

# Proving Bonnets Theorem using Riemann-Stieljes Integration

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## Abstract

The Riemann-Stieljes integration is an innovative way to prove Bonnets theorem about integrals, students who deal with the subject of mathematics can use this method to get a solution easily and faster as compared to other methods. The process of Riemann Integration which is taught in Real Analysis classes is a specific case of the Riemann-Stieltjes Integration. Thus, many of the terms and properties used to describe Riemann Integration are discussed in this project and they are extended to the Riemann-Stieltjes integral. This project therefore provides a careful introduction to the theory of Riemann-Stieltjes integration, and explains the properties of this integral. After doing so, we present some applications in functional analysis, where we used the fact that continuous functions on a closed interval are Riemann-Stieltjes Integrable with respect to any function of bounded variation, and this was used in proving the Riesz Representation Theorem. To show versatility of the Riemann-Stieltjes Integral, we also present some applications in Probability Theory, where the integral generates a formula for the Expectation, regardless of its underlying distribution. Other applications considered are population growth, and Mechanics.

This paper is focused on proving the Bonnets theorem of integrals in less time and easily.

## Introduction

### What is Bonnets theorem of integrals?

Let  $f$  be an increasing function on  $[a,b]$ ,  $g$  be continuous on  $[a,b]$ ,  $B \in \mathbb{R}$  be such that  $f(b-) \leq B$ , and  $f(x) \geq 0$  for all  $x \in [a,b]$ . Then there exists an  $x_0 \in [a,b]$  such that  $\int_a^b f(x)g(x)dx = B$ .

$$\int_a^b f(x)g(x)dx$$

### What is Riemann-Stieljes integration?

The Riemann–Stieltjes integral of a real-valued function  $f$  of a real variable with respect to a real

function  $g$  is denoted by:  $\int_a^b f(x)dg(x)$

and defined to be the limit, as the norm of the partition (i.e. the length of the longest subinterval)  $P = \{a = x_0 < \dots < x_n = b\}$

of the interval  $[a,b]$  approaches zero, of the approximating sum

$$S(P, f, g) = \sum_{i=0}^{n-1} f(c_i)(g(x_{i+1}) - g(x_i))$$

Where  $c_i$  is in the  $i$ -th subinterval  $[x_i, x_{i+1}]$ . The two functions  $f$  and  $g$  are respectively called the integrand and the integrator. Typically,  $g$  is taken to be monotone (or at least of bounded variation) and right-semicontinuous. We specifically do not require  $g$  to be continuous, which allows for integrals that have point mass terms. The "limit" is here understood to be a number  $A$  (the value of the Riemann–Stieltjes integral) such that for every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for every partition  $P$  with  $\text{mesh}(P) < \delta$ , and for every choice of points  $c_i$  in  $[x_i, x_{i+1}]$ ,

$$|S(P, f, g) - A| < \varepsilon$$

## Equation Solving Methodology

Defining  $\phi(x) = \int_a^x \phi(t) d\phi(x) = \int_a^x \phi(t) dt$  we have  $\phi$  absolutely continuous, and it is a general property of Riemann-Stieltjes integrals,

$$\int_a^b \phi(x) f(x) dx = \int_a^b \phi df. \int_a^b \phi(x) f(x) dx = \int_a^b \phi df$$

Applying integration by parts we obtain,

$$\begin{aligned} \int_a^b \phi(x) f(x) dx &= \int_a^b \phi df \\ &= \phi(b)f(b) - \phi(a)f(a) \\ &\quad - \int_a^b \phi df. \int_a^b \phi(x) f(x) dx = \int_a^b \phi df = \phi(b)f(b) - \phi(a)f(a) - \int_a^b \phi df \end{aligned}$$

Since  $\phi(a) = 0$ , it follows that

$$\int_a^b \phi(x) f(x) dx = \phi(b)f(b) - \int_a^b \phi df. (*) (*) \int_a^b \phi(x) f(x) dx = \phi(b)f(b) - \int_a^b \phi df$$

Since  $\phi$  is continuous it is bounded on  $[a, b]$

Let  $A = \inf_{x \in [a, b]} \phi(x)$  and  $B = \sup_{x \in [a, b]} \phi(x)$

Since  $A \leq \phi(x) \leq B$  for all  $x \in [a, b]$  and  $f$  is decreasing we have

$$A(f(a) - f(b)) \leq - \int_a^b \phi df \leq B(f(a) - f(b)). (**)(**) A(f(a) - f(b)) \leq - \int_a^b \phi df \leq B(f(a) - f(b))$$

From (\*) and (\*\*) it follows that

$$A(f(a) - f(b)) + \phi(b)f(b) \leq \int_a^b \phi(x) f(x) dx \leq B(f(a) - f(b)) + \phi(b)f(b), \text{ and}$$

$$Af(a) + (\phi(b) - A)f(b) \leq \int baf(x)h(x)dx \leq Bf(a) - (B - \phi(b))f(b). Af(a) + (\phi(b) - A)f(b) \\ \leq \int abf(x)h(x)dx \leq Bf(a) - (B - \phi(b))f(b)$$

Since

$A \leq \phi(b) \leq B$  and  $f$  is non-negative, we have

$$(\phi(b) - A)f(b) \geq 0 \quad (\phi(b) - A)f(b) \geq 0 \text{ and} \\ (B - \phi(b))f(b) \geq 0 \quad (B - \phi(b))f(b) \geq 0$$

Therefore,

$$Af(a) \leq \int ba\phi(x)f(x)dx \leq Bf(a)$$

## Literature review

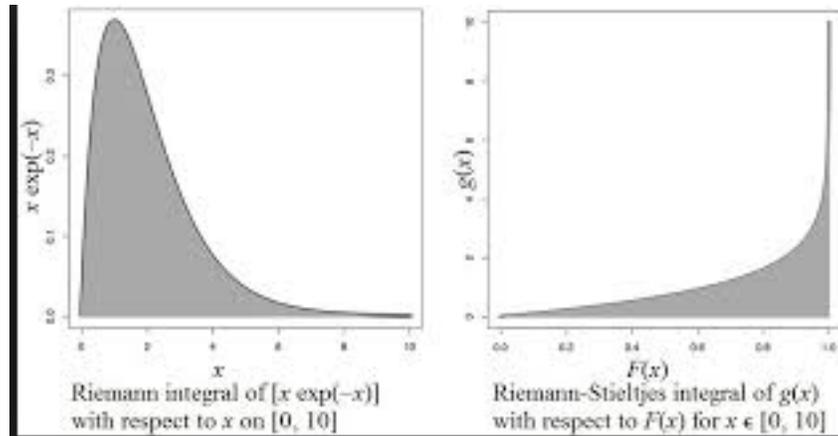
For the knowledge of Bonnet's theorem, I have referred several books of 9th, 10th and engineering standards. The study was drawing my interest thoroughly to the concepts which are hard to study and understand. An advantage of this alternative to Bonnet's proof is that it is based directly on the completeness of the real numbers rather than on the result of Weierstrass.

The Riemann integration serves as an instructive and useful precursor of the Lebesgue integral, and an invaluable tool in unifying equivalent forms of statistical theorems that apply to discrete and continuous probability.

It was Riemann in 1853 who developed the definition of integration and provided that necessary and sufficient conditions for a bounded function defined on  $[a, b]$  to be integrable. Later on, in 1894 Stieltjes gives a modification of the Riemann Integral which helps give insight into the Riemann-Stieltjes Integral. It serves as an instructive and useful precursor of the Lebesgue integral. The researcher approach to Riemann-Stieltjes Integration was summarized below which was a modification of the Riemann Integral.

Integration has been an important part of mathematics since the 17th Century, when Gotfried Wilhelm Leibniz and Isaac Newton constructed the Fundamental Theorem of Calculus, which says that the area is essentially the same as taking an anti-derivative. In the 19th Century, Cauchy investigated integrals of continuous functions. The integral that he used was later redefined by Riemann, which he used to investigate integrals of discontinuous functions. His definition was far easier to understand and to teach than all of the previous ones, which resulted in the Riemann Integral becoming the standard integral to be taught to students in mathematics class which helps, give an insight to the Riemann-Stieltjes Integral (a generalization of Riemann Integral). An important generalization of the Riemann-Stieltjes Integral is the Lebesgue-Stieltjes Integral which generalizes the Riemann-Stieltjes integral in a way analogous to how the Lebesgue integral generalizes the Riemann integral. If improper Riemann-Stieltjes integrals are allowed, the Lebesgue integral is not strictly more general than the Riemann-Stieltjes integral. The Lebesgue integral was based on measure theory and, when mastered, proved a very powerful tool in mathematical applications.

## Graphs and Figures



**Riemann integral of  $[x \exp(-x)]$   
with respect to  $x$  on  $[0,10]$**

**Riemann-Stieltjes integral of  $g(x)$   
with respect to  $F(x)$  for  $x \in [0,10]$**

### Purpose enhancement

There are two theorems included in the Bonnets theorem. The two theorems are given below:

The fundamental theorem of calculus is a theorem that links the concept of differentiating a function with the concept of integrating a function.

The first part of the theorem, sometimes called the first fundamental theorem of calculus, states that one of the antiderivatives, say  $F$ , of some function  $f$  may be obtained as the integral of  $f$  with a variable bound of integration. This implies the existence of antiderivatives for continuous functions.

Conversely, the second part of the theorem, sometimes called the second fundamental theorem of calculus, states that the integral of a function  $f$  over some interval can be computed by using any one, say  $F$ , of its infinitely many antiderivatives. This part of the theorem has key practical applications, because explicitly finding the antiderivative of a function by symbolic integration avoids numerical integration to compute integrals. This provides generally a better numerical accuracy.

We use the Riemann Stieltjes integration to prove this theorem faster and in an easy way.

### Conclusion

The above project relies on the principles of mathematics and is better and innovative than other projects of the same kind because it is being addressed in a new way containing all the parameters of bonnets theorem and is easy and smart way to learn and implement on it. This system covers mostly all of the content required to prove the bonnets theorem. This helps the user to reduce the time taken, saves lots of effort which makes it mostly best in time and space complexity.

## Recommendation

Proving the Bonnets theorem using Riemann stieltjes integration is use full for mathematics domain. The scope for further study is to solve all types of questions related to the theorem, that are recommendation for new researcher using this paper.

## Acknowledgement

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