α, β Colouring of Graphs and Related Aspects

A.A.Bhange [#1], H.R. Bhapkar[*2]

[#] *MIT ADT University, Pune, India*

***Abstract*— This paper defines the perfect colouring of planar graph. We intend to study the relation between chromatic number, chromatic index, semi perfect colouring and perfect colouring of graphs. If G is a tree graph on n ≥ 3 vertices then α = χ'(G) + 1 and β = α + 1. Precisely, we prove related results for some particular planar graphs.**

***Keywords*— Perfect Colouring, Semi perfect Colouring, chromatic number, chromatic index.**

I. Introduction

A. *Graphs*

A graph G is a set of ordered pair (V, E) where V is a non-empty finite set of elements called vertices or nodes and E is a set of elements of V, known as edges of G. [7]

A graph G is a planar graph if it is possible to represent it in the plane such that no two edges of the graph intersect except at the end points. [8]

A vertex colouring problems for a graph G is to colour all the vertices of the graph with different colours in such a way that any two adjacent vertices of G have assigned different colours. The least number of colours needed to colour the graph is called its chromatic number. It is denoted by the symbol χ(G), where G is a graph.

The edge colouring of a graph G (V, E) is a mapping, which assigns a colour to every edge, satisfying condition that no two edges sharing a common vertex have the same colour. [6]

Face or region colouring of a planar graph assigns a colour to each face or region so that no two faces that share a boundary have the same colour.

B. *Theorem (Four Colour Map Theorem)*

Every planar map can be coloured with four or fewer colours.

The Four Colour theorem was first stated 200 years ago and finally proved conclusively in 1976. The professor of mathematics, Augustus De Morgan (1806-71) and his friend William Rowan Hamilton studied this theorem and gave first proof. In 1879, Alfred Kempe, published a short paper on colouring of maps. He added some other ideas of colouring. In 1879, Alfred Kempe published this proof in the American Journal of Mathematics in simple versions. In 1980, Tait P.G. offered independent solution to this problem. After collaborating with John Koch on the problem of reducibility, in 1976, Kenneth Appel and Wolfgang Haken gave the complete proof to the four colour conjecture by reducing the testing problem to an unavoidable set with 1936 configurations. Because of the computer based proof, many Mathematicians were not agreeing with this proof. However, many proofs written by different Mathematicians have been found to be faulty. So all we have been waiting for the simple proof of this theorem. [2]-[4]

This problem is stated by Douglas B. West. He has published it in his book. He state that "The vertices and edges of a graph G can be coloured with Δ(G) + 2 colours such that adjacent vertices have different colours, incident edges have different colours and incident edge and vertices have different colours." [9]

C. *Semi Perfect Colouring Of the Graph*

Semi Perfect Coloring is assigning colors such that adjacent vertices have different colors, incident edges have different colors and incident edge and vertices have different colors. If α minimum number of colors are required to color any planar graph G by semi perfect coloring, then it is denoted by SPC (G) = α. [3]

D. *Perfect Coloring of Planar Graphs*

It is a type of coloring in which we have to assign different color to a region, boundary edges and boundary vertices according to some law. [4]

E. *Conjecture*

How many minimum colours will be required to colour planar graph such that

- Adjacent vertices have different colors.
- Incident edges have different colors.
- Adjacent regions have different colors.
- A region, boundary edges and boundary vertices of that region have different colors.

This type of coloring is known as Perfect Coloring of graph.

If β number of colours are required to colour any graph G by perfect colouring, then it is denoted by PC (G) = β. We have proved this open problem partially. [5,6]

II. Main Results

A. *Theorem : If G is null graph with n vertices then* $\alpha = \chi(G)$.

Proof: Let G be a null graph with n vertices say $V_1, V_2, \ldots V_n$. So there are n distinct vertices without edges, So $\chi(G) = 1$. As there are no edges thus $\alpha = 1$.
Thus $\alpha = \chi(G)$.

B. *Theorem : If G is null graph with n vertices then* $\beta = \chi(G) + 1$.

Proof: Let G be a null graph with n vertices say $V_1, V_2, \ldots V_n$. So there are n distinct vertices without edges, So $\chi(G) = 1$.
Also there is an unbounded region containing n vertices so $\beta = 2$.
Thus $\beta = \chi(G) + 1$.

C. *Theorem : If G is a tree graph on $n \geq 3$ vertices then $\alpha = \chi'(G) + 1$ and $\beta = \alpha + 1$.*

Proof: Let G be any tree with $n \geq 3$ vertices. Let u be maximum degree vertex in G. i.e. u=max d (v_i) for all i. Let H be a subgraph of G containing u such that for all $x \in V$ and d(x , u)=1.i.e. H is a star sub-graph of G.

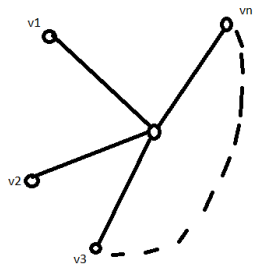

Fig. 1 Star Graph H

Let H be a star graph on n vertices say $V_1, V_2 \ldots Vn$. Subsequently, graph H has one vertex of degree n-1 and n-1 pendent vertices. All n-1 edges have common vertex V_n so$\chi'(G)$=n-1.Also give different colour to the common vertex V_n. So α=n.
Thus $\alpha = \chi'(G)+1$.Similarly apply same procedure for all remaining vertices and edges of G. Hence $\alpha = \chi'(G)+1$.

1) *Corollary: If G is a star graph on $n \geq 3$ vertices then $\alpha = \chi'(G) + 1$ and $\beta = \chi'(G) + 2$.*

Proof: From the above result $\alpha = n$ and $\chi'(G) = n-1$. The graph G is not closed graph so there is one open region only. Assign separate colour to it. $\beta = \alpha + 1 = n + 1$.Thus $\beta = \chi'(G) + 2$.

D. *Theorem: If G is a path graph on $n \geq 3$ vertices then $\alpha = \chi(G) + \chi'(G) - 1$ and $\beta = \chi(G) + \chi'(G)$.*

Proof : Let G be a path graph on $n \geq 3$ vertices say $V_1, V_2, \ldots V_n$, such that V_i is adjacent to V_i +1, for i = 1, 2, ... n-1. So V_1, V_n are pendent vertices. Assign alternate colour 1 to even number of vertices and colour 2 to odd number of vertices so $\chi(G) = 2$. In the similar way colour the edges too, will get$\chi'(G) = 2$.

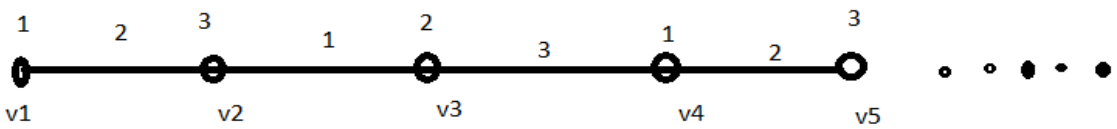

Figure 2: Path Graph G

Assign colour 1 to vertex V_1, colour 2 to edge e1 = $\{V_1, V_2\}$ and colour 3 to vertex V_2 in graph G. An edge $e_2 = \{V_2, V_3\}$ is adjacent to e1 and incident at V_2. So assign colour to an edge e_2 different from colours 2 and 3.
Therefore assign a colour 1 to an edge e_2, colour 2 to vertex V_3 and colour 3 to an edge $e_3 = \{V_3, V_4\}$.The same sequence of colours is repeated finitely many times. Hence, 3 colours are required to colour chain graph according to semi perfect colouring. Hence α=3.
Graph G has only one region. So assign a different colour to this region.so β=4.So we can conclude $\alpha = \chi(G) + \chi'(G) - 1$ and $\beta = \chi(G) + \chi'(G)$ for any path graph.

E. *Theorem 5. If C_n is a cycle graph with even vertices ($n \geq 4$) then*

$$\alpha(C_{3n}) = \chi + 1 = \chi' + 1, \alpha(C_{3n+1}) = \chi + 2 = \chi' + 2 \text{ and } \alpha(C_{3n+2}) = \chi + 2 = \chi' + 2.$$

Proof: Let G be a graph with 6 vertices. In G we have six vertices as A, B, C, D, E, F
And edges e_1, e_2, e_3, e_4, e_5, e_6 as shown in figure 3.

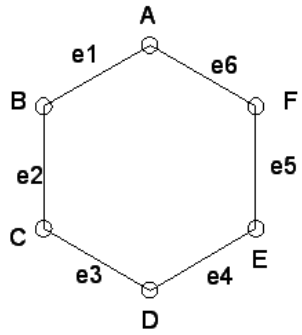

Fig. 3: Cycle Graph G with six vertices

Assign colour 1, 2, 1, 2, 1, 2 to vertices A, B, C, D, E, F respectively, so $\chi(G) = 2$.
In the similar way assign colours 1, 2, 1,2,1,2 to edges e_1, e_2, e_3, e_4, e_5, e_6 respectively. So $\chi'(G) = 2$.
Consider the closed path A -e_1-B- e_2 -C-e_3-D- e_4 –E- e_5-F- e_6-A. Now, assign colours successively 1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3, 1 to every element of the path. Therefore α=3.
Hence $(C_{3n}) = \chi(G) + 1 = \chi'(G) + 1$; when n is even.
In the similar way we can prove $\alpha(C_{3n+1}) = \chi + 2 = \chi' + 2$ and $\alpha(C_{3n+2}) = \chi + 2 = \chi' + 2$.

1) Corollary: If C_n is a cycle graph with even vertices ($n \geq 4$) then
$\beta(C_{3n}) = \chi + 3 = \chi' + 3$, $\beta(C_{3n+1}) = \chi + 4 = \chi' + 4$ *and* $\beta(C_{3n+2}) = \chi + 4 = \chi' + 4$

Proof: Using above theorem we can calculate semi perfect colouring α. In addition to it for perfect colouring β there are two regions one bounded and the other unbounded. Thus $\beta = \alpha + 2$. Hence the result.

F. Theorem 7: If G is a rose graph with $m \geq 2$ loops then $\alpha = \chi'(G) + 1$ and $\beta = \chi'(G) + 2$.

Proof: Without loss of generality, assume that H is a rose graph with five loops as given in fig 5. Assign five different colours to five edges as they are having common vertex. So $\chi'(G)=5$
Again assign colour 1 to vertex and five different colours to five edges, hence α=6.
Thus $\alpha = \chi'(G)+1$.

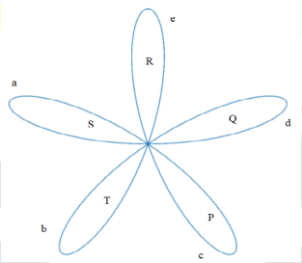

Fig. 5: Rose Graph H with Five Loops

Now assign colour of edge a to region P, colour of edge b to region Q, colour of edge c to region R, colour of edge d to region S, colour of edge e to region T. The vertex V and these five loops are the boundaries of an infinite region. So assign a different colour to this infinite region. Thus we required seven different colours for the perfect colouring of H. Hence β=7
Thus $\beta = \chi'(G) + 2$.

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