**M-open Sets in Bitopological Spaces**

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**Abstract**

The aim of this paper is to introduce and investigate the concept of $\tau_i\tau_j$-$M^*$ closed sets which are introduced in a bitopological space in analogy with $M^*$ closed sets in topological spaces. Also $M^*$ closure and $M^*$ interior operators in bitopological spaces are introduced. In addition, several properties of these notions and connections to several other known ones are provided.

**Keywords and phrases:** $\tau_i\tau_j$-$M^*$ closed set, $\tau_i\tau_j$-$M^*$ $cl(A)$, $\tau_i\tau_j$-$M^*$ $int(A)$.

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1. **Introduction and Preliminaries**

Levine in 1963 initiated a new types of open set called semiopen set [9]. A subset $A$ of a space $(X, \tau)$ is called regular open (resp., regular closed) [12] if $A = int(cl(A))$ (resp., $A = cl(int(A))$). The delta interior [4] of a subset $A$ of $(X, \tau)$ is the union of all regular open sets of $X$ contained in $A$ and is denoted by $\delta int(A)$. A subset $A$ of a space $(X, \tau)$ is called $\delta$-open [11] if $A = \delta int(A)$. The complement of $\delta$-open set is called $\delta$-closed. Alternatively, a set $A$ of $(X, \tau)$ is called $\delta$-closed [4] if $A = \delta cl(A)$, where $\delta cl(A) = \{x \in X: A \cap int(cl(U)) \neq \phi, U \in \tau \text{ and } x \in U\}$. A subset $A$ of a space $X$ is called $\theta$-open [1] if $A = \theta int(A)$, where $\theta int(A) = \cup \{int(U): U \subseteq A, U \in \tau^c\}$, and a subset $A$ is called $\theta$-semiopen [2] (resp., $\delta$-preopen [11] , $e$-open [5], $M$-open [6], $M^*$-open [3], $\delta$-semiopen [10], $\delta$-open [12], $e^*$-open [5] and $a$-open [5]) if $A \subseteq cl(\theta int(A))$ (resp., $A \subseteq int(\delta cl(A))$, $A \subseteq cl(\delta int(A)) \cup int(\delta cl(A))$) and $A \subseteq cl(\theta int(A)) \cup int(\delta cl(A)) \cup int(\delta cl(A))$, $A \subseteq int(\delta cl(A))$, $A = \delta int(A)$, $A \subseteq cl(\delta cl(A))$ and $A \subseteq int(cl(A) \cap int(A))$, where $int()$, $cl()$, $\theta int()$, $\delta int()$ and $\delta cl()$ are the interior, closure, $\theta$-interior, $\delta$-interior and $\delta$-closure operations, respectively. The notion of bitopological spaces (in short, Bts’s) was first introduced by Kelly [7].

Throughout this paper, Let $(X, \tau_1, \tau_2)$ or simply $X$ be a Bts and $i, j \in \{1, 2\}$. A subset $S$ of a Bts $X$ is said to be $\tau_{1,2}$-open [8] if $S = A \cup B$ where $A \in \tau_1$ and $B \in \tau_2$. A subset $S$ of $X$ is said to be $\tau_{1,2}$-closed if the complement of $S$ is $\tau_{1,2}$-open. and $\tau_{1,2}$-clopen if $S$ is both $\tau_{1,2}$-open and $\tau_{1,2}$-closed. For a subset $A$ of $X$, the interior (resp., closure) of $A$ with respect to $\tau_i$ will be denoted by $int_i(A)$ (resp., $cl_i(A)$) for $i = 1, 2$. In this paper, we introduce and investigate the concept of $\tau_i\tau_j$-$M^*$ closed sets which are introduced in a bitopological spaces in analogy with $M^*$ closed sets in topological spaces [3]. Also introduce $M^*$ closure and $M^*$ interior operators in bitopological spaces. In addition, several properties of these notions and connections to several other known ones are provided.

**Definition 1.1** Let $(X, \tau_1, \tau_2)$ be a Bts. A subset $A$ of $X$ is called $\tau_i\tau_j$-$M$-open (briefly, $\tau_i\tau_j$-$M$-o) if $A \subseteq \tau_j(cl(\tau_i int(A))) \cup \tau_i int(\delta cl(A))$ and $A$ is $\tau_i\tau_j$-$M$-closed (in short, $\tau_i\tau_j$-$M$-c) if $X \setminus A$ is $\tau_i\tau_j$-$M$-o. A is pairwise $M$-open if it is both $\tau_i\tau_j$-$M$-o and $\tau_j\tau_i$-$M$-o. Let $(X, \tau_1, \tau_2)$ be a Bts. A subset $A$ of $X$ is called $\tau_i\tau_j$-$\theta$-semiopen (briefly, $\tau_i\tau_j$-$\theta$-so) if $A \subseteq cl_j(\theta int_i(A))$, $\tau_i\tau_j$-$\delta$-preopen (briefly, $\tau_i\tau_j$-$\delta$-po) if $A \subseteq int_i(\delta cl_j(A))$, $\tau_i\tau_j$-$e$-open (briefly, $\tau_i\tau_j$-$e$-o) if $A \subseteq cl_j(\delta int_i(A)) \cup$
Lemma 1.1 [5]
1. $A$ is open iff $A = \text{int}_\theta(A)$.
2. $\text{int}_\theta(A)$ is the union of all $\theta$-o sets of $X$ whose closure are contained in $A$.
3. For any subset $A$ of $X$, $A \subseteq cl(A) \subseteq cl_\delta(A) \subseteq cl_\theta(A)$ (resp).

Lemma 1.2 [5] Let $A$ be a subset of a space $(X, \tau)$. Then the following statements are hold.
1. $\text{pint}(\delta-cl(A)) = \delta-cl(A) \cap \text{int}(cl(A))$ and $\text{pcl}(\delta-pint(A)) = \delta-pcl(A) \cap \text{int}(cl(A))$.
2. $\text{pint}_\theta(\delta-pcl(A)) = \delta-pcl(A) \cap \text{int}(cl_\theta(A))$ and $\text{pcl}_\theta(\delta-pint(A)) = \delta-pint(A) \cap \text{int}(cl_\theta(A))$.
3. $\text{sc}_\theta(\text{int}_\theta(A)) = \text{sc}(\text{int}_\theta(A)) = \text{int}(\text{sc}(\text{int}_\theta(A)))$.
4. $\text{sint}_\theta(\text{cl}(\text{int}_\theta(A))) = \text{int}(\text{cl}(\text{int}_\theta(A)))$.

Lemma 1.3 [3] Let $U$ be an open subset of $(X, \tau)$. Then for any subset $A$, $(U \cap cl(A)) \subseteq \text{cl}(U \cap cl(A))$.

2 $\tau_1 \tau_j - M^*$-open Sets

Definition 2.1 Let $(X, \tau_1, \tau_2)$ be a Bts. A subset $A$ of $X$ is called $\tau_1 \tau_j - M^*$-open (briefly, $\tau_1 \tau_j - M^*$-o) if $A \subseteq \text{int}_1(\text{cl}_j(\theta \text{int}_1(A)))$ and $A$ is $\tau_1 \tau_j - M^*$-closed (briefly, $\tau_1 \tau_j - M^*$-c) if $X/A$ is $\tau_1 \tau_j - M^*$-o. $A$ is pairwise $M^*$-o if it is both $\tau_1 \tau_2 - M^*$-o and $\tau_2 \tau_1 - M^*$-o. Clearly $A$ is $\tau_1 \tau_j - M^*$-c iff $A \supseteq cl_j(\text{int}_1(\theta \text{cl}_j(A)))$. We denote the family of all $\tau_1 \tau_j - M^*$-c sets by $D_{M^*}(\tau_1, \tau_j)$.

Definition 2.2 Let $(X, \tau_1, \tau_2)$ be a Bts. A subset $A$ of $X$ is called $\tau_1 \tau_j - \delta$-semi open (briefly, $\tau_1 \tau_j - \delta$-so) if $A \subseteq \text{cl}_j(\delta \text{int}_1(A))$, $\tau_1 \tau_j - \delta$-open (briefly, $\tau_1 \tau_j - \delta$-o) if $A = \delta \text{int}_1(A)$, $\tau_1 \tau_j - e^*$-open (briefly, $\tau_1 \tau_j - e^*$-o) if $A \subseteq cl_j(\text{int}_1(\delta \text{cl}_j(A)))$, $\tau_1 \tau_j - a$-open (briefly, $\tau_1 \tau_j - a$-o) if $A \subseteq \text{int}(\text{cl}_1(\delta \text{int}_1(A)))$.

Lemma 2.1 Let $A$ be a subset of a space $(X, \tau_1, \tau_2)$. Then the following statements are hold:
1. Every $\tau_1-\theta$-o set is a $\tau_1 \tau_j - M^*$-o set.
2. Every $\tau_1-\tau$-o set is a $\tau_1 \tau_j - M^*$-o set.
3. Every $\tau_1 \tau_j - M^*$-o set is a $\tau_1 \tau_j - \theta$-so set.

Proof. (1) Let $A$ be a $\tau_1 \tau_j - \theta$-o set. Then $A = \theta \text{int}_1(A)$ and by Lemma 1.1 $\theta \text{int}_1(A) \subseteq \text{int}_1(A) \subseteq A$. Hence, $A \subseteq \text{int}_1(A)$. Since $A = \text{int}_1(A) \subseteq cl_j(\theta \text{int}_1(A))$ then $A = \text{int}_1(A) \subseteq \text{int}_1(\text{cl}_j(\theta \text{int}_1(A)))$. Thus $A$ is $\tau_1 \tau_j - M^*$-o.

(2) Obvious from the Definition.

(3) Let $A$ be $\tau_1 \tau_j - M^*$-o. Then $A \subseteq \text{int}_1(\text{cl}_j(\theta \text{int}_1(A)) \subseteq \text{cl}_j(\theta \text{int}_1(A))$. Hence $A$ is $\tau_1 \tau_j - \theta$-so.

But the converse of the above results need not be true as shown by the following example.

Example 2.1 Let $X = \{a, b, c, d\}$, $\tau_1 = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{d, c\}, \{a, d, c\}, \{b, c, d\}\}$ and $\tau_2 = \{\phi, X, \{a\}, \{b, d\}, \{a, c\}, \{a, b, d\}\}$. Then (1) the set $\{a, b, d\}$ is a $\tau_1 \tau_2 - M^*$-o set that is neither a $\tau_1-\theta$-o set nor $\tau_1 \tau_2 - \theta$-o set; (2) the set $\{d\}$ is a $\tau_1 \tau_2 - M^*$-o (resp. $\tau_1 \tau_2 - \delta$-o) set that is not $\tau_1 \tau_2 - \theta$-o; (3) the set $\{d\}$ is a $\tau_1 \tau_2 - \theta$-o set that is not a $\tau_1 \tau_2 - M^*$-o set; (4) the set $\{c\}$ is a $\tau_1 \tau_2 - e^*$-o (resp. $\tau_1 \tau_2 - \delta$-o) set that is not a $\tau_1 \tau_2 - M^*$-o and $\tau_1 \tau_2 - \theta$-so set; (5) the set $\{c\}$ is a $\tau_1 \tau_2 - e^*$-o set that is not a $\tau_1 \tau_2 - \delta$-o set.

Example 2.2 Let $X = \{a, b, c, d\}$, $\tau_1 = \{\phi, X, \{a\}, \{b\}\}$ and $\tau_2 = \{\phi, X, \{c\}\}$. Then the set $\{a, b\}$ is a $\tau_1 \tau_2 - \theta$-o set that is not a $\tau_1 \tau_2 - M^*$-o set.

Example 2.3 Let $X = \{a, b, c, d\}$, $\tau_1 = \{\phi, X, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $\tau_2 = \{\phi, X, \{a, b, d\}\}$. Then (1) the set $\{a, b\}$ is a $\tau_1 \tau_2 - a$-o (resp. $\tau_1 \tau_2 - e^*$-o) set that is not a $\tau_1 \tau_2 - M^*$-o set.
\(\tau_1 \tau_2 - \delta \text{-o} \) (resp. \(\tau_1 \tau_2 - \delta \text{-so} \) set; (2) the set \(\{a, c\}\) is a \(\tau_1 \tau_2 - \theta \text{-so} \) set that is not a \(\tau_1 \tau_2 - M^* \text{-o} \) set; (3) the set \(\{c\}\) is a \(\tau_1 \tau_2 - M \text{-o} \) set that is not a \(\tau_1 \tau_2 - \delta \text{-so} \) set.

**Example 2.4** Let \(X = \{a, b, c\}, \ \tau_1 = \{\phi, X, \{a\}, \{b\}\} \) and \(\tau_2 = \{\phi, X, \{a\}, \{a, b\}\} \) Then (1) the set \(\{a, c\}\) is a \(\tau_1 \tau_2 - \delta \text{-po} \) set that is not a \(\tau_1 \tau_2 - \alpha \text{-o} \) set; (2) the set \(\{b, c\}\) is a \(\tau_1 \tau_2 - e^* \text{-o} \) (resp. \(\tau_1 \tau_2 - \delta \text{-so} \) set that is not a \(\tau_1 \tau_2 - e \text{-o} \) (resp. \(\tau_1 \tau_2 - \alpha \text{-o} \) set.

**Remark 2.1** According to Definition 2.1, Lemma 2.1 and the above examples, the following diagram holds for a subset \(A\) of a space \(X:\)

![Diagram](image)

Note: \(A \rightarrow B\) denotes \(A\) implies \(B\), but not conversely.

**Theorem 2.1**
1. Arbitrary unions of \(\tau_1 \tau_j - M^* \text{-o} \) sets are \(\tau_1 \tau_j - M^\# \text{-o} \) sets.
2. The intersection of an \(\tau_1 \tau_j - M^* \text{-o} \) set with an \(\tau_1 \text{-o} \) set is an \(\tau_1 \tau_j - M^* \text{-o} \) set.
3. Arbitrary intersection of \(\tau_1 \tau_j - M^* \text{-c} \) sets are \(\tau_1 \tau_j - M^* \text{-c} \) sets.

**Proof.** (1) Let \(\{A_\alpha, \alpha \in I\}\) be a family of \(\tau_1 \tau_j - M^* \text{-o} \) sets. Then \(A_\alpha \subseteq \text{int}_i(cl_j(\text{int}_i(A_\alpha)))\), \(\forall \alpha \in I\), hence \(U_\alpha A_\alpha \subseteq U_\alpha \text{int}_i(cl_j(\text{int}_i(A_\alpha))) \subseteq \text{int}_i(cl_j(\text{int}_i(U_\alpha A_\alpha)))\) for all \(\alpha \in I\). Thus \(U_\alpha A_\alpha\) is \(\tau_1 \tau_j - M^* \text{-o} \) set.

(2) and (3) are obvious.

**Theorem 2.2** Let \(A\) be an \(\tau_1 \tau_j - M^* \text{-o} \) subset of \((X, \tau_1, \tau_2)\) and \(U \in \tau_1 \cap \tau_2\). Then \(U \cap A\) is \(\tau_1 \tau_j - M^* \text{-o} \) set.

**Proof.** As \(A\) is an \(\tau_1 \tau_j - M^* \text{-o} \) set, \(A \subseteq \text{int}_i(cl_j(\text{int}_i(A)))\). Thus, \(U \cap A \subseteq U \cap \text{int}_i(cl_j(\text{int}_i(A))) \subseteq cl_j(U \cap \text{int}_i(\text{int}_i(A)))\) by Lemma 2.1 \(\subseteq cl_j(\text{int}_i(cl_j(\text{int}_i(U \cap A)))\) by Lemma 2.1 Therefore, \(U \cap A\) is \(\tau_1 \tau_j - M^* \text{-o} \).

**Theorem 2.3** Let \(A\) and \(B\) be subset of \((X, \tau_1, \tau_2)\) such that \(A \subseteq B\). If \(A\) is an \(\tau_1 \tau_j - M^* \text{-o} \) set in \((X, \tau_1, \tau_2)\), then \(A\) is an \(\tau_1 \tau_j - M^* \text{-o} \) set in \((B, \tau_1 / B, \tau_2 / B)\).

**Proof.** \(A \subseteq \text{int}_i(cl_j(\text{int}_i(A))) \subseteq cl_j(\text{int}_i(\text{int}_i(A))) \cap B = \text{int}_i/b(cl_j/b(\text{int}_i/b(A)))\). Hence \(A\) is an \(\tau_1 \tau_j - M^* \text{-o} \) set in \((B, \tau_1 / B, \tau_2 / B)\).

**Example 2.5** In Example 2.1, \(A = \{c\} \in \tau_1 / B\) where \(B = \{a, b, c\}\). Hence \(A\) is an \(\tau_1 \tau_j - M^* \text{-o} \) set in \((B, \tau_1 / B, \tau_2 / B)\), but not an \(\tau_1 \tau_j - M^* \text{-o} \) set in \((X, \tau_1, \tau_2)\).

**Definition 2.3** Let \((X, \tau_1, \tau_2)\) be a Bts and \(i, j \in \{1, 2\}\) be fixed integers. For each subset \(E\) of \(X\), define \(\tau_1 \tau_j - M^* cl(E) = \cap \{A: E \subseteq A \in D_{M^*} (\tau_i, \tau_j)\}\).

**Theorem 2.4** Let \(A\) and \(B\) be subsets of \((X, \tau_1, \tau_2)\) then
1. \(\tau_1 \tau_j - M^* cl(X) = X\) and \(\tau_1 \tau_j - M^* cl(\phi) = \phi\).
2. \(A \subseteq \tau_1 \tau_j - M^* cl(A)\).
3. If \(B\) is any \(\tau_1 \tau_j - M^* \text{-c}\) set containing \(A\) then \(\tau_1 \tau_j - M^* cl(A) \subseteq B\).

**Proof.** Follows from the Definition 2.3.
Theorem 2.5 Let $A$ and $B$ be subsets of $(X, \tau_1, \tau_2)$ and $i, j \in \{1, 2\}$ be fixed integers. If $A \subseteq B$, then $\tau_i \tau_j \cdot M^* \text{cl}(A) \subseteq \tau_i \tau_j \cdot M^* \text{cl}(B)$.

Proof. Let $A \subseteq B$, by Definition 2.3, $\tau_i \tau_j \cdot M^* \text{cl}(B) = \bigcap \{F : B \subseteq F \in D_{M^*}(\tau_i, \tau_j)\}$. If $B \subseteq F \in D_{M^*}(\tau_i, \tau_j)$, since $A \subseteq B$, $A \subseteq B \subseteq F \in D_{M^*}(\tau_i, \tau_j)$. We have $\tau_i \tau_j \cdot M^* \text{cl}(A) \subseteq F$. Therefore $\tau_i \tau_j \cdot M^* \text{cl}(A) \subseteq \bigcap \{F : B \subseteq F \in D_{M^*}(\tau_i, \tau_j)\} = \tau_i \tau_j \cdot M^* \text{cl}(B)$. That is $\tau_i \tau_j \cdot M^* \text{cl}(A) \subseteq \tau_i \tau_j \cdot M^* \text{cl}(B)$.

Definition 2.4 Let $A$ be a subset of $(X, \tau_1, \tau_2)$. Then (1) The intersection of all $\tau_i \tau_j \cdot M^* \cdot c$ set containing $A$ is called the $\tau_i \tau_j \cdot M^* \cdot c$-closure of $A$, denoted by $\tau_i \tau_j \cdot M^* \text{cl}(A)$. (2) The union of all $\tau_i \tau_j \cdot M^* \cdot o$ sets contained in $A$ is called the $\tau_i \tau_j \cdot M^* \cdot o$-interior of $A$, denoted by $\tau_i \tau_j \cdot M^* \text{int}(A)$.

Theorem 2.6 Let $A$ and $B$ be subset of $(X, \tau_1, \tau_2)$ and $x \in X$. Then
1. $A$ is $\tau_i \tau_j \cdot M^* \cdot c$ iff $\tau_i \tau_j \cdot M^* \text{cl}(A) = A$.
2. $A$ is $\tau_i \tau_j \cdot M^* \cdot o$ iff $\tau_i \tau_j \cdot M^* \text{int}(A) = A$.
3. $x \in \tau_i \tau_j \cdot M^* \text{cl}(A)$ iff for every $\tau_i \tau_j \cdot M^* \cdot o$ set $U$ containing $x$, $U \cap A \neq \emptyset$.
4. $x \in \tau_i \tau_j \cdot M^* \text{int}(A)$ iff there exist an $\tau_i \tau_j \cdot M^* \cdot o$ set $U$ such that $x \in U \subseteq A$.
5. If $A \subseteq B$, then $\tau_i \tau_j \cdot M^* \text{int}(A) \subseteq \tau_i \tau_j \cdot M^* \text{int}(B)$ and $\tau_i \tau_j \cdot M^* \text{cl}(A) \subseteq \tau_i \tau_j \cdot M^* \text{cl}(B)$.

Theorem 2.7 Let $\{A_\alpha : \alpha \in \Delta\}$ be a family of subsets of $(X, \tau_1, \tau_2)$. Then
1. $\tau_i \tau_j \cdot M^* \text{int}(\bigcap \{A_\alpha : \alpha \in \Delta\}) \subseteq \bigcap \{\tau_i \tau_j \cdot M^* \text{int}(A_\alpha : \alpha \in \Delta)\}$.
2. $\bigcup \{\tau_i \tau_j \cdot M^* \text{int}(A_\alpha : \alpha \in \Delta)\} \subseteq \tau_i \tau_j \cdot M^* \text{int}(\bigcup \{A_\alpha : \alpha \in \Delta\})$.

Theorem 2.8 The following hold for a subset of a space $(X, \tau_1, \tau_2)$.
1. $A$ is $\tau_i \tau_j \cdot M^* \cdot o$ iff $A = A \cap \text{int}_i(cl_j(\text{int}_i(A)))$.
2. $A$ is $\tau_i \tau_j \cdot M^* \cdot c$ iff $A = A \cup \text{cl}_j(\text{int}_i(\text{cl}_j(A)))$.

Proof. (1) Let $A$ be an $\tau_i \tau_j \cdot M^* \cdot o$. Then $A \subseteq \text{int}_i(cl_j(\text{int}_i(A)))$. Hence $A \cap \text{int}_i(cl_j(\text{int}_i(A))) = A$. Conversely, $A = A \cap \text{int}_i(cl_j(\text{int}_i(A)))$. Then, $A \subseteq \text{int}_i(cl_j(\text{int}_i(A)))$. Hence $A$ is $\tau_i \tau_j \cdot M^* \cdot o$.

(2) Similar to the proof of (1).

Theorem 2.9 The following hold for a subset of a space $(X, \tau_1, \tau_2)$.
1. $\tau_i \tau_j \cdot M^* \text{int}(A) = A \cap \text{int}_i(cl_j(\text{int}_i(A)))$.
2. $\tau_i \tau_j \cdot M^* \text{cl}(A) = A \cup \text{cl}_j(\text{int}_i(\text{cl}_j(A)))$.

Proof. (1) Since $\tau_i \tau_j \cdot M^* \text{int}(A)$ is $\tau_i \tau_j \cdot M^* \cdot o$, $\tau_i \tau_j \cdot M^* \text{int}(A) \subseteq \text{int}_i(cl_j(\text{int}_i(\tau_i \tau_j \cdot M^* \text{int}(A)))) \subseteq \text{int}_i(cl_j(\text{int}_i(A)))$. Also, $A \cap \tau_i \tau_j \cdot M^* \text{int}(A) \subseteq A \cap \text{int}_i(cl_j(\text{int}_i(A)))$. Hence, $\tau_i \tau_j \cdot M^* \text{int}(A) \subseteq A \cap \text{int}_i(cl_j(\text{int}_i(A)))$. Conversely, since, $\text{int}_i(cl_j(\text{int}_i(A) \cap \text{int}_i(cl_j(\text{int}_i(A)))) \subseteq \text{int}_i(cl_j(\text{int}_i(A) \cap \text{int}_i(cl_j(\text{int}_i(A)))) \subseteq \text{int}_i(cl_j(\text{int}_i(A) \cap \text{int}_i(cl_j(\text{int}_i(A)))) \subseteq \text{int}_i(cl_j(\text{int}_i(A) \cap \text{int}_i(cl_j(\text{int}_i(A)))) = \text{int}_i(cl_j(\text{int}_i(A) \cap \text{int}_i(cl_j(\text{int}_i(A)))) \subseteq A \cap \text{int}_i(cl_j(\text{int}_i(A)))$. This implies that, $A \cap \text{int}_i(cl_j(\text{int}_i(A)))$ is an $\tau_i \tau_j \cdot M^* \cdot o$ set contained in $A$. Hence $A \cap \text{int}_i(cl_j(\text{int}_i(A))) \subseteq \tau_i \tau_j \cdot M^* \text{int}(A)$. Therefore $\tau_i \tau_j \cdot M^* \text{int}(A) = A \cap \text{int}_i(cl_j(\text{int}_i(A)))$.

Theorem 2.10 For a subset $A$ of a Bts $(X, \tau_1, \tau_2)$.
1. $A$ is $\tau_i \tau_j \cdot M^* \cdot o$ set iff $A = \tau_i \tau_j \cdot M^* \text{cl}(A)$.
2. $A$ is $\tau_i \tau_j \cdot M^* \cdot c$ set iff $A = \tau_i \tau_j \cdot M^* \text{cl}(A)$.

Theorem 2.11 Let $A$ and $B$ be subset of a space $(X, \tau_1, \tau_2)$. Then the following are hold:
1. $\tau_i \tau_j \cdot M^* \text{cl}(X \backslash A) = X \backslash \tau_i \tau_j \cdot M^* \text{int}(A)$.
2. $\tau_i \tau_j \cdot M^* \text{int}(X \backslash A) = X \backslash \tau_i \tau_j \cdot M^* \text{cl}(A)$.
3. If $A \subseteq B$, then $\tau_i \tau_j \cdot M^* \text{cl}(A) \subseteq \tau_i \tau_j \cdot M^* \text{cl}(B)$ and $\tau_i \tau_j \cdot M^* \text{int}(A) \subseteq \tau_i \tau_j \cdot M^* \text{int}(B)$.
4. $\tau_i \tau_j \cdot M^* \text{cl}(\tau_i \tau_j \cdot M^* \text{cl}(A)) = \tau_i \tau_j \cdot M^* \text{cl}(A)$ and $\tau_i \tau_j \cdot M^* \text{int}(\tau_i \tau_j \cdot M^* \text{int}(A)) = \tau_i \tau_j \cdot M^* \text{int}(A)$. 

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5. \( \tau_I \tau_J \cdot M^* \text{cl}(A) \cup \tau_I \tau_J \cdot M^* \text{cl}(B) \subseteq \tau_I \tau_J \cdot M^* \text{cl}(A \cup B) \) and \( \tau_I \tau_J \cdot M^* \text{int}(A) \cup \tau_I \tau_J \cdot M^* \text{int}(B) \subseteq \tau_I \tau_J \cdot M^* \text{int}(A \cup B) \).

6. \( \tau_I \tau_J \cdot M^* \text{cl}(A) \cap \tau_I \tau_J \cdot M^* \text{cl}(B) \supseteq \tau_I \tau_J \cdot M^* \text{cl}(A \cap B) \) and \( \tau_I \tau_J \cdot M^* \text{int}(A) \cap \tau_I \tau_J \cdot M^* \text{int}(B) \supseteq \tau_I \tau_J \cdot M^* \text{int}(A \cap B) \).

**Proof.** (1) By Theorem 2.10, \( \tau_I \tau_J \cdot M^* \text{cl}(X \setminus A) = (X \setminus A) \cup (cl_I \{ \text{int}_I(\theta cl_I((X \setminus A))) \} = (X \setminus A) \cup ((X \setminus \{ \text{int}_I(\theta cl_I((X \setminus A))) \}) = (X \setminus (A \cap \text{cl}_I(\{ \text{int}_I(\theta cl_I((X \setminus A))) \})) = X \setminus (\tau_I \tau_J \cdot M^* \text{int}(A)).

(2) and (3) follows from the Definition 2.4.

(4) By Theorem 2.10(1), \( \tau_I \tau_J \cdot M^* \text{cl}(\tau_I \tau_J \cdot M^* \text{cl}(A)) = cl_I \{ \text{int}_I(\theta cl_I(\tau_I \tau_J \cdot M^* \text{cl}(A)) = cl_I \{ \text{int}_I(\theta cl_I((A \cup cl_I(\{ \text{int}_I(\theta cl_I(\{ \text{int}_I(\theta cl_I((X \setminus A)))) \})) \subseteq cl_I \{ \text{int}_I(\theta cl_I((A \cup \theta cl_I(\{ \text{int}_I(\theta cl_I(\{ \text{int}_I(\theta cl_I((X \setminus A)))) \})) \subseteq cl_I \{ \text{int}_I(\theta cl_I(\{ \text{int}_I(\theta cl_I((X \setminus A)))) \})) \subseteq \tau_I \tau_J \cdot M^* \text{cl}(A). \)

But \( \tau_I \tau_J \cdot M^* \text{cl}(A) \subseteq \tau_I \tau_J \cdot M^* \text{cl}(\tau_I \tau_J \cdot M^* \text{cl}(A)). \)

Hence \( \tau_I \tau_J \cdot M^* \text{cl}(A) = \tau_I \tau_J \cdot M^* \text{cl}(\tau_I \tau_J \cdot M^* \text{cl}(A)). \)

(5) By Theorem 2.10(2), \( \tau_I \tau_J \cdot M^* \text{cl}(A) \tau_I \tau_J \cdot M^* \text{cl}(B) \) = \( (A \cup cl_I(\{ \text{int}_I(\theta cl_I(A))) \cup (B \cup cl_I(\{ \text{int}_I(\theta cl_I(B)))) \) = \( (A \cup cl_I(\{ \text{int}_I(\theta cl_I(A))) \cup cl_I(\{ \text{int}_I(\theta cl_I(\{ \text{int}_I(\theta cl_I(\{ \text{int}_I(\theta cl_I(\{ \text{int}_I(\theta cl_I((X \setminus A)))) \})) \} \subseteq \tau_I \tau_J \cdot M^* \text{cl}(A) \subseteq \tau_I \tau_J \cdot M^* \text{cl}(\tau_I \tau_J \cdot M^* \text{cl}(A)). \)

Hence \( \tau_I \tau_J \cdot M^* \text{cl}(A) \subseteq \tau_I \tau_J \cdot M^* \text{cl}(\tau_I \tau_J \cdot M^* \text{cl}(A)). \)

(6) By Theorem 2.10(3), \( \tau_I \tau_J \cdot M^* \text{int}(A \cap B) = (A \cap B) \cup int_I(cl_I(\{ \text{int}_I(A \cap B)))) \) = \( (A \cap int_I(cl_I(\{ \text{int}_I(A \cap B)))) \cup (B \cap int_I(cl_I(\{ \text{int}_I(B)))) \) = \( \tau_I \tau_J \cdot M^* \text{int}(A) \cap \tau_I \tau_J \cdot M^* \text{int}(B). \)

**Lemma 2.2** Let \( A \) be subset of a space \( (X, \tau_1, \tau_2) \). Then,

1. \( \tau_I \tau_J \cdot M^* \text{cl}(A) = (A \cup s \text{int}_I(\{ \theta cl_I(A))) \).
2. \( \tau_I \tau_J \cdot M^* \text{int}(A) = (A \cap s \text{cl}_I(\{ \theta int_I(A))) \).

**Proof.** (1) From Lemma 1.2(4) \( (A \cup s \text{int}_I(\{ \theta cl_I(A))) = (A \cup cl_I(\{ \text{int}_I(\theta cl_I(A))) \) = \( \tau_I \tau_J \cdot M^* \text{cl}(A) \).

(2) From Lemma 1.2(3) \( (A \cap s \text{cl}_I(\{ \theta int_I(A))) = (A \cap cl_I(\{ \theta int_I(A))) \) = \( \tau_I \tau_J \cdot M^* \text{int}(A) \).

**Theorem 2.12** The following are equivalent for a subset \( A \) of \( (X, \tau_1, \tau_2) \).

1. \( A \) is an \( \tau_I \tau_J \cdot M^* \)-o set.
2. \( A \subseteq (s \theta cl_I(\{ \theta int_I(A))) \).
3. \( (s \theta cl_I(\{ \theta int_I(A))) \).

**Proof.**

(1) \( \Rightarrow \) (2): Let \( A \) be an \( \tau_I \tau_J \cdot M^* \)-o set. Then by Theorem 2.11, \( A = \tau_I \tau_J \cdot M^* \text{int}(A) \). By Lemma 1.3, \( A = (A \cup s \theta cl_I(\{ \theta int_I(A))) \) = \( A \subseteq s \theta cl_I(\{ \theta int_I(A))) \). Hence \( A \subseteq s \theta cl_I(\{ \theta int_I(A))) \).

(2) \( \Rightarrow \) (1): \( A \subseteq (s \theta cl_I(\{ \theta int_I(A))) \). This implies that \( A \subseteq A \cup s \theta cl_I(\{ \theta int_I(A)) = \tau_I \tau_J \cdot M^* \text{int}(A) \).

Hence \( A \subseteq s \tau_I \tau_J \cdot M^* \text{int}(A) \) and hence \( A = \tau_I \tau_J \cdot M^* \text{int}(A) \) and it is \( \tau_I \tau_J \cdot M^* \)-o.

(2) \( \Rightarrow \) (3): \( A \subseteq s \theta cl_I(\{ \theta int_I(A)) \). Then \( s \theta cl_I(\{ \theta int_I(A)) \subseteq \theta cl_I(\{ \theta int_I(A)) \). But \( \theta int_I(A) \subseteq A \). Hence \( \theta cl_I(\{ \theta int_I(A)) \subseteq s \theta cl_I(\{ \theta int_I(A)) \).

(3) \( \Rightarrow \) (2): Let \( s \theta cl_I(\{ \theta int_I(A)) \subseteq \theta cl_I(\{ \theta int_I(A)) \). Then \( \theta cl_I(\{ \theta int_I(A)) \subseteq s \theta cl_I(\{ \theta int_I(A)) \). But \( A \subseteq s \theta cl_I(\{ \theta int_I(A)) \).

And therefore \( \theta cl_I(\{ \theta int_I(A)) \).

**Theorem 2.13** Let \( A \) be subset of a space \( (X, \tau_1, \tau_2) \). Then the following are equivalent:

1. \( A \) is an \( \tau_I \tau_J \cdot M^* \)-c set.
2. \( A \supseteq (s \theta int_I(\theta cl_I(A))) \).
3. \( (s \theta int_I(\theta cl_I(A))) \).

**Definition 2.5** A subset \( A \) of a Bts \( (X, \tau_1, \tau_2) \) is said to be locally \( \tau_I \tau_J \cdot M^* \)-closed (briefly, \( l-\tau_I \tau_J \cdot M^* \)-c) if \( A = U \cap A \) for each \( u \in \tau_I \tau_J \cdot o \) and \( F \in \tau_I \tau_J \cdot M^* C(X) \).

**Theorem 2.14** Let \( H \) be a subset of a space \( (X, \tau_1, \tau_2) \). Then \( H \) is \( l-\tau_I \tau_J \cdot M^* \)-c if and only if \( H = U \cap \tau_I \tau_J \cdot M^* \text{cl}(H) \).
Proof. Let \( H \) be an \( l - \tau_i \tau_j - M^* -c \) set. Then \( H = U \cap F \) for each \( u \in \tau_i \tau_j -o \) and \( F \in \tau_i \tau_j - M^* \mathcal{C}(X) \). Hence \( H \subseteq \tau_i \tau_j - M^* \mathcal{C}(H) \subseteq \tau_i \tau_j - M^* \mathcal{C}(F) = F \). Thus \( U \cap H \subseteq U \cap \tau_i \tau_j - M^* \mathcal{C}(H) \subseteq U \cap \tau_i \tau_j - M^* \mathcal{C}(F) = H \). This implies that \( H \subseteq U \cap \tau_i \tau_j - M^* \mathcal{C}(H) \subseteq U \cap \tau_i \tau_j - M^* \mathcal{C}(F) = H \). Hence \( H = U \cap \tau_i \tau_j - M^* \mathcal{C}(H) \). Converse is obvious, since \( \tau_i \tau_j - M^* \mathcal{C}(H) \in \tau_i \tau_j - M^* \mathcal{C}(X) \).

**Theorem 2.15** Let \( A \) be \( l - \tau_i \tau_j - M^* -c \) subset of a Bts \((X, \tau_1, \tau_2)\). Then the following hold:

1. \( \tau_i \tau_j - M^* \mathcal{C}(A) \setminus A \) is an \( \tau_i \tau_j - M^* -c \) set.
2. \( A \cup (X \setminus \tau_i \tau_j - M^* \mathcal{C}(A)) \) is an \( \tau_i \tau_j - M^* -o \) set.
3. \( A \subseteq \tau_i \tau_j - M^* \text{int}(A \cup (X \setminus \tau_i \tau_j - M^* \mathcal{C}(A))) \).

**Theorem 2.16** Let \( A \) be subsets of \((X, \tau_1, \tau_2)\). If \( \tau_1 \subseteq \tau_2 \) and \( M^* -o \) \((X, \tau_1) \subseteq M^* -o \) \((X, \tau_2)\), then \((1,2) - M^* \mathcal{C}(A) \subseteq (2,1) - M^* \mathcal{C}(A)\).

Proof. By Definition 2.3, \((1,2) - M^* \mathcal{C}(A) = \bigcap \{ F : A \subseteq F \in D_M^*(1,2) \} \). Since \( \tau_1 \subseteq \tau_2 \) and \( M^* -o \) \((X, \tau_1) \subseteq M^* -o \) \((X, \tau_2)\), \( D_M^*(\tau_2, \tau_1) \subseteq D_M^*(\tau_1, \tau_2) \) implies \( D_M^*(1,2) \subseteq D_M^*(2,1) \). Therefore \( \bigcap \{ F : A \subseteq F \in D_M^*(1,2) \} \subseteq \bigcap \{ F : A \subseteq F \in D_M^*(2,1) \} \) is \((2,1) - M^* \mathcal{C}(A) \). Hence, \((1,2) - M^* \mathcal{C}(A) \subseteq (2,1) - M^* \mathcal{C}(A)\).

**Theorem 2.17** Let \( A \) be subsets of \((X, \tau_1, \tau_2)\) and \( \tau_i, \tau_j \in \{1,2\} \) be fixed integers, then \( A \subseteq \tau_i \tau_j - M^* \mathcal{C}(A) \subseteq \tau_j - \mathcal{C}(A) \).

Proof. By Definition 2.3, it follows that \( A \subseteq \tau_i \tau_j - M^* \mathcal{C}(A) \). Now to prove that \( \tau_i \tau_j - M^* \mathcal{C}(A) \subseteq \tau_j - \mathcal{C}(A) \), by Definition of closure, \( \tau_j - \mathcal{C}(A) = \bigcap \{ F : X \subseteq A \subseteq F \text{ and } F \in \tau_j \} \). If \( A \subseteq F \) and \( F \) is \( \tau_j - \mathcal{C} \) set then \( F \) is \( \tau_i \tau_j - M^* -c \) as every \( \tau_i - \mathcal{C} \) set is \( \tau_i \tau_j - M^* -c \).

Therefore \( \tau_i \tau_j - M^* \mathcal{C}(A) \subseteq \bigcap \{ F : X \subseteq A \subseteq F \text{ and } F \in \tau_j \} \) and \( \tau_j - \mathcal{C}(A) \).

**Example 2.6** Let \( X = \{a, b, c, d\} \), \( \tau_1 = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{d, c\}, \{a, d\}, \{b, c, d\}\} \) and \( \tau_2 = \{\phi, X, \{a\}, \{b, d\}, \{a, c\}, \{c, \{a, b, d\}\} \) and \( M^* -c \) sets are \( \{\phi, X, \{a\}, \{b\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{c, \{a, b, d\}\} \). Take \( A = \{b, c\} \). Then \( \tau_2 - \mathcal{C}(A) = (X \setminus \{b, c\}) \). Now \( A \subseteq \tau_2 - \mathcal{C}(A) \) but \( A \not\subseteq (2,1) - M^* \mathcal{C}(A) \). Also \( (1,2) - M^* \mathcal{C}(A) \subseteq (2,1) - M^* \mathcal{C}(A) \).

**Theorem 2.18** Let \( A \) be subsets of \((X, \tau_1, \tau_2)\) and \( i, j \in \{1,2\} \) be fixed integers, if \( A \) is \( \tau_i \tau_j - M^* -c \) then \( \tau_i \tau_j - M^* \mathcal{C}(A) = A \).

Proof. Let \( A \) be a \( \tau_i \tau_j - M^* -c \) subset of \((X, \tau_1, \tau_2)\). We know that \( A \subseteq \tau_i \tau_j - M^* \mathcal{C}(A) \) also \( \beta \subseteq A \) and \( A \) is \( \tau_i \tau_j - M^* -c \) by Theorem 2.4 (3) \( \tau_i \tau_j - M^* \mathcal{C}(A) \subseteq A \). Hence \( \tau_i \tau_j - M^* \mathcal{C}(A) = A \).

**Theorem 2.9** The operator \( \tau_i \tau_j - M^* -closure \) in Definition 3(\textit{a}), (1) is the Kuratowski closure operator on \( X \).

Proof. (1) Let \( \tau_i \tau_j - M^* \mathcal{C}(\phi) = \phi \) by Theorem 2.4 (1).

(2) Suppose \( E \subseteq \tau_i \tau_j - M^* \mathcal{C}(E) \) for any subset \( E \) of \( X \) by Theorem 2.4 (2).

(3) Suppose \( E \) and \( F \) are two subsets of \((X, \tau_1, \tau_2)\). It follows from Theorem 2.5 that, \( \tau_i \tau_j - M^* \mathcal{C}(E) \subseteq \tau_i \tau_j - M^* \mathcal{C}(E \cup F) \) and that, \( \tau_i \tau_j - M^* \mathcal{C}(F) \subseteq \tau_i \tau_j - M^* \mathcal{C}(E \cup F) \). Hence we have \( \tau_i \tau_j - M^* \mathcal{C}(E) \cup \tau_i \tau_j - M^* \mathcal{C}(F) \subseteq \tau_i \tau_j - M^* \mathcal{C}(E \cup F) \). Now if \( x \notin \tau_i \tau_j - M^* \mathcal{C}(E) \cup \tau_i \tau_j - M^* \mathcal{C}(F) \) then \( x \notin \tau_i \tau_j - M^* \mathcal{C}(E \cup F) \) it follows that there exist \( A, B \in D_M^*(\tau_i, \tau_j) \) such that \( E \subseteq A \), \( x \notin A \) and \( F \subseteq B \), \( x \notin B \). Hence, \( E \cup F \subseteq A \cup B \), \( x \notin A \cup B \). Since \( A \cup B \) is \( \tau_i \tau_j - M^* -c \). If \( A, B \in D_M^*(\tau_i, \tau_j) \), then \( A \cup B \in D_M^*(\tau_i, \tau_j) \). So \( X \notin \tau_i \tau_j - M^* \mathcal{C}(E \cup F) \). Then we have \( \tau_i \tau_j - M^* \mathcal{C}(E \cup F) \subseteq \tau_i \tau_j - M^* \mathcal{C}(E) \cup \tau_i \tau_j - M^* \mathcal{C}(F) \). From the above discussions we have \( \tau_i \tau_j - M^* \mathcal{C}(E \cup F) = \tau_i \tau_j - M^* \mathcal{C}(E) \cup \tau_i \tau_j - M^* \mathcal{C}(F) \).

(4) Let \( E \) be any subset of \((X, \tau_1, \tau_2)\) by the Definition of \( \tau_i \tau_j - M^* -closure \), \( \tau_i \tau_j - M^* \mathcal{C}(E) = \bigcap \{ A \subseteq X : E \subseteq A \in D_M^*(\tau_i, \tau_j) \} \). If \( E \subseteq A \in D_M^*(\tau_i, \tau_j) \), then \( \tau_i \tau_j - M^* \mathcal{C}(E) \subseteq A \). Since \( A \) is a \( \tau_i \tau_j - M^* -c \) set containing \( \tau_i \tau_j - M^* \mathcal{C}(E) \) by Theorem 2.4 (3),
Definition 2.6 Let \( i,j \in \{1,2\} \) be two fixed integers. Let \( \tau_M(\tau_{ij}) \) be a topology on \( X \) generated by \( \tau_{ij} \) - \( M^* \)-closure in the usual manner. That is \( \tau_M(\tau_{ij}) = \{ E \subseteq X : \tau_{ij} \tau_{M^*} \text{cl}(E) = E \} \).

**Theorem 2.20** let \((X,\tau_i,\tau_j)\) be a Bts and \( i,j \in \{1,2\} \) be two fixed integers, then \( \tau_j \subseteq \tau_M(\tau_{ij}) \).

**Proof.** Let \( G \in \tau_j \), it follows that \( G^c \) is \( \tau_j \)-c by Theorem 2.17 is \( \tau_{ij} \tau_{M^*} \text{c} \). Therefore \( \tau_{ij} \text{cl}(G^c) = G^c \), by Theorem 2.18 that \( G \in \tau_M(\tau_{ij}) \) and hence \( \tau_j \subseteq \tau_M(\tau_{ij}) \).

**Theorem 2.21** Let \((X,\tau_1,\tau_2)\) be a Bts and \( i,j \in \{1,2\} \) be two fixed integers, if a subset \( E \) of \( X \) is \( \tau_{ij} \tau_{M^*} \text{c} \), then \( E \) is \( \tau_M \text{c} \).

**Proof.** Let a subset \( E \) of \( X \) be \( \tau_{ij} \tau_{M^*} \text{c} \). By Theorem 2.18, \( \tau_{ij} \tau_{M^*} \text{cl}(E) = E \). That is \( \tau_{ij} \tau_{M^*} \text{cl}(E^c) = (E^c)^c \), it follows that \( E^c \in \tau_M(1,2) \). Therefore \( E \) is \( \tau_M(\tau_{ij}) \text{c} \).

**Theorem 2.22** If \( \tau_1 \subseteq \tau_2 \) and \( M^* \circ (X,\tau_1) \subseteq M^* \circ (X,\tau_2) \) in \((X,\tau_1,\tau_2)\) then \( \tau_M(1,2) \subseteq \tau_M(1,2) \).

**Proof.** Let \( G \in \tau_M(2,1) \), then \( (2,1) \tau_{M^*} \text{cl}(G^c) = G^c \). To prove that \( G \in \tau_M(1,2) \). That is to prove \((1,2) \tau_{M^*} \text{cl}(G^c) = G^c \). Now \((1,2) \tau_{M^*} \text{cl}(G^c) = \bigcap \{ F \subseteq X : G^c \subseteq F \in D_M(1,2) \} \). Since \( \tau_1 \subseteq \tau_2 \) and \( M^* \circ (X,\tau_1) \subseteq M^* \circ (X,\tau_2) \), by Theorem 2.21 \( D_M(2,1) \subseteq D_M(1,2) \). Thus \( \bigcap \{ F \subseteq X : G^c \subseteq F \in D_M(1,2) \} \subseteq \bigcap \{ F \subseteq X : G^c \subseteq F \in D_M(2,1) \} \). This is \((1,2) \tau_{M^*} \text{cl}(G^c) \subseteq (2,1) \tau_{M^*} \text{cl}(G^c) \) is true by Theorem 2.17 (3). Then we have \((1,2) \tau_{M^*} \text{cl}(G^c) = G^c \). That is \( G \in \tau_M(1,2) \) and hence \( \tau_M(2,1) \subseteq \tau_M(1,2) \).

**References**


