Notions via \( r \)-fuzzy \( \tilde{e} \)-open Sets

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Abstract

In this paper the concept of \( r \)-fuzzy \( \tilde{e} \)-border, \( r \)-fuzzy \( \tilde{e} \)-exterior and \( r \)-fuzzy \( \tilde{e} \)-frontier in the sense of Ramadan [3] and Sostak [7] are introduced. Some interesting properties and characterizations of them are investigated. Interrelations among the concepts introduced are studied with relevant examples.

Keywords and phrases: \( r \)-fuzzy \( \tilde{e} \)-open, \( r \)-fuzzy \( \tilde{e} \)-interior, \( r \)-fuzzy \( \tilde{e} \)-closure, \( r \)-fuzzy \( \tilde{e} \)-border, \( r \)-fuzzy \( \tilde{e} \)-exterior, \( r \)-fuzzy \( \tilde{e} \)-frontier.


1. Introduction

The concept of fuzzy set was introduced by Zadeh [10] in his classical paper. Fuzzy sets have applications in many fields such as information [6] and control [8]. In 1985, Sostak [7] established a new form of fuzzy topological structure. The concept of fuzzy \( e \)-open set was introduced and studied by Seenivasan [5]. The concept of fuzzy \( e \)-space was introduced and studied by [5]. The concept of \( g \)-border, \( g \)-frontier were studied in [1]. In this paper, the concepts of \( r \)-fuzzy \( \tilde{e} \)-border, \( r \)-fuzzy \( \tilde{e} \)-exterior, \( r \)-fuzzy \( \tilde{e} \)-frontier in the sense of Sostak [7] and Ramadan [3] are introduced.

Throughout this paper, let \( X \) be a non-empty set, \( I=[0,1] \) and \( I_0=(0,1] \).

2. Preliminaries

Definition 2.1 [4] A function \( T:I^X \rightarrow I \) is called a smooth topology on \( X \) if it satisfies the following conditions:

(i) \( T(0)=T(1)=1 \).

(ii) \( T(\mu_i \wedge \mu_z) \geq T(\mu_i) \wedge T(\mu_z) \) for any \( \mu_i, \mu_z \in I^X \).

(iii) \( T(\bigwedge_{i \in I} \mu_i) \geq \bigwedge_{i \in I} T(\mu_i) \) for any \( \{\mu_i\}_{i \in I} \in I^X \).

The pair \( (X,T) \) is called a smooth topological space.

Remark 2.1 Let \( (X,T) \) be a smooth topological space. Then, for each \( r \in I_0 \), \( T_r = \{ \mu \in I^X ; T(\mu) \geq r \} \) is Chang's fuzzy topology on \( X \).
Proposition 2.1 [4] Let \((X,T)\) be a smooth topological space. For each \(\lambda \in I^X, r \in I_0\) an operator \(C_r : I^X \times I_0 \to I^X\) is defined as follows:

\[ C_r(\lambda, r) = \bigwedge \{ \mu : \mu \geq \lambda, T(\bar{r} - \mu) \geq r \} \] .

For \(\lambda, \mu \in I^X\) and \(r, s \in I_0\) it satisfies the following conditions:

1. \(C_r(\bar{0}, r) = \bar{0}\).
2. \(\lambda \leq C_r(\lambda, r)\).
3. \(C_r(\lambda, r) \vee C_r(\mu, r) = C_r(\lambda \vee \mu, r)\).
4. \(C_r(\lambda, r) \leq C_r(\lambda, s)\) if \(r \leq s\).
5. \(C_r(C_r(\lambda, r), r) = C_r(\lambda, r)\).

Proposition 2.2. [3] Let \((X,T)\) be a smooth topological space. For each \(\lambda \in I^X, r \in I_0\) an operator \(I_r : I^X \times I_0 \to I^X\) is defined as follows:

\[ I_r(\lambda, r) = \sqrt[\wedge]{\{ \mu : \mu \leq \lambda, T(\bar{r} - \mu) \geq r \}} \] .

For each \(\lambda, \mu \in I^X\) and \(r, s \in I_0\) it satisfies the following conditions:

1. \(I_r(\bar{1} - \lambda, r) = \bar{1} - C_r(\lambda, r)\).
2. \(I_r(\bar{1}, r) = \bar{1}\).
3. \(I_r(\lambda, r) \leq \lambda\).
4. \(I_r(\lambda, r) \wedge I_r(\mu, r) = I_r(\lambda \wedge \mu, r)\).
5. \(I_r(\lambda, r) \geq I_r(\lambda, s)\) if \(r \leq s\).
6. \(I_r(I_r(\lambda, r), r) = I_r(\lambda, r)\).

Definition 2.2 [2] Let \((X, \tau)\) be a fuzzy topological space, \(\lambda \in I^X\) and \(r \in I_0\).

Then

1. A fuzzy set \(\lambda\) is called \(r\)-fuzzy regular open (for short, \(r\)-fro) if \(\lambda = I_r(C_r(\lambda, r), r)\).
2. A fuzzy set \(\lambda\) is called \(r\)-fuzzy regular closed (for short, \(r\)-frc) if \(\lambda = C_r(I_r(\lambda, r), r)\).

Definition 2.3 [2] Let \((X, \tau)\) be a fts. For \(\lambda, \mu \in I^X\) and \(r \in I_0\).

1. The \(r\)-fuzzy \(\delta\) closure of \(\lambda\), denoted by \(\delta - C_r(\lambda, r)\), and is defined by \(\delta - C_r(\lambda, r) = \bigwedge \{ \mu \in I^X : \mu \geq \lambda, \mu \text{ is } r\text{-frc} \}\).
2. The \(r\)-fuzzy \(\delta\) interior of \(\lambda\), denoted by \(\delta - I_r(\lambda, r)\), and is defined by \(\delta - I_r(\lambda, r) = \sqrt[\vee]{\{ \mu \in I^X : \mu \leq \lambda, \mu \text{ is } r\text{-feo} \}}\).

Definition 2.4 [9] Let \((X, \tau)\) be a fuzzy topological space, \(\lambda \in I^X\) and \(r \in I_0\).

Then

1. A fuzzy set \(\lambda\) is called \(r\)-fuzzy \(e\) open (for short, \(r\)-feo) if \(\lambda \leq I_r(\delta - C_r(\lambda, r), r) \vee C_r(\delta - I_r(\lambda, r), r)\).
2. A fuzzy set \(\lambda\) is called \(r\)-fuzzy regular closed (for short, \(r\)-frc) if \(\lambda \geq I_r(\delta - C_r(\lambda, r), r) \wedge C_r(\delta - I_r(\lambda, r), r)\).
**Definition 2.5** [9] Let \((X, \tau)\) be a fts. For \(\lambda, \mu \in I^X\) and \(r \in I_0\).

1. The \(r\)-fuzzy \(e\) closure of \(\lambda\), denoted by \(fe - C_r(\lambda, r)\), and is defined by \(fe - C_r(\lambda, r) = \bigwedge\{\mu \in I^X | \mu \geq \lambda, \mu \text{ is } r\text{-feco}\}\).
2. The \(r\)-fuzzy \(e\) interior of \(\lambda\), denoted by \(fe - I_r(\lambda, r)\), and is defined by \(fe - I_r(\lambda, r) = \bigvee\{\mu \in I^X | \mu \leq \lambda, \mu \text{ is } r\text{-feco}\}\).

**Lemma 2.1** [9] In a fuzzy topological space \(X\),
1. Any union of \(r\)-fuzzy \(e\)-open sets is a \(r\)-fuzzy \(e\)-open set.
2. Any intersection of \(r\)-fuzzy \(e\)-closed sets is a \(r\)-fuzzy \(e\)-closed set.

### 3. \(r\)-fuzzy \(e\)-open sets

In this section, the concept of \(r\)-feco-border, \(r\)-feco-frontier and \(r\)-feco-exterior are introduced and its properties are studied by providing necessary examples.

**Definition 3.1** Let \((X, T)\) be a smooth topological space. For \(\lambda, \mu \in I^X\) and \(r \in I_0\).

1. \(\lambda\) is called \(r\)-fuzzy \(e\)-open (briefly \(r\)-feco) if \(fe - I_r(\lambda, r) \geq \mu\), whenever \(\lambda \geq \mu\) and \(\mu \in I^X\) is \(r\)-feco.
2. \(\lambda\) is called \(r\)-fuzzy \(e\)-closed (briefly \(r\)-feco) if \(fe - C_r(\lambda, r) \leq \mu\), whenever \(\lambda \leq \mu\) and \(\mu \in I^X\) is \(r\)-feco.
3. The \(r\)-fuzzy \(e\)-interior of \(\lambda\), denoted by \(fe - I_r(\lambda, r)\) is defined as \(fe - I_r(\lambda, r) = \bigvee\{\mu: \mu \leq \lambda, \mu \text{ is } r\text{-feco}\}\).
4. The \(r\)-fuzzy \(e\)-closure of \(\lambda\), denoted by \(fe - C_r(\lambda, r)\) is defined as \(fe - C_r(\lambda, r) = \bigwedge\{\mu: \mu \geq \lambda, \mu \text{ is } r\text{-feco}\}\).

**Proposition 3.1** Let \((X, T)\) be a smooth topological space. For each \(\lambda, \mu \in I^X\) and \(r \in I_0\), the following statements hold:

1. \(fe - I_r(\lambda, r)\) is the largest \(r\)-feco set such that \(fe - I_r(\lambda, r) \leq \lambda\).
2. If \(\lambda\) is \(r\)-feco, then \(fe - I_r(\lambda, r) = \lambda\).
3. If \(\lambda\) is \(r\)-feco, then \(fe - I_r(fe - I_r(\lambda, r), r) = fe - I_r(\lambda, r)\).
4. \(\bar{I} - fe - I_r(\lambda, r)\) is defined as \(fe - C_r(\bar{I} - \lambda, r)\).
5. \(\bar{I} - fe - C_r(\lambda, r) = fe - I_r(\bar{I} - \lambda, r)\).
6. If \(\lambda \leq \mu\), then \(fe - I_r(\lambda, r) \leq fe - I_r(\mu, r)\).
7. \(fe - I_r(\lambda \lor \mu, r) = fe - I_r(\lambda, r) \lor fe - I_r(\mu, r)\).
8. \(fe - I_r(\lambda \land \mu, r) = fe - I_r(\lambda, r) \land fe - I_r(\mu, r)\).
9. If \(\lambda \leq \mu\), then \(fe - C_r(\lambda, r) \leq fe - C_r(\mu, r)\).
Proof. Proof of (1) and (2) is trivial. Proof of (3) follows from (2).

Let \( f \) be a function and \( \lambda, \mu \in I^X \). The \( -e \)-border of \( \lambda \) and \( r \) is defined as:

\[
\tilde{e}_r(\lambda, r) = \tilde{\lambda} \cap \{ (x, y) : \mu(x, y) \leq \lambda \}
\]

where \( \tilde{\lambda} = \bigcap \{ (x, y) : \mu(x, y) \geq \lambda \} \).

(4) \( \tilde{e}_r(\lambda, r) = \tilde{\lambda} - \bigcap \{ \mu : \mu \leq \lambda, \mu \text{ is } r-f \tilde{e} \text{ -closed} \} = \tilde{\lambda} - e_r(\lambda, r) \).

(5) \( \tilde{e}_r(\lambda, r) = \tilde{\lambda} - \bigcap \{ \mu : \mu \geq \lambda, \mu \text{ is } r-f \tilde{e} \text{ -open} \} = \tilde{\lambda} - e_r(\lambda, r) \).

(6) Since \( \lambda \leq \mu \), \( f \tilde{e}_r(\lambda, r) \leq f \tilde{e}_r(\lambda, r) \).

(7) \( f \tilde{e}_r(\lambda, r) \leq f \tilde{e}_r(\lambda, r) \).

(8) \( f \tilde{e}_r(\lambda, r) \leq f \tilde{e}_r(\lambda, r) \).

(9) \( f \tilde{e}_r(\lambda, r) \leq f \tilde{e}_r(\lambda, r) \).

Definition 3.2 Let \((X, T)\) be a smooth topological space. For each \( \lambda \in I^X \) and \( r \in I_0 \), the \( r \)-fuzzy \( e \)-border of \( \lambda \), denoted by \( f e - b_r(\lambda, r) \), is defined as:\n
\[
fe - b_r(\lambda, r) = \lambda - f e - I_r(\lambda, r).
\]

Definition 3.3 Let \((X, T)\) be a smooth topological space. For each \( \lambda \in I^X \) and \( r \in I_0 \), the \( r-f \tilde{e} \)-border of \( \lambda \), denoted by \( f \tilde{e} - b_r(\lambda, r) \), is defined as:\n
\[
\tilde{e} - b_r(\lambda, r) = \lambda - \tilde{e} - I_r(\lambda, r).
\]

Proposition 3.2 Let \((X, T)\) be a smooth topological space. For each \( \lambda, \mu \in I^X \) and \( r \in I_0 \), the following statements hold:

1. \( f e - b_r(\lambda, r) \leq f \tilde{e} - b_r(\lambda, r) \).
2. If \( \lambda \) is \( r \)-fuzzy \( e \)-open, then \( f e - b_r(\lambda, r) = 0 \).
3. \( f e - I_r(f e - b_r(\lambda, r), r) \leq \lambda \).
4. \( f e - b_r(\lambda, r) \leq f \tilde{e} - C_r(\lambda, r) \).
5. \( f e - b_r(\lambda, r) \leq f \tilde{e} - b_r(\lambda, r) \).
6. \( f e - b_r(\lambda, r) \leq f \tilde{e} - b_r(\lambda, r) \).
7. \( f e - I_r(f e - b_r(\lambda, r), r) \leq f \tilde{e} - I_r(\lambda, r) \).
8. \( f e - I_r(\lambda, r) \leq f \tilde{e} - I_r(\lambda, r) \).
Proof. (1) Now, \( f\tilde{e} - I_r(\lambda, r) \leq f\tilde{e} - I_T(\lambda, r) \). This implies that \( \lambda - f\tilde{e} - I_r(\lambda, r) \geq f\tilde{e} - I_T(\lambda, r) \). Hence \( f\tilde{e} - b_r(\lambda, r) \geq f\tilde{e} - b_T(\lambda, r) \).

(2) If \( \lambda \) is \( r \)-\( f\tilde{e} \)-open then \( \lambda = f\tilde{e} - b_r(\lambda, r) \). Hence \( f\tilde{e} - I_r(\lambda, r) = \lambda - f\tilde{e} - I_r(\lambda, r) = 0 \).

(3) Now, \( f\tilde{e} - I_r(\lambda, r), r = f\tilde{e} - I_r(\lambda) - f\tilde{e} - I_r(\lambda, r) \leq \lambda - f\tilde{e} - I_r(\lambda, r) \leq \lambda \). Thus \( f\tilde{e} - I_T(\lambda, r) \leq \lambda \).

(4) \( f\tilde{e} - b_T(\lambda, r) = \lambda - f\tilde{e} - I_T(\lambda, r) = \lambda - (1 - f\tilde{e} - C_T(1 - \lambda, r)) = f\tilde{e} - C_T(1 - \lambda, r) - (1 - \lambda) \leq f\tilde{e} - C_T(1 - \lambda, r) \). Hence, \( f\tilde{e} - b_T(\lambda, r) \leq f\tilde{e} - C_T(1 - \lambda, r) \).

(5) \( f\tilde{e} - b_T(\lambda \land \mu, r) = (\lambda \land \mu) - f\tilde{e} - I_T(\lambda \land \mu, r) \leq (\lambda \land \mu - f\tilde{e} - I_T(\lambda, r) \land f\tilde{e} - I_T(\mu, r) = (\lambda - f\tilde{e} - I_T(\lambda, r)) \land (\mu - f\tilde{e} - I_T(\mu, r) = f\tilde{e} - b_T(\lambda, r) \land f\tilde{e} - b_T(\mu, r) \). Hence, \( f\tilde{e} - b_T(\lambda \land \mu, r) \leq f\tilde{e} - b_T(\lambda, r) \land f\tilde{e} - b_T(\mu, r) \).

Proof of (7) and (8) is trivial.

Definition 3.4 Let \( (X, T) \) be a smooth topological space. For \( \lambda \in I^X \) and \( r \in I_0 \), the \( r \)-\( \text{fuzzy} \) \( e \)-\( \text{frontier} \) of \( \lambda \), denoted by \( fe - Fr_r(\lambda, r) \), is defined as \( fe - Fr_r(\lambda, r) = fe - C_T(\lambda, r) - fe - I_T(\lambda, r) \).

Definition 3.5 Let \( (X, T) \) be a smooth topological space. For \( \lambda \in I^X \) and \( r \in I_0 \), the \( r \)-\( \text{fuzzy} \) \( e \)-\( \text{frontier} \) of \( \lambda \), denoted by \( f\tilde{e} - Fr_r(\lambda, r) \), is defined as \( f\tilde{e} - Fr_r(\lambda, r) = f\tilde{e} - C_T(\lambda, r) - f\tilde{e} - I_T(\lambda, r) \).

Proposition 3.3 Let \( (X, T) \) be a smooth topological space. For \( \lambda, \mu \in I^X \) and \( r \in I_0 \), the following statements hold:

1. \( fe - Fr_r(\lambda, r) \leq f\tilde{e} - Fr_r(\lambda, r) \).
2. \( f\tilde{e} - b_T(\lambda, r) \leq fe - Fr_r(\lambda, r) \).
3. \( f\tilde{e} - Fr_r(\lambda, r) = f\tilde{e} - Fr_r(1 - \lambda, r) \).
4. \( f\tilde{e} - Fr_r(\tilde{e} - I_T(\lambda, r), r) \leq f\tilde{e} - Fr_r(\lambda, r) \).
5. \( f\tilde{e} - Fr_r(\tilde{e} - C_T(\lambda, r), r) \leq f\tilde{e} - Fr_r(\lambda, r) \).
6. \( f\tilde{e} - I_T(\lambda, r) \geq \lambda - f\tilde{e} - Fr_r(\lambda, r) \).
7. \( f\tilde{e} - Fr_r(\lambda \land \mu, r) \leq f\tilde{e} - Fr_r(\lambda, r) \land f\tilde{e} - Fr_r(\mu, r) \).
8. \( f\tilde{e} - Fr_r(\lambda \land \mu, r) \geq f\tilde{e} - Fr_r(\lambda, r) \land f\tilde{e} - Fr_r(\mu, r) \).
Proof. (1) Now, \( \tilde{f} \tilde{e} - I_T(\lambda, r) \leq \tilde{f} \tilde{e} - I_T(\lambda, r) \). It follows that \( \tilde{f} \tilde{e} - C_T(\lambda, r) = \tilde{f} \tilde{e} - I_T(\lambda, r) \geq \tilde{f} \tilde{e} - C_T(\lambda, r) = \tilde{f} \tilde{e} - I_T(\lambda, r) \). Hence, \( \tilde{f} \tilde{e} - F_{r_T}(\lambda, r) \geq \tilde{f} \tilde{e} - F_{r_T}(\lambda, r) \).

(2) Now, \( \tilde{f} \tilde{e} - b_T(\lambda, r) = \lambda - \tilde{f} \tilde{e} - I_T(\lambda, r) \leq \tilde{f} \tilde{e} - C_T(\lambda, r) = \tilde{f} \tilde{e} - F_{r_T}(\lambda, r) \). Hence, \( \tilde{f} \tilde{e} - b_T(\lambda, r) \leq \tilde{f} \tilde{e} - F_{r_T}(\lambda, r) \).

(3) \( \tilde{f} \tilde{e} - F_{r_T}(\lambda, r) = \tilde{f} \tilde{e} - C_T(\lambda, r) - \tilde{f} \tilde{e} - I_T(\lambda, r) = (\tilde{I} - \tilde{f} \tilde{e} - I_T(\lambda, r)) = \tilde{f} \tilde{e} - C_T(\tilde{I} - \tilde{t} - r_T(\lambda, r)) = \tilde{f} \tilde{e} - F_{r_T}(\tilde{I} - \tilde{t} - r_T(\lambda, r)) \). Hence, \( \tilde{f} \tilde{e} - F_{r_T}(\lambda, r) = \tilde{f} \tilde{e} - F_{r_T}(1 - \tilde{t} - r_T(\lambda, r)) \).

(4) \( \tilde{f} \tilde{e} - F_{r_T}(\lambda, r) = \tilde{f} \tilde{e} - C_T(\lambda, r) - \tilde{f} \tilde{e} - I_T(\lambda, r) = \tilde{f} \tilde{e} - I_T(\lambda, r) \). Hence \( \tilde{f} \tilde{e} - F_{r_T}(\lambda, r) \leq \tilde{f} \tilde{e} - F_{r_T}(\lambda, r) \).

(5) \( \tilde{f} \tilde{e} - F_{r_T}(\lambda, r) = \tilde{f} \tilde{e} - C_T(\lambda, r) - \tilde{f} \tilde{e} - I_T(\lambda, r) = \tilde{f} \tilde{e} - F_{r_T}(\lambda, r) \). Hence \( \tilde{f} \tilde{e} - F_{r_T}(\lambda, r) \leq \tilde{f} \tilde{e} - F_{r_T}(\lambda, r) \).

(6) \( \tilde{f} \tilde{e} - F_{r_T}(\lambda, r) = \tilde{f} \tilde{e} - I_T(\lambda, r) \). Hence \( \tilde{f} \tilde{e} - F_{r_T}(\lambda, r) \leq \tilde{f} \tilde{e} - I_T(\lambda, r) \).

(7) \( \tilde{f} \tilde{e} - F_{r_T}(\lambda, r) = \tilde{f} \tilde{e} - I_T(\lambda, r) \). Hence \( \tilde{f} \tilde{e} - F_{r_T}(\lambda, r) \leq \tilde{f} \tilde{e} - I_T(\lambda, r) \).

Definition 3.6 Let \((X, T)\) be a smooth topological space. For \(\lambda \in I^X\) and \(r \in I_0\), the \(r\)-fuzzy \(e\)-exterior of \(\lambda\), denoted by \(f e - Ext_T(\lambda, r)\), is defined as \(f e - Ext_T(\lambda, r) = f e - I_T(\tilde{1} - \tilde{r} - r_T(\lambda, r))\).

Definition 3.7 Let \((X, T)\) be a smooth topological space. For \(\lambda, \mu \in I^X\) and \(r \in I_0\), the \(r\)-fuzzy \(e\)-exterior of \(\lambda\), denoted by \(f e - Ext_T(\lambda, r)\), is defined as \(f e - Ext_T(\lambda, r) = f e - I_T(\tilde{1} - \tilde{r} - r_T(\lambda, r))\).

Proposition 3.4 Let \((X, T)\) be a smooth topological space. For \(\lambda, \mu \in I^X\) and \(r \in I_0\), the following statement

\[
\begin{align*}
(1) & \quad \tilde{f} \tilde{e} - Ext_T(\lambda, r) \leq \tilde{f} \tilde{e} - Ext_T(\lambda, r). \\
(2) & \quad \tilde{f} \tilde{e} - Ext_T(\lambda, r) = \tilde{f} \tilde{e} - I_T(\tilde{1} - \tilde{r} - r_T(\lambda, r)) = \tilde{1} - \tilde{f} \tilde{e} - C_T(\lambda, r). \\
(3) & \quad \tilde{f} \tilde{e} - Ext_T(\tilde{f} \tilde{e} - Ext_T(\lambda, r)) = \tilde{f} \tilde{e} - I_T(\tilde{f} \tilde{e} - C_T(\lambda, r)). \\
(4) & \quad \text{If } \lambda \leq \mu, \text{ then } \tilde{f} \tilde{e} - Ext_T(\lambda, r) \geq \tilde{f} \tilde{e} - Ext_T(\mu, r). \\
(5) & \quad \tilde{f} \tilde{e} - Ext_T(\tilde{1}, r) = \tilde{0}. \\
(6) & \quad \tilde{f} \tilde{e} - Ext_T(\tilde{0}, r) = \tilde{1}. \\
(7) & \quad \tilde{f} \tilde{e} - I_T(\lambda, r) \leq \tilde{f} \tilde{e} - Ext_T(\lambda, r). 
\end{align*}
\]
\begin{align*}
(8) & \quad f^\sim e - \text{Ext}_T(\lambda \lor \mu, r) \leq f^\sim e - \text{Ext}_T(\lambda, r) \land f^\sim e - \text{Ext}_T(\mu, r). \\
(9) & \quad f^\sim e - \text{Ext}_T(\lambda \land \mu, r) \geq f^\sim e - \text{Ext}_T(\lambda, r) \lor f^\sim e - \text{Ext}_T(\mu, r).
\end{align*}

**Proof.** (1) Now, $f^\sim e - C_T(\lambda, r) \leq f^\sim e - C_T(\lambda, r)$. This implies $\bar{1} - f^\sim e - C_T(\lambda, r) \geq \bar{1} - f^\sim e - C_T(\lambda, r)$. Hence, $f^\sim e - I_T(\bar{1} - \lambda, r) \geq f^\sim e - I_T(\bar{1} - \lambda, r)$. Therefore, $f^\sim e - \text{Ext}_T(\lambda, r) \geq \text{Ext}_T(\lambda, r)$.

(2) Proof of (2) is trivial.

(3) $f^\sim e - \text{Ext}_T( f^\sim e - \text{Ext}_T(\lambda, r), r) = f^\sim e - \text{Ext}_T( f^\sim e - I_T(\bar{1} - \lambda, r), r) = f^\sim e - \text{Ext}_T(1 - f^\sim e - C_T(\lambda, r), r) = f^\sim e - I_T(1 - (1 - f^\sim e - C_T(\lambda, r), r)) = f^\sim e - I_T(1 - f^\sim e - C_T(\lambda, r), r) = f^\sim e - I_T(f^\sim e - C_T(\lambda, r), r)$. 

(4) If $\lambda \leq \mu$, then $f^\sim e - C_T(\lambda, r) \leq f^\sim e - C_T(\mu, r)$. This implies $\bar{1} - f^\sim e - C_T(\lambda, r) \geq \bar{1} - f^\sim e - C_T(\mu, r)$. Therefore, $f^\sim e - I_T(1 - \lambda, r) \geq f^\sim e - I_T(1 - \mu, r)$. Hence, $f^\sim e - \text{Ext}_T(\lambda, r) \geq f^\sim e - \text{Ext}_T(\mu, r)$.

(5) $f^\sim e - \text{Ext}_T(\bar{1}, r) = f^\sim e - I_T(\bar{1} - 1, r) = f^\sim e - I_T(\bar{r}, r) = T$. Since $\bar{r}$ itself is a r-fuzzy-open set.

(6) $f^\sim e - \text{Ext}_T(\bar{0}, r) = f^\sim e - I_T(\bar{1} - 0, r) = f^\sim e - I_T(\bar{0}, r) = \bar{1}$. Since $\bar{1}$ itself is a r-fuzzy-open set.

(7) $f^\sim e - \text{Ext}_T(f^\sim e - \text{Ext}_T(\lambda, r), r) = f^\sim e - \text{Ext}_T(f^\sim e - I_T(\bar{1} - \lambda, r), r) = f^\sim e - \text{Ext}_T(f^\sim e - C_T(\lambda, r), r) = f^\sim e - I_T(f^\sim e - C_T(\lambda, r), r) \geq f^\sim e - I_T(\lambda, r)$. Hence, $f^\sim e - I_T(\lambda, r) \leq f^\sim e - \text{Ext}_T(f^\sim e - \text{Ext}_T(\lambda, r), r)$.

(8) $\quad f^\sim e - \text{Ext}_T(\lambda \lor \mu, r) = e - I_T(\bar{1} - (\lambda \lor \mu, r) = e - I_T(\bar{1} - \lambda, r) \land e - I_T(\bar{1} - \mu, r) = e - I_T(\bar{1} - \lambda, r) \land e - I_T(\bar{1} - \mu, r)$.

(9) $f^\sim e - \text{Ext}_T(\lambda \land \mu, r) = f^\sim e - I_T(\bar{1} - (\lambda \land \mu, r) = f^\sim e - I_T((\bar{1} - \lambda) \lor (\bar{1} - \mu, r) \lor f^\sim e - I_T(\bar{1} - \mu, r)$.

**References**


