y = \{x, y, z\} \) defined as follows

\[
\lambda(x) = 0.1, \quad \lambda(y) = 0.1, \quad \lambda(z) = 0.1; \\
\mu(x) = 0.9, \quad \mu(y) = 0.9, \quad \mu(z) = 0.9.
\]

Then \( \tau, \eta : I^X \to I \) defined as

\[
\tau(\lambda) = \begin{cases} 1, \\ \frac{1}{2}, & \text{if } \lambda = 0, \\ 0, & \text{otherwise} \end{cases}
\]

are fuzzy topologies on \( X \) and \( Y \). Consider the identity mapping \( f : (X, \tau) \to (Y, \eta) \). Take \( \tau = \frac{1}{2} \). For any \( \frac{1}{2} \)-fo set \( \mu \) in \((Y, \eta)\), \( f^{-1}(\mu) = \mu \) is \( \frac{1}{2} \)-f \( \delta \) so set in \((X, \tau)\). Then \( f f \) is fuzzy \( \delta \) s-cts, but \( f \) is not f \( \delta \) -cts, since \( f^{-1}(\mu) \) is not \( \frac{1}{2} \)-f \( \delta \) o set.

**Example 2.5** Let \( \lambda, \mu, \omega \) be fuzzy subsets of \( X = Y = \{x, y, z\} \) defined as follows

\[
\lambda(x) = 0.3, \quad \lambda(y) = 0.4, \quad \lambda(z) = 0.5; \\
\mu(x) = 0.6, \quad \mu(y) = 0.9, \quad \mu(z) = 0.5; \\
\omega(x) = 0.7, \quad \omega(y) = 1, \quad \omega(z) = 0.5.
\]

Then \( \tau, \eta : I^X \to I \) defined as
\[
\tau(\lambda) = \{1, \frac{1}{2}\},
\]

= 0 or \( \lambda - \frac{1}{2} \), if \( \lambda = 0\), otherwise, \( \eta(\mu) = \{1, \frac{1}{2}\}, \) = 0 or \( \lambda - \frac{1}{2} \), if \( \lambda = 0\), otherwise, are fuzzy topologies on \( X \) and \( Y \). Consider the identity mapping \( f : (X, \tau) \to (Y, \eta) \). Take \( \iota = \frac{1}{2} \).

For any \( \frac{1}{2} \)-\( f \)-open set \( \omega \) in \( (Y, \eta) \), \( f^{-1}(\omega) = \omega \) is \( \frac{1}{2} \)-\( f \)-open set in \( (X, \tau) \). Then \( f \) is \( f \)-\( M \)-cts, but \( f \) is neither \( f \theta \)-s-cts nor \( f \delta \)-s-cts, since \( f^{-1}(\omega) \) is neither \( \frac{1}{2} \)-\( f \theta \)-so nor \( \frac{1}{2} \)-\( f \delta \)-so set.

**Example 2.6** Let \( \lambda \), \( \mu \) and \( \omega \) be fuzzy subsets of \( X = Y = \{x, y, z\} \) defined as follows
\[
\begin{align*}
\lambda(x) &= 0.3, \quad \lambda(y) = 0.4, \quad \lambda(z) = 0.5; \\
\mu(x) &= 0.6, \quad \mu(y) = 0.5, \quad \mu(z) = 0.5; \\
\omega(x) &= 0.7, \quad \omega(y) = 0.6, \quad \omega(z) = 0.5.
\end{align*}
\]

Then \( \tau, \eta : I^X \to I \) defined as
\[
\tau(\lambda) = \{1, \frac{1}{2}\},
\]

= 0 or \( \lambda - \frac{1}{2} \), if \( \lambda = 0\), otherwise, \( \eta(\mu) = \{1, \frac{1}{2}\}, \) = 0 or \( \lambda - \frac{1}{2} \), if \( \lambda = 0\), otherwise, are fuzzy topologies on \( X \) and \( Y \). Consider the identity mapping \( f : (X, \tau) \to (Y, \eta) \). Take \( \iota = \frac{1}{2} \).

For any \( \frac{1}{2} \)-\( f \)-open set \( \omega \) in \( (Y, \eta) \), \( f^{-1}(\omega) = \omega \) is \( \frac{1}{2} \)-\( f \)-open set in \( (X, \tau) \). Then \( f \) is \( f \)-\( M \)-cts and \( f \theta \)-s-cts, but \( f \) is not \( f \theta \)-cts, since \( f^{-1}(\omega) \) is not \( \frac{1}{2} \)-\( f \theta \)-so set.

**Example 2.7** Let \( \lambda \), \( \mu \) and \( \omega \) be fuzzy subsets of \( X = Y = \{x, y, z\} \) defined as follows
\[
\begin{align*}
\lambda(x) &= 0.3, \quad \lambda(y) = 0.5, \quad \lambda(z) = 0.5; \\
\mu(x) &= 0.5, \quad \mu(y) = 0.5, \quad \mu(z) = 0.5; \\
\omega(x) &= 0.7, \quad \omega(y) = 0.6, \quad \omega(z) = 0.5.
\end{align*}
\]

Then \( \tau, \eta : I^X \to I \) defined as
\[
\tau(\lambda) = \{1, \frac{1}{2}\},
\]

= 0 or \( \lambda - \frac{1}{2} \), if \( \lambda = 0\), otherwise, \( \eta(\mu) = \{1, \frac{1}{2}\}, \) = 0 or \( \lambda - \frac{1}{2} \), if \( \lambda = 0\), otherwise, are fuzzy topologies on \( X \) and \( Y \). Consider the identity mapping \( f : (X, \tau) \to (Y, \eta) \). Take \( \iota = \frac{1}{2} \).

For any \( \frac{1}{2} \)-\( f \)-open set \( \lambda \) in \( (Y, \eta) \), \( f^{-1}(\lambda) = \lambda \) is \( \frac{1}{2} \)-\( f \)-open in \( (X, \tau) \). Then \( f \) is \( f \)-cts, but \( f \) is not \( f \theta \)-cts and \( f \delta \)-cts, since \( f^{-1}(\lambda) \) is neither \( \frac{1}{2} \)-\( f \theta \)-so nor \( \frac{1}{2} \)-\( f \delta \)-so set.

**Theorem 2.1** Let \( (X, \tau_1) \) and \( (Y, \tau_2) \) be \( \theta \)-s-cts and \( f : X \to Y \) be a mapping. Then the following statements are equivalent:

(i) \( f \) is \( f \)-\( M \)-cts mapping.

(ii) \( f^{-1}(\mu) \) is \( f \)-\( M \)-cts in \( X \) for each \( \mu \in I^Y \), \( \iota \in I_0 \) with \( \tau_2(1-\mu) \geq \iota \).
(iii) \( f(MC_2(\lambda, t)) \leq C_2(f(\lambda), t), \quad \forall \lambda \in I^X \) and \( r \in I_0 \).
(iv) \( MC_1(f^{-1}(\mu), t) \leq f^{-1}(C_2(\mu, t)), \forall \mu \in I^Y \) and \( r \in I_0 \).
(v) \( I_1(\theta C_1(f^{-1}(\mu), t), t) \land C_1(\delta I_1(f^{-1}(\mu), t), t) \leq f^{-1}(C_2(\mu, t)), \quad \forall \mu \in I^Y \) and \( r \in I_0 \).
(vi) \( f^{-1}(I_2(\mu, t)) \leq MI_1(f^{-1}(\mu), t), \) for each \( \mu \in I^Y \) and \( r \in I_0 \).

**Proof.** (i) \( \Rightarrow \) (ii): Let \( \mu \in I^Y, t \in I_0 \) with \( \tau_2(\bar{1} - \mu) \geq t \). Since \( f \) is \( fM \) -cts mapping, \( f^{-1}(\bar{1} - \mu) \) is an \( \iota \)-f-M o set of \( X \). But \( f^{-1}(\bar{1} - \mu) = \bar{1} - f^{-1}(\mu) \). Therefore \( f^{-1}(\mu) \) is an \( \iota \)-f-M c set of \( X \).
(ii) \( \Rightarrow \) (iii): Let \( \lambda \in I^X, \) \( r \in I_0 \), since \( \tau_2(\bar{1} - C_2(f(\lambda), t)) \geq t \). Then by (ii), \( f^{-1}(C_2(f(\lambda), t)) \) is an \( \iota \)-f-M c set of \( X \).

Since \( \lambda \leq f^{-1}(f(\lambda)) \leq f^{-1}(C_2(f(\lambda), t)), \)
we have \( MC_1(\lambda, t) \leq f^{-1}(C_2(f(\lambda), t)) \). Hence \( f(MC_1(\lambda, t)) \leq C_2(f(\lambda), t) \).

(iii) \( \Rightarrow \) (iv): For all \( \mu \in I^Y, t \in I_0 \), let \( \lambda = f^{-1}(\mu) \). By (iii), we have

\[
    f(MC_1(f^{-1}(\mu), t)) \leq C_2(f(f^{-1}(\mu), t)) \leq C_2(\mu, t).
\]

It implies \( MC_1(f^{-1}(\mu), t) \leq f^{-1}(C_2(\mu, t)) \).

(iv) \( \Rightarrow \) (i): Let \( \mu \in I^Y, t \in I_0 \) with \( \tau_2(\mu) \geq t \). By (iv),

\[
    MC_1(f^{-1}(\bar{1} - \mu), t) \leq f^{-1}(C_2(\bar{1} - \mu, t)) = f^{-1}(\bar{1} - \mu).
\]

By Theorem 1.4, we have \( f^{-1}(\bar{1} - \mu) \geq \bar{1} - (MI_1(f^{-1}(\mu), t)) \). Hence \( f^{-1}(\mu) \) is \( \iota \)-f-M o set in \( X \).

(ii) \( \Rightarrow \) (v): For all \( \mu \in I^Y, t \in I_0 \), since \( \tau_2(\bar{1} - C_2(\mu, t)) \geq t \). Then by (ii), we see that \( f^{-1}(C_2(\mu, t)) \) is \( \iota \)-f-M c in \( X \).

Hence

\[
    f^{-1}(C_2(\mu, t)) \geq I_1(\theta C_1(f^{-1}(C_2(\mu, t), t), t) \land C_1(\delta I_1(f^{-1}(C_2(\mu, t), t), t), t) \geq I_1(\theta C_1(f^{-1}(\mu), t), t) \land C_1(\delta I_1(f^{-1}(\mu), t), t).\]

(v) \( \Rightarrow \) (ii): For all \( \mu \in I^Y, r \in I_0 \), with \( \tau_2(\bar{1} - \mu) \geq t \). Then by (v),

\[
    I_1(\theta C_1(f^{-1}(\mu), t), t) \land C_1(\delta I_1(f^{-1}(\mu), t), t) \leq f^{-1}(C_2(\mu, t)) = f^{-1}(\mu).\]

Hence \( f^{-1}(\mu) \) is \( \iota \)-f-M c in \( X \).

(iv) \( \Rightarrow \) (vi): It is easily proved from Theorem 1.4.

(vi) \( \Rightarrow \) (i): Let \( \mu \) be \( \iota \)-fuzzy open set of \( Y \). Then \( \mu = I_1(\mu, t) \).

By (vi), \( f^{-1}(\mu) \leq MI_1(f^{-1}(\mu), t). \)

On the other hand, by Theorem (1.5),

\[
    f^{-1}(\mu) \geq MI_1(f^{-1}(\mu), t).\]

Thus, \( f^{-1}(\mu) = MI_1(f^{-1}(\mu), t) \), that is, \( f^{-1}(\mu) \) is \( \iota \)-f-M o set.

**Theorem 2.2** For a map \( f : (X, \tau_1) \rightarrow (Y, \tau_2) \) the succeeding statements are equivalent:

(i) \( f \) is \( fM \) -cts mapping.

(ii) \( f^{-1}(\mu) \) is \( \iota \)-f-M c in \( X \) for each \( \lambda \in I^X, t \in I_0 \) with \( \tau_2(\bar{1} - \mu) \geq t \).

(iii) \( I_1(\theta C_1(f^{-1}(\mu), t), t) \land C_1(\delta I_1(f^{-1}(\mu), t), t) \leq f^{-1}(C_2(\mu, t)), \quad \forall \mu \in I^Y \) and \( r \in I_0 \).
(iv) \( f^{-1}(I_2(r, t)) \leq C_1(\theta I_1(f^{-1}(\mu), t)) \vee I_1(\delta C_1(f^{-1}(\mu), t), t) \) for each \( \mu \in I^y \) and \( r \in I_0 \).

**Proof.** (i) \( \Rightarrow \) (ii): Let \( \mu \in I^y, t \in I_0 \) with \( \tau_2(1-\mu) \geq t \). Since \( f \) is fuzzy \( M \)-continuous mapping, \( f^{-1}(1-\mu) \) is an \( \iota\text{-}fm \) o set of \( X \). But \( f^{-1}(1-\mu) = 1 - f^{-1}(\mu) \). Therefore \( f^{-1}(\mu) \) is an \( \iota\text{-}fm \) c set of \( X \).

(ii) \( \Rightarrow \) (iii): For all \( \mu \in I^y, t \in I_0 \), since \( \tau_2(1-C_2(\mu, t)) \geq t \). Then by (ii), we see that \( f^{-1}(C_2(\mu, t)) \) is \( \iota\text{-}fm \) c in \( X \). Hence

\[
f^{-1}(C_2(\mu, t)) \geq I_1(\theta C_1(f^{-1}(C_2(\mu, t)), t), t) \land C_1(\delta I_1(f^{-1}(C_2(\mu, t)), t), t)
\]

\[
\geq I_1(\theta C_1(f^{-1}(\mu), t), t) \land C_1(\delta I_1(f^{-1}(\mu), t), t).
\]

(iii) \( \Rightarrow \) (iv): For all \( \mu \in I^y, r \in I_0 \), with \( \tau_2(1-\mu) \geq t \). Then by (iii),

\[
I_1(\theta C_1(f^{-1}(\mu), t), t) \land C_1(\delta I_1(f^{-1}(\mu), t), t) \leq f^{-1}(C_2(\mu, t)).
\]

Thus (iv) is proved.

(iv) \( \Rightarrow \) (i): Let \( \mu \) be \( \iota\text{-}fuzzy \) open set of \( Y \). Then \( \mu = I_1(\mu, t) \). By (iv),

\[
f^{-1}(\mu) \leq C_1(\theta I_1(f^{-1}(\mu), t)) \land I_1(\delta C_1(f^{-1}(\mu), t), t).
\]

That is, \( f^{-1}(\mu) \) is \( \iota\text{-}fm \) o set.

**Definition 2.2** A fuzzy set \( \lambda \) in a fts \((X, \tau)\) is called \( \iota\text{-}fuzzy \) dense if there exists no \( \iota\text{-}fc \) set \( \mu \) in \((X, \tau)\) such that \( \lambda < \mu < 1 \).

**Definition 2.3** A fuzzy set \( \lambda \) in a fts \((X, \tau)\) is called \( \iota\text{-}fuzzy \) nowhere dense if there exists no non-zero \( \iota\text{-}fo \) set \( \mu \) in \((X, \tau)\) such that \( \mu < C_1(\lambda, t) \). That is, \( I_1(C_1(\lambda, t), t) = 0 \), in \((X, \tau)\).

**Lemma 2.1** For a sfts \((X, \tau)\), every \( \iota\text{-}fuzzy \) dense set is \( \iota\text{-}fo \).

**Proposition 2.1** Let \((X, \tau_1)\) and \((Y, \tau_2)\) be fts's and \( f : X \to Y \) be a mapping. An \( \delta\text{-}cts \) mapping \( f \) is \( \delta\text{-}p\text{-}cts \) if for any fuzzy subset \( \lambda \) of \( X \) is \( \iota\text{-}fuzzy \) nowhere dense.

**Proof.** Let \( \mu \in \tau_2 \). Since \( f \) is an \( \delta\text{-}cts \) mapping, then \( f^{-1}(\mu) \) is an \( \iota\text{-}fm \) o set in \((X, \tau_1)\). Put \( f^{-1}(\mu) = \lambda \) is \( \iota\text{-}fm \) o set in \( X \). Hence
\[ \lambda \leq C_\gamma (\theta I_\gamma (\lambda, t), t) \vee_{\tau_2} I_\gamma (\Delta C_\gamma (\lambda, t), t). \]

But \( \theta I_\gamma (\lambda, t) \leq I_\gamma (\lambda, t) \leq C_\gamma (\lambda, t) \), then \( \theta I_\gamma (\lambda, t) \leq I_\gamma (C_\gamma (\lambda, t), t) \). Since \( \lambda \) is \( \tau \)-fuzzy nowhere dense and Lemma Error! Reference source not found. we have \( \theta I_\gamma (\lambda, t) = 0 \). Therefore \( f \) is \( f \delta \) p-cts.

**Definition 2.4** Let \((X, \tau_1) \) and \((Y, \tau_2) \) be sfts's and \( f : (X, \tau_1) \rightarrow (Y, \tau_2) \) be a mapping. Then \( f \) is called fuzzy \( \theta \)-open map if the image of every \( \tau \)-fuzzy open set of \( X, \tau_1 \) is \( \tau \)-fuzzy \( \theta \)-open set in \((Y, \tau_2) \).  

**Definition 2.5** Let \((X, \tau_1) \) and \((Y, \tau_2) \) be sfts's and \( f : (X, \tau_1) \rightarrow (Y, \tau_2) \) be a mapping. Then \( f \) is called fuzzy \( \theta \)-bicontinuous if \( f \) is fuzzy \( \theta \)-open map and \( \theta \)-continuous map.

**Theorem 2.3** Let \((X, \tau_1) \) and \((Y, \tau_2) \) be sfts's and \( f : (X, \tau_1) \rightarrow (Y, \tau_2) \) be a fuzzy \( \theta \)-bicontinuous mapping. Then the inverse image of each \( \tau \)-fM o set in \((Y, \tau_2) \) under \( f \) is \( \tau \)-fM o set in \((X, \tau_1) \).

**Proof.** Let \( f \) be a fuzzy \( \theta \)-bicontinuous mapping and \( \mu \) be a \( \tau \)-fM o set in \((Y, \tau_2) \). Then
\[
\mu \leq C_{\tau_2} (\theta I_{\tau_2} (\mu, t), t) \vee_{\tau_2} I_{\tau_2} (\Delta C_{\tau_2} (\mu, t), t).
\]

\[
f^{-1}(\mu) \leq f^{-1} (C_{\tau_2} (\theta I_{\tau_2} (\mu, t), t)) \vee_{\tau_2} f^{-1} (I_{\tau_2} (\Delta C_{\tau_2} (\mu, t), t)).
\]

\[
\leq C_{\tau_2} (f^{-1} (\theta I_{\tau_2} (\mu, t), t)) \vee_{\tau_2} f^{-1} (I_{\tau_2} (\Delta C_{\tau_2} (\mu, t), t)).
\]

Since \( f \) is an fuzzy \( \theta \)-bicontinuous mapping, then \( f \) is fuzzy \( \theta \)-open map and \( \theta \)-continuous map. Therefore \( f \) is \( f \theta \) s-cts map and \( f \theta \) p-cts map. Hence
\[
f^{-1}(\mu) \leq C_{\tau_2} (\theta I_{\tau_2} (f^{-1} (\theta I_{\tau_2} (\mu, t), t)), t) \vee_{\tau_2} I_{\tau_2} (\Delta C_{\tau_2} (f^{-1} (I_{\tau_2} (\Delta C_{\tau_2} (\mu, t), t)), t), t).
\]

\[
\leq C_{\tau_2} (\theta I_{\tau_2} (f^{-1} (\mu, t), t)) \vee_{\tau_2} I_{\tau_2} (\Delta C_{\tau_2} (f^{-1} (\mu, t), t), t).
\]

This shows that \( f^{-1}(\mu) \) is \( \tau \)-fM o set in \((X, \tau_1) \).

**Remark 2.2** Let \((X, \tau_1) \) and \((Y, \tau_2) \) be sfts's and \( f : (X, \tau_1) \rightarrow (Y, \tau_2) \) be a fuzzy \( \theta \)-bicontinuous mapping. Then the inverse image of each \( \tau \)-fM o set in \((X, \tau_1) \) under \( f \) is \( \tau \)-fM o set in \((Y, \tau_2) \).

**Remark 2.3** Let \((X, \tau_1) \) and \((Y, \tau_2) \) be fts's and \( f : X \rightarrow Y \) be a mapping. The composition of two \( f \) M -cts mappings need not be \( f \) M -cts as shown by the following example.

**Example 2.8** Let \( \lambda, \omega \) and \( \mu \) be fuzzy subsets of \( X = Y = Z = \{a, b, c\} \) defined as follows:
\[
\lambda(x) = 0.4, \quad \lambda(y) = 0.5, \quad \lambda(z) = 0.2; \\
\omega(x) = 0.7, \quad \omega(y) = \tilde{1}, \quad \omega(z) = 0.5. \\
\mu(x) = 0.5, \quad \mu(y) = 0.4, \quad \mu(z) = 0.7.
\]

Then \( \tau_1, \tau_2 \) and \( \tau_3 : I^X \rightarrow I \) defined as
\[
\tau_1(\lambda) = \{1, \}
\]
= 0 or 1, \( \frac{1}{2} \), if \( \omega = 0 \), otherwise, \( \tau_2(\omega) = \{1, = 0 or 1, \frac{1}{2} \), if \( \mu = 0 \), otherwise, are fuzzy topologies on \( X \), \( Y \) and \( Z \). Consider the identity mapping \( f : (X, \tau_1) \rightarrow (Y, \tau_2) \) and \( g : (Y, \tau_2) \rightarrow (Z, \tau_3) \). Take \( t = \frac{1}{2} \). For any \( \frac{1}{2} \)-fo set \( \omega \) in \( (Y, \tau_2) \), \( f^{-1}(\omega) = \omega \) is \( \frac{1}{2} \)-fMo set in \( (X, \tau_1) \). Also, for any \( \frac{1}{2} \)-fuzzy open set \( \mu \) in \( (Z, \tau_3) \), \( g^{-1}(\mu) = \mu \) is \( \frac{1}{2} \)-fMo in \( (Y, \tau_2) \). Thus \( f \) is \( fM \)-cts and \( g \) is \( fM \)-cts. But \( g \circ f \) is not \( fM \)-cts, as \( \mu \) is \( \frac{1}{2} \)-fo set in \( (Z, \tau_3) \), \( (g \circ f)^{-1}(\mu) = f^{-1}(g^{-1}(\mu)) = \mu \) is not \( \frac{1}{2} \)-fMo in \( (X, \tau_1) \).

**Example 2.9** Let \( (X, \tau_1) \), \( (Y, \tau_2) \) and \( (Z, \tau_3) \) be sfts's. If \( f : (X, \tau_1) \rightarrow (Y, \tau_2) \) and \( g : (Y, \tau_2) \rightarrow (Z, \tau_3) \) are mappings, then

(i) \( g \circ f \) is \( fM \)-cts mapping if \( f \) is \( fM \)-cts and \( g \) is \( f \)-cts.

(ii) \( g \circ f \) is \( fM \)-cts mapping if \( f \) is fuzzy \( \theta \)-bicontinuous and \( g \) is \( fM \)-cts mapping.

**Proof.** (i) Let \( \mu \in \tau_1 \). Since \( g \) is \( f \)-cts, then \( g^{-1}(\mu) \) is an \( \frac{1}{2} \)-fo set in \( (Y, \tau_2) \). Since \( f \) is \( fM \)-cts, then \( f^{-1}(g^{-1}(\mu)) = (g \circ f)^{-1}(\mu) \) is \( \frac{1}{2} \)-fMo set in \( \tau_1 \). Hence \( g \circ f \) is \( fM \)-cts.

(ii) Let \( \mu \in \tau_3 \). Since \( g \) is \( fM \)-cts, then \( g^{-1}(\mu) \) is an \( \frac{1}{2} \)-fMo set in \( (Y, \tau_2) \). Since \( f \) is fuzzy \( \theta \)-bicontinuous, by Theorem Error! Reference source not found., \( (g \circ f)^{-1}(\mu) \) is \( \frac{1}{2} \)-fMo set in \( \tau_1 \). Hence \( g \circ f \) is \( fM \)-cts.

**Conclusion:** In this paper, \( fM \)-cts, \( f \theta \)-cts and \( f \delta s \)-cts in sfts's. Also, some of their fundamental properties are studied. Moreover, we investigate the relationships between \( f \)-cts, \( f \theta s \)-cts, \( f \theta \)-cts, \( f \delta s \)-cts, \( f \delta \)-cts, \( f a \)-cts, \( fM \)-cts, \( f e \)-cts and \( f e^* \)-cts mappings.

**References**


