REGULAR WEAKLY HOMEOMORPHISM IN IDEAL TOPOLOGICAL SPACES

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Abstract

In this paper we introduce and study two new homeomorphisms namely $I_{rw}$-homeomorphism and $I_{rw}^*$-homeomorphism and study some of their properties in ideal topological spaces.

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1. Introduction


In this paper, we introduce the concepts of $I_{rw}$-homeomorphism and study the relationship with $I$ -homeomorphisms. Also we introduce new class of maps $I_{rw}$-homeomorphism which forms a subclass of $I_{rw}$-homeomorphism. This class of maps is closed under compositions of maps. We prove that the set of all $I_{rw}$-homeomorphisms form a group under the composition of maps.

2. Preliminaries

Let $(X, \tau)$ be a topological space with no separation properties assumed. For a subset $A$ of a topological space $(X, \tau)$, $\text{cl}(A)$ and $\text{int}(A)$ denote the closure and interior of $A$ in $(X, \tau)$, respectively. An ideal $I$ on a topological space $(X, \tau)$ is a non-empty collection of subsets of $X$ which satisfies the following properties: (1) $A \in I$ and $B \subseteq A$ implies $B \in I$, (2) $A \in I$ and $B \in I$ implies $A \cup B \in I$. An ideal topological space is a topological space $(X, \tau)$ with an ideal $I$ on $X$ and is denoted by $(X, \tau, I)$. For a subset $A \subseteq X$, $A^*(I, \tau) = \{x \in X \mid A \cap U \notin I \text{ for every } U \in \tau(X, x)\}$ is called the local function of $A$ with respect to $I$ and $\tau$ [3, 4]. We simply write $A^*$ instead of $A^*(I, \tau)$ in case there is no chance for confusion. For every ideal topological space $(X, \tau, I)$, there exists a topology $\tau^*(I)$, finer than $\tau$, generated by the base $\beta(I, \tau) = \{U - J \mid U \in \tau \text{ and } J \in I\}$. It is known in [3] that $\beta(I, \tau)$ is not always a topology. When there is no ambiguity, $\tau^*(I)$ is denoted by $\tau^*$. For a
subset $A \subseteq X$, $\text{cl}^*(A)$ and $\text{int}^*(A)$ will, respectively, denote the closure and interior of $A$ in $(X, \tau^*)$.

**Definition 2.1**  
(i) A subset $A$ of a space $(X, \tau)$ is said to be regular open [5] if $A = \text{int}(\text{cl}(A))$ and $A$ is said to be regular closed [5] if $A = \text{cl}(\text{int}(A))$.  
(ii) A subset $A$ of a space $(X, \tau)$ is said to be regular semiopen [2] if there is a regular open set $U$ such that $U \subseteq A \subseteq \text{cl}(U)$. The complement of a regular semiopen set is said to be regular semiclosed.  
(iii) A subset $A$ of a space $(X, \tau)$ is said to be $\text{rw}$-closed [9] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular semiopen. $A$ is said to be $\text{rw}$-open if $X - A$ is $\text{rw}$-closed.

**Definition 2.2**  
(1) A subset $A$ of an ideal space $(X, \tau, I)$ is said to be $I$-open [1] if $A \subseteq \text{int}(A^*)$. The complement of an $I$-open set is said to be $I$-closed.  
(2) A subset $A$ of an ideal space $(X, \tau, I)$ is said to be a regular weakly closed set with respect to the ideal $I$ ($I_{rw}$-closed) [6] if $A^* \subseteq U$ whenever $A \subseteq U$ and $U$ is regular semiopen. $A$ is called a regular weakly open set ($I_{rw}$-open) if $X - A$ is an $I_{rw}$-closed set.

### 3. $I_{rw}$-homeomorphism in ideal topological space

We introduce the following definitions

**Definition 3.1** A map $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ is said to be 
(i) $I_{rw}$-closed if the image $f(A)$ is $I_{rw}$-closed set in $(Y, \sigma)$ for each closed set $A$ in $(X, \tau, I)$.  
(ii) $I_{rw}$-continuous [7] if $f^{-1}(A)$ is $I_{rw}$-closed set in $(X, \tau, I)$ for each closed set $A$ in $(Y, \sigma)$.

**Definition 3.2** A bijective function $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ is called 
(i) $I_{rw}$-irresolute if the inverse image $f^{-1}(A)$ is $I_{rw}$-closed set in $(X, \tau, I)$ for each $I_{rw}$-closed set $A$ in $(Y, \sigma, I)$.  
(ii) $I_{rw}^*$-homeomorphism if both $f$ and $f^{-1}$ are $I_{rw}$-irresolute.  
(iii) $I_{rw}^*$-homeomorphism if both $f$ and $f^{-1}$ are $I_{rw}$-continuous.

We say the spaces $(X, \tau, I)$ and $(Y, \sigma, J)$ are $I_{rw}$-homeomorphic if there exists a $I_{rw}$-homeomorphism from $(X, \tau, I)$ onto $(Y, \sigma, J)$.

We denote the family of all $I_{rw}$-homeomorphisms (resp. $I_{rw}^*$-homeomorphisms) of an ideal topological space $(X, \tau, I)$ onto itself by $I_{rw}^h(X, \tau, I)$ (resp. $I_{rw}^{*h}(X, \tau, I)$).

**Theorem 3.1** Every $I$-homeomorphism is a $I_{rw}$-homeomorphism but not conversely.

**Proof.** Let $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ be a $I$-homeomorphism. Then $f$ and $f^{-1}$ are $I$-continuous and $f$ is bijection. As every $I$-continuous function is $I_{rw}^*$-continuous, we have $f$ and $f^{-1}$ are $I_{rw}$-continuous. Therefore, $f$ is $I_{rw}$-homeomorphism.

The converse of the above theorem need not be true, as seen from the following example.

**Example 3.1** Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{c\}, \{a, c\}\}$, $\sigma = \{X, \phi, \{c\}, \{a, b\}\}$ and $I = \{\phi, \{c\}\}$. Then the identity map on $X$ is a $I_{rw}$-homeomorphism, but it is not $I$-homeomorphism. Since the inverse image of the open set $\{c\}$ in $(Y, \sigma)$ is $\{c\}$ which is not
Theorem 3.2 For any bijection \( f : (X, \tau, I) \rightarrow (Y, \sigma, J) \), the following statements are equivalent:
(i) \( f^{-1} : (Y, \sigma, J) \rightarrow (X, \tau, I) \) is \( I_{rw} \)-continuous.
(ii) \( f \) is a \( I_{rw} \)-open map.
(iii) \( f \) is a \( I_{rw} \)-closed map.

Proof. (i) \( \Rightarrow \) (ii): Let \( U \) be a \( I \)-open set of \((X, \tau, I)\). By assumption \((f^{-1})^{-1}(U) = f(U)\) is a \( I_{rw} \)-open set in \((Y, \sigma, J)\) and so \( f \) is a \( I_{rw} \)-open map.
(ii) \( \Rightarrow \) (iii): Let \( F \) be a \( I \)-closed set of \((X, \tau, I)\). Then \( F^c \) is \( I \)-open in \((X, \tau, I)\). Since \( f \) is \( I_{rw} \)-open, \( f(F^c) \) is \( I_{rw} \)-open in \( Y \). But \( f(F^c) = (f(F))^c \), we have \( f(F) \) is \( I_{rw} \)-closed in \( Y \) and so \( f \) is a \( I_{rw} \)-closed map.
(iii) \( \Rightarrow \) (i): Suppose \( F \) is a \( I \)-closed set in \((X, \tau, I)\). By assumption \( f(F) = (f^{-1})^{-1}(F) \) is \( I_{rw} \)-closed set in \((Y, \sigma, J)\) and so \( f^{-1} \) is \( I_{rw} \)-continuous.

Theorem 3.3 Let \( f : (X, \tau, I) \rightarrow (Y, \sigma, J) \) be a bijective and \( I_{rw} \)-continuous, then the following statements are equivalent:
(i) \( f \) is a \( I_{rw} \)-open map.
(ii) \( f \) is a \( I_{rw} \)-homeomorphism.
(iii) \( f \) is a \( I_{rw} \)-closed map.

Proof. Proof follows from the Definitions 3.1, 3.2 and Theorem 3.2.

Remark 3.1 The composition of two \( I_{rw} \)-homeomorphism need not be a \( I_{rw} \)-homeomorphism as seen from the following example.

Example 3.2 Let \( X = Y = Z = \{a, b, c\} \), \( \tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\} \), \( I = \{\emptyset, \{c\}\} \), \( \sigma = \{\emptyset, \{c\}, \{a, b\}, Y\} \) and \( \eta = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Z\} \) respectively. Let \( f : (X, \tau, I) \rightarrow (Y, \sigma, I) \) and \( g : (Y, \sigma, I) \rightarrow (Z, \eta) \) be identity map respectively. Then both \( f \) and \( g \) are \( I_{rw} \)-homeomorphisms but their composition \( g \circ f : (X, \tau, I) \rightarrow (Z, \eta) \), is not a \( I_{rw} \)-homeomorphism, because for the open sets \( \{b\} \) of \((X, \tau, I)\), \( g \circ f(\{b\}) = g(f(\{b\})) = g(\{b\}) = \{b\} \) which is not a \( I_{rw} \)-open in \((Z, \eta)\). Therefore, \( g \circ f \) is not a \( I_{rw} \)-open and not a \( I_{rw} \)-homeomorphism.

Definition 3.3 A map \( f : (X, \tau, I) \rightarrow (Y, \sigma, J) \) is called \( I^*_rw \)-open if \( f(U) \) is \( I_{rw} \)-open in \((Y, \sigma, J)\) for every \( I_{rw} \)-open set \( U \) of \((X, \tau, I)\).

Theorem 3.4 For any bijection \( f : (X, \tau, I) \rightarrow (Y, \sigma, J) \), the following statements are equivalent:
(i) \( f^{-1} : (Y, \sigma, J) \rightarrow (X, \tau, I) \) is \( I_{rw} \)-irresolute.
(ii) \( f \) is a \( I^*_rw \)-open map.
(iii) \( f \) is a \( I^*_rw \)-closed map.
Proof. (i) \(\Rightarrow\) (ii) Let \(U\) be a \(I_{rw^-}\)-open in \((X,\tau,I)\). By (i) \((f^{-1})^{-1}(U) = f(U)\) is \(I_{rw^-}\)-open in \((Y,\sigma,J)\). Hence (ii) holds.

(ii) \(\Rightarrow\) (iii) Let \(V\) be \(I_{rw^-}\)-closed in \((X,\tau,I)\). Then \(X - V\) is \(I_{rw^-}\)-open and by (ii) \(f(X - V) = Y - f(V)\) is \(I_{rw^-}\)-open in \((Y,\sigma,J)\). That is \(f(V)\) is \(I_{rw^-}\)-closed in \(Y\) and so \(f\) is \(I_{rw^-}\)-closed map.

(iii) \(\Rightarrow\) (i) Let \(W\) be \(I_{rw^-}\)-closed in \((X,\tau,I)\). By (iii), \(f(W)\) is \(I_{rw^-}\)-closed in \((Y,\sigma,J)\). But \(f(W) = (f^{-1})^{-1}(W)\). Thus (i) holds.

**Theorem 3.5** For any spaces \(I_{rw^-}^*\)-homeomorphism, then their composition \(gof: (X,\tau,I) \rightarrow (Z,\eta,k)\) is also \(I_{rw^-}^*\)-homeomorphism.

**Proof.** Let \(U\) be a \(I_{rw^-}\)-open set in \((Z,\eta,k)\). Since \(g\) is \(I_{rw^-}\)-irresolute, \(g^{-1}(U)\) is \(I_{rw^-}\)-open in \((Y,\sigma,J)\). Since \(f\) is \(I_{rw^-}\)-irresolute, \(f^{-1}(g^{-1}(U)) = (gof)^{-1}(U)\) is \(I_{rw^-}\)-open set in \((X,\tau,I)\). Therefore, \((gof)^{-1}\) is \(I_{rw^-}\)-irresolute. Also, for a \(I_{rw^-}\)-open set \(G\) in \((X,\tau,I)\), we have \((gof)(G) = g(f(G)) = g(W)\), where \(W = f(G)\). By hypothesis \(f(G)\) is \(I_{rw^-}\)-open in \((Y,\sigma,J)\) and so again by hypothesis \(g(f(G))\) is a \(I_{rw^-}\)-open set in \((Z,\eta,k)\). That is \((gof)(G)\) is a \(I_{rw^-}\)-open set in \((Z,\eta,k)\) and therefore, \(gof\) is \(I_{rw^-}\)-irresolute. Also, \(gof\) is a bijection. Hence \(gof\) is \(I_{rw^-}\)-homeomorphism.

**Theorem 3.6** The set \(I_{rw^-}^*\)-homeomorphism, then their composition \(gof: (X,\tau,I) \rightarrow (Z,\eta,k)\) is also \(I_{rw^-}^*\)-homeomorphism.

**Proof.** Define a binary operation \(\ast: I_{rw^-}^*\)-homeomorphism, then their composition \(gof: (X,\tau,I) \rightarrow (Z,\eta,k)\) is also \(I_{rw^-}^*\)-homeomorphism.

**Proof.** Let \(U\) be a \(I_{rw^-}\)-open set in \((Z,\eta,k)\). Since \(g\) is \(I_{rw^-}\)-irresolute, \(g^{-1}(U)\) is \(I_{rw^-}\)-open in \((Y,\sigma,J)\). Since \(f\) is \(I_{rw^-}\)-irresolute, \(f^{-1}(g^{-1}(U)) = (gof)^{-1}(U)\) is \(I_{rw^-}\)-open set in \((X,\tau,I)\). Therefore, \((gof)^{-1}\) is \(I_{rw^-}\)-irresolute. Also, for a \(I_{rw^-}\)-open set \(G\) in \((X,\tau,I)\), we have \((gof)(G) = g(f(G)) = g(W)\), where \(W = f(G)\). By hypothesis \(f(G)\) is \(I_{rw^-}\)-open in \((Y,\sigma,J)\) and so again by hypothesis \(g(f(G))\) is a \(I_{rw^-}\)-open set in \((Z,\eta,k)\). That is \((gof)(G)\) is a \(I_{rw^-}\)-open set in \((Z,\eta,k)\) and therefore, \(gof\) is \(I_{rw^-}\)-irresolute. Also, \(gof\) is a bijection. Hence \(gof\) is \(I_{rw^-}\)-homeomorphism.

**Theorem 3.7** The set \(I_{rw^-}^*\)-homeomorphism, then their composition \(gof: (X,\tau,I) \rightarrow (Z,\eta,k)\) is also \(I_{rw^-}^*\)-homeomorphism.

**Proof.** Let \(U\) be a \(I_{rw^-}\)-open set in \((Z,\eta,k)\). Since \(g\) is \(I_{rw^-}\)-irresolute, \(g^{-1}(U)\) is \(I_{rw^-}\)-open in \((Y,\sigma,J)\). Since \(f\) is \(I_{rw^-}\)-irresolute, \(f^{-1}(g^{-1}(U)) = (gof)^{-1}(U)\) is \(I_{rw^-}\)-open set in \((X,\tau,I)\). Therefore, \((gof)^{-1}\) is \(I_{rw^-}\)-irresolute. Also, for a \(I_{rw^-}\)-open set \(G\) in \((X,\tau,I)\), we have \((gof)(G) = g(f(G)) = g(W)\), where \(W = f(G)\). By hypothesis \(f(G)\) is \(I_{rw^-}\)-open in \((Y,\sigma,J)\) and so again by hypothesis \(g(f(G))\) is a \(I_{rw^-}\)-open set in \((Z,\eta,k)\). That is \((gof)(G)\) is a \(I_{rw^-}\)-open set in \((Z,\eta,k)\) and therefore, \(gof\) is \(I_{rw^-}\)-irresolute. Also, \(gof\) is a bijection. Hence \(gof\) is \(I_{rw^-}\)-homeomorphism.

**Theorem 3.8** The set \(I_{rw^-}^*\)-homeomorphism, then their composition \(gof: (X,\tau,I) \rightarrow (Z,\eta,k)\) is also \(I_{rw^-}^*\)-homeomorphism.

**Proof.** Using the map \(f\), we define a map \(\psi_f: I_{rw^-}^*\)-homeomorphism, then their composition \(gof: (X,\tau,I) \rightarrow (Z,\eta,k)\) is also \(I_{rw^-}^*\)-homeomorphism.

**Proof.** Using the map \(f\), we define a map \(\psi_f: I_{rw^-}^*\)-homeomorphism, then their composition \(gof: (X,\tau,I) \rightarrow (Z,\eta,k)\) is also \(I_{rw^-}^*\)-homeomorphism, then their composition \(gof: (X,\tau,I) \rightarrow (Z,\eta,k)\) is also \(I_{rw^-}^*\)-homeomorphism, then their composition \(gof: (X,\tau,I) \rightarrow (Z,\eta,k)\) is also \(I_{rw^-}^*\)-homeomorphism, then their composition \(gof: (X,\tau,I) \rightarrow (Z,\eta,k)\) is also \(I_{rw^-}^*\)-homeomorphism, then their composition \(gof: (X,\tau,I) \rightarrow (Z,\eta,k)\) is also \(I_{rw^-}^*\)-homeomorphism.
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