DEVELOPING AN ALTERNATE EXTENDED CONCEPTUAL MEAN-VARIANCE MODEL WITH CONSTRAINTS BASED ON LITERATURE

Biplab Mahapatra, Sanjay Mohapatra, Soumya Guha Deb
Xavier Institute of Management, Bhubaneswar, India

ABSTRACT-The purpose of this paper is to critically review the literature related to Markowitz Mean Variance Model (MV model henceforth) to gain an understanding of the criticisms the MV model is subject to. Second, the paper identifies the realistic constraints, from literature, faced by investors in practice, which are not included in the original MV model. Third, the paper presents an extended model incorporating some of the constraints and suggests a genetic algorithm based approach to solve the proposed model.

Keywords-**Mean Variance, Portfolio optimization, Constraints, Evolutionary Algorithm**

INTRODUCTION

In MV, Markowitz uses only two constraints [1]. First, the total weight assigned to all assets in a portfolio will be 1(one) i.e. the full investable amount will be invested, known as Budget Constraint. Second, no short selling is allowed i.e. the weights cannot be –ve, also known as the non-negativity constraint. However, investors, as well as, Portfolio Managers face quite a large number of constraints in practice compared to the few considered in the original MV model. Allocation of limited investable funds in right proportions to various available investable assets to maximize return and minimize risk is a critical task of the Portfolio Managers as well as investors. Availability of hundreds of investable assets belonging to different asset classes like bonds, equity, real estate, commodities and derivatives further compounds the problem. The asset allocation problem can be broken into two steps. First finding out the proportion of fund to be allocated to different asset classes. Second, within each asset class identifying individual assets in which to invest. Optimal allocation of funds to maximize return and minimize risk is a combinatorial problem and is subject to combinatorial explosion [2].

The process itself is quite complex given the innumerable options available within each asset class. The inclusion of multiple boundary constraints derived primarily from behavioral aspects of individual investors, makes the optimization problem too complex for the portfolio manager. Historically approaches used to achieve the optimization are quadratic programming, mixed linear Integer model, etc. However, many of these classical methods fail to handle the complexity in the face of multiple boundary constraints. Over the last few decades, finance researchers attempted to apply more advanced approaches like metaheuristics including Simulated Annealing, Particle Swarm optimization [2, 3] Neural Network, Tabu Search, Genetic Algorithm, Hill Climbing to name a few [2, 4, 3] to solve this Portfolio Optimization problem.

The purpose of this study is primarily threefold. First, we present a critical review of the literature concerning Markowitz's Mean-variance model (MV model henceforth). Second, we identify and present several other practical constraints faced by investors and portfolio managers on a day-to-day basis, which are not included in the standard MV model. Third, we propose an extended version of the MV model and suggest a possible way to solve the extended model.

The remaining part of the paper is organized as follows: section 2 discusses the MV model and the criticisms it faces. Section 3 discusses the standard constraints used in the original models vis-à-vis other relevant practical constraints. In section 4 we propose an extended version of the standard MV model and suggest the methods to solve the problem followed by the conclusion section wherein we highlight potential future research to extend and enrich the model.

MEAN-VARIANCE MODEL

Markowitz's Mean variance Model(1952)

In MV model [1], the objective is to minimize risk given an expected return or maximize return given an acceptable risk. The problem is defined as a constrained quadratic optimization problem [1]. For n assets,

$$MinimizeVariance_{portfolio} = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} \quad (1)$$

Subject to

$$ExpectedReturn_{Portfolio} = \sum_{i=1}^{n} w_i \mu_i \quad (2)$$

$$\sum_{i=1}^{n} w_i = 1 \quad , i = 1, 2, \ldots n \quad (3)$$

$$w_i \geq 0 \quad (4)$$

Where

σ_{ij} = covariance between the two assets i and j (i =1,2,….,n; j = 1,2,…..,n)

μ_i= Expected Return of assets i (i = 1, 2… n) and
w_i = percentage of investable fund allocated to a particular asset

The objective function is to minimize the variance of the portfolio as indicated in equation (1) subject to the following boundary constraints Portfolio return is equal to the expected return as expressed in equation (2)

- The total investable amount is fully invested as expressed in equation (3) and
- None of assets' weight expressed as$w_i$is negative as expressed in equation (4)

The formulated model, a single objective problem, is to minimize the risk subjected to the desired return as fixed by the user.

MV model is based on following assumptions:

a) Investors are rational: Investors will invest with the sole aim of maximizing profit and not for the purpose of gambling.

b) Investors are risk averse: Given an expected return, the investors will try to reduce the risk to the minimum amount possible. That's the reason for minimizing the risk based on certain expected return. "*Variance is an undesirable thing*" – Markowitz.

c) No short selling is allowed.

d) Taxes and transaction costs are not there.

e) Returns of assets follow multivariate normal distribution.

f) Risk can be measured by variance.

g) Securities can be divided any number of times (even fractions) and traded.

Criticism of Markowitz's Portfolio Theory : related literature and discussion

Markowitz's portfolio theory is subject to widespread criticism from academicians and practitioners alike. We highlight some of them here as follows:

Returns are not Multivariate Normal. In MV model asset returns are assumed to follow a Gaussian multivariate normal distribution. However, it has been shown that this assumption does not hold good in practice [5]. The distributions of assets' returns are more often than not found to be leptokurtic or fat tailed i.e. a higher probability of extreme values [6]. This implies the need of higher moments for an appropriate explanation.

Need for Estimation. The Mean-variance model is dependent on correct estimations of expected returns and risks. Given that the assumption about normality does not hold, the need to accurately predict the returns and risks of the underlying assets, portfolio managers wish to include in the portfolio becomes necessary [7]. However, in reality, the challenges to do accurate estimations aremultifold. Many competing approaches have been attempted in this context with varying degree of success using Neural Network [8], Support Vector Machines (SVM) [9, 10, 11]etc. None of these approaches is free of estimation errors and shortcomings. These estimation errors maximize the risk. Portfolio optimizers using mean-variance model are "Error Maximizers" [12]. It is a fact that mean-variance portfolios are often outperformed by equally weighted portfolios [13, 14, 15].

Large number of Covariance calculations. Another aspect is the determination of the covariance between the assets. With an increase in the universe of the investable assets, the number of covariance calculation becomes excessive and quite challenging. [16]

Dynamic nature of variance and covariance amongst assets causing frequent portfolio rebalancing: Change in the values of the variance and return impacts the optimal asset allocation. In practice, this will prompt for Portfolio Rebalancing. Whereas frequent portfolio rebalancing in practice will attract transaction cost, which will affect the overall portfolio return. This is not in line with assumptions of negligible tax and transaction costs in the original MV model, [1]. These costs can have significant adverse impact on the realized portfolio return So frequent portfolio rebalancing is not practical as it comes at a cost. The question remains about the optimal frequency of portfolio rebalancing.

Impractical Assets weights. Optimization using MV model may suggest weights, which are not practical. For example, fractional asset units might not be simply available in the market for trading. Equities are typically traded in whole, Derivatives are typically traded in lots. Again derivative lot sizes may vary with the type of underlying instrument. This factor, i.e. Transaction Lot constraints, is not incorporated in the MV model. Considering this will help in choosing the right allocations, which are practically applicable. Researchers have attempted to consider this to bring the optimization process closer to reality. [17, 18, 19].

Variance as Risk Measure. Academicians question variance as a risk measure. Even Markowitz [20]suggests advantages of semi-variance over variance. In his own language, "analyses based on S [semivariance] tend to produce better portfolios than those based on V [variance]".This is because variance penalizes both the downside and upside deviations from the mean equally. However, from an investor's perspective, downside deviations are really risk but upside deviations should be pleasant surprises and should not be penalized. Different risk measures suggested by the researchers, like Value at Risk (VaR), lower Partial Moments [21, 22], MAD [23], Semi Variance [20, 24], Conditional Value at risk (CVaR) try and address this notion of true risk not being captured by variance.

CONSTRAINTS USED IN MODERN PORTFOLIO SELECTION: RELATED LITERATURE

For conducting literature review and deriving a conceptual framework based on the research gap identified, we consider the existing literature on the Portfolio Optimization with a focus on evolutionary algorithms. We establish distinct boundaries as follows:

a) Consider only the articles published in peer reviewed scientific journals and written in English language

b) Consider Publications with focus on Portfolio Optimization Literatures and Evolutionary algorithm

c) We focus only on journal publications, book series, as they are easily accessible

We search for articles using keywords Evolutionary Algorithm, Portfolio Optimization, Constraints, Mean Variance for the period of 2013-2017. The databases we use for this study are Elsevier (www.sciencedirect.com), Springer (www.springerlink.com). We also use library services like Ebsco (www.ebsco.com). To make the search complete we incorporate papers mentioned in Ponsichet. al. [25] with a specific focus on the Portfolio Optimization and constraints. Using these criteria, we finally select 55 papers to review in this study.

Constraints in original MV Model

As mentioned earlier the original MV model has the following constraints

1) No short selling, also known as non-negativity constraint, and
2) Budget constraints i.e. allocation of the full available fund

Initially Markowitz [1]use no-short selling constraint in his model. Subsequently almost all the researchers use non-negativity constraint as well as Budget constraint in their problem formulations [26, 27, 28, 29, 30, 31, 32, 33, 34] [35, 36, 37, 38, 39, 40, 41] However, Black et. al. [42]drop this constraint and expanded the model to allow for short selling.

Additional perceived and practical constraints relevant for Investors

In practice, other than the two mentioned above, investors face many other constraints. Following Kolm et al. [43], we classify the additional constraints used in literature in different categories as follows

a) Regulatory Constraints
b) Guideline constraints
c) Trading Constraints
d) Risk Management constraints
e) Discretionary Constraints

Regulatory Constraints. Market regulators, for example SEBI in India, SEC in USA etc., impose the regulatory constraints to protect the investors from the unscrupulous market participants. These constraints differ from country to country as well as across asset classes. Investors, as well as Portfolio Managers, cannot violate these being on the right side of the law even if these restrict their capabilities. Bajeuxet. al [44] mention about different regulatory constraints including constraint on short sales, weight constraint in the context of active portfolio management. The paper also indicates a form of regulatory constraint i.e. lock-in period constraint, where the Portfolio Manager or Investors need to hold the equities for a particular period. 5-10-40 Constraint.is a variation of weight constraint and is as per German Investment Law [45] applicable to Mutual Fund. As per the rule "*a maximum of 10 percent of a fund's net assets may be invested in securities from a single issuer, and that investments of more than 5 percent with a single issuer may not make up more than 40 percent of the whole portfolio.*" [46]. Derigset. al. [47]use this constraint in the formulation of the model and solve using Simulated Annealing. Branke et al [48] use this constraint along with budget constraints, no short sale constraints, and cardinality constraints and solve tis using enveloped multi-objective genetic algorithm.

Guideline Constraints. Kolmet. al. [43]suggest that the restrictions imposed by the client are of paramount importance and should be respected like regulatory constraints. A socially responsible investor may impose restrictions as not to invest in particular sectors like tobacco sectors or liquor sectors or in companies dealing with gambling like Casinos. She may also impose restrictions on the maximum number of securities to be included in her portfolio.

Guideline constraints are of following types:

Invest only in specific equities, which are part of Benchmark Index.In the literature, we have not seen any specific mention of this constraint but researchers always focus on specific indexes. To name a few Deng et al [49]use Hang Seng 31 in Hong Kong, DAX 100 in Germany, FTSE 100 in UK, S&P 100 in USA and Nikkei 225 in Japan, Pai et al. [50] use equities from BSE200 as well as Nikkei225.

Maximum number of Securities in the Portfolio.This constraint, known as Cardinality Constraint, may be imposed by the organization with which the Portfolio Manager is associated with or by the customers. Empirically it is shown that the number of equities in the portfolio, which reduces the diversifiable risk to negligible, is 20 – 25 equities. [2]. The evidence is there that the number has increased to 50 in recent years [51]. Adding 20-25 equities (or 50 equities as pointed out by Campbell) will eliminate the equity-specific risks. This constraint is one of the most used constraints by the researchers in their model. [2, 4, 52, 53, 54, 55, 18, 56, 19]. Cardinality constraint helps in multiple ways. First, managing and tracking the limited number of equities is practically feasible. With the increase in equity numbers in the portfolio, reduction in tracking capabilities takes place. Second is the concept of Model Insurance [43]. The probabilities of error-free risk and return estimates are practically zero. Underestimation or overestimation of risk will lead to an inefficient portfolio. Specifically, underestimation of risk is detrimental as it will expose the portfolio to extreme bets. This constraint helps in reducing that exposure and in turn provides a model insurance. Cardinality constraint is expressed in the below form:

$C_l \leq \sum_{k=1}^{N} z_k \leq C_uand0 \leq C_l \leq C_u \leq N$

Where

Cl = Lower limit i.e. the minimum number of equities in the Portfolio

Cu = Upper limit i.e. the maximum number of equities in the Portfolio

z = Equities, binary value of 0 or 1 to indicate presence or absence of an equity

N = the maximum number of equities

Researchers employ different form of cardinality constraints like only the lower limits with greater than equality constraints [57, 58, 49, 59, 60], only the upper limits [53, 61, 62, 63] also both the upper and lower limit as described in the above formulation [54, 64, 65, 66, 67]. Some researchers use equality constraints [4, 55, 68, 48, 18, 69, 70, 71]. Some researchers [72]also use this as additional objective instead of as a constraint.

Pre-assignment constraints.An instruction to include a predefined set of equities or a broad asset class in the portfolio is known as pre-assignment constraint. The Manager's job will be to determine the optimum percentage. Chang et al. [4] mentioned about this constraints. Gaspero et al. [73] use this constraint for the first time. Subsequently other researchers [74, 69] use this constraint in their model.

Trading Constraints. Certain restrictions are the result of the way the market operates. For example trading of a fraction of a security is not possible in the equity market, fractional lot cannot be sold in case of options or futures, or the number of securities can be sold in the market without an impact. Some forms of trading constraints areas follows:

Round lot Constraint For some asset classes, the trading happens in lots like options. Similarly, for securities, a fractional portion cannot be traded. Therefore, to hold these assets in the portfolio, the restrictions imposed by the rules need to be followed. This restriction is known as Round lot constraint. Researchers have applied this constraint in their attempt to develop realistic optimized portfolios. Many researchers [17, 57, 58, 64, 54, 18, 62, 69] [75] [76]include this constraint while trying to create optimized portfolio using the genetic algorithm.

$$wiC = LiPi$$

Where,
wi = weight of the asset
C = total investable capital
Li = tradable lot size of the asset
Pi = Price of the Lot

Trading maximum numbers.Portfolio Managers or individual investor follow the constraints like not to trade more than a particular amount for a particular type of equity. This is to prevent negative impact for equities having low volume trading. The unfavorable movement of the market is termed as slippage cost. This is part of the transaction cost. The impact due to the slippage cost is non-linear and can be considerably high as far as up to two orders of magnitude of the negotiated cost. [2]. Therefore, the manager may impose a restriction on themselves [43].

Buy-in threshold. Also known as Floor constraint.Trading with small numbers will negatively affect the return considering the transaction cost. Therefore putting a constraint from trading perspective makes sense. The mathematical formulation of the constraints is as shown below

$$a_i \leq w_i, 0 \leq a_i \leq 1$$

Where
ai = lower limit or floor constraint on weights,
wi = weight in the portfolio

Floor constraint is used by researchers [57, 58, 61, 77, 54, 64, 62], But generally both floor and ceiling constraints are used [53, 78, 79] extensively

Transaction cost. Transaction cost is considered as nil in MV model but in reality, it affects the investors' return in case of both single period portfolio as well as multi-period portfolio. Expected asset return seems to vary with time [80, 81], so there is a need to rebalance the portfolio over the investment horizon to capture the shifting investment opportunities [82, 83]. While doing the portfolio rebalancing the investor will incur transaction cost and that will affect portfolio return. Even for a single period, Transaction costs are to be paid for both buying and selling activities. except in IPO (Initial Public Offer) scenario, where transaction cost in the buying process is not required.

As per Maringeret. al. [84]Transaction costs can be of different types like Fixed Transaction cost, Proportional Transaction cost, Proportional transaction cost with a minimum threshold, proportional transaction cost + fixed transaction cost. Out of these types of transaction costs typically proportional transaction cost is more common. Mathematically formulation following Maringeret. al. [84]is:

$$C_i = \begin{cases} C_f & (Fixedcostonly) \\ c_p.n_i.S_{0i} & (Proportionalcostonly) \\ max\{C_f, c_p.n_i.S_{0i}\} & (Proportionalcostwithlowerlimit) \\ C_f + c_p.n_i.S_{0i} & (Proportionalcost + fixedcost) \end{cases}$$

Where,
n_i =Natural, positive numbers of assets i={$1, \ldots \ldots, N$}
S_{0i} = Present price of the asset
C_f = Fixed cost
c_p = Proportional cost

Chen et. al. [59], Silva et. al [63], Ruiz-Torrubiano et. al. [85], Najafiet. al [86] use Transaction costs in the model. Liu et. al. [87], Guoet. al. [88]use transaction cost but the model is for multi-period. Meghwaniet. al. [89]use Transaction cost as additional objective.

Risk Management Constraints. Portfolio optimizers employing the mean-variance model are "Error Maximizers" [12]. Error in the input is maximized and in turn, increases risk. One way to reduce the risk is by imposing a limit on the exposure. A portfolio manager likes to put constraints on her choices like a limit on the exposure to a particular sector or securities possessing common characteristics. The constraint classified under this category can be part of Guideline Constraints also as the line between these two classifications is fuzzy. Given the scope for investment across the world, portfolio managers or investors may impose a constraint on geographic location wise exposure also. This way the risk of exposure is limited. Some of the constraints, which can be classified in this group, are:

Ceiling Constraint- % Exposure to a particular asset or security. Accurate estimations of the future return and the risk of individual assets are practically infeasible. Also in case of wrong estimations, the risk of some of the assets will be overestimated and some will be underestimated. These estimation errors maximize the risk. Portfolio optimizers employing the mean-variance model are "Error Maximizers" [12].

One way to reduce the effect of the estimation errors is to use constrained optimization, specifically the ceiling constraints, through which the exposure of the portfolio to a particular asset can be limited as suggested by Gupta et al. [90]. This also helps in reducing the volatility of the portfolio and increases realized portfolio performance. Imposing upper weight constraints helps in reducing the risk effectively [91]. Researchers use this constraint along with floor constraint in the formulation of the model. [4, 53, 55, 77, 68, 50, 18, 49, 65] [74, 66, 59, 63, 69]

Ceiling Constraint - % Exposure to a particular Sector. Also known as Class constraints, helps in limitingexposure to assets having similar traits [4] and provides a sense of "Model Insurance" [43]. Researchers have attempted adding this constraint to the formulation of portfolio optimization [77, 50, 18, 56].

Ceiling Constraint - % Exposure to a particular geography. This constraint can limit the exposure from the third perspective, which is Economy in the EIC model, in a particular geography. This type of factor will be applicable for portfolios exposed to multiple geographies like emerging markets, developed markets, etc. Bajeuxet. al. [44]indicate about the rules put in place related to the percentage allowed to be invested in specific geography. They discuss these in the context of European countries.

Sectoral Capitalization Constraint.Market Capitalization is used by Oh et al [92] while optimizing Portfolio using Genetic Algorithm. They use Market Capitalization to select the sectors and then included the competitive stocks to reduce the risk. Solemni et al [18], Lwinet. al. [93]use this constraint in such a way that the combined weight of the stocks selected from a sector is related with the sectoral proportion with respect to the overall market capitalization.

Discretionary Constraints. A portfolio manager likes to put constraints on his/her choices like a limit on the exposure to a particular sector or securities possessing common characteristics. The constraints, classified under this category, can be part of Guideline Constraints also as the line between these two classifications is fuzzy. In his book, Graham et al [94] suggested some of the criteria for selecting stocks for Defensive Investors, like

a) Adequate Size of the Enterprise.

b) A sufficiently Strong Financial Condition: He suggested some measure like "two-to-one current ratio, long-term debt should not exceed the net current assets etc".

c) Earnings Stability: Some earnings for the common stocks in the last ten years is suggested by Graham.

d) Dividend Record: Uninterrupted dividend in the past twenty years.

e) Earnings Growth: "A minimum increase of at least one-third in EPS in the past ten years using three years averages at the beginning and end"

f) "*Moderate Price/Earnings ratio*": P/E should be less than 15 where earning is considered as average earnings of the past three years.

g) "*Moderate Ratio of Prices to assets*"

These are some of the examples of discretionary constraints. A portfolio Manager or investors can put such constraints while selecting securities. The list is no way exhaustive. During our literature review, we did not find any researcher using these constraints.

TOWARDS A CONCEPTUAL MODEL

Gaps in Research based on literature review

Arone et al [26] modify the MV model and use downside risk measure instead of variance as risk measure. But they do not use any additional constraints. Loraschi et al. [27] also do not extend the constraints further. Change, et al. [4] uses cardinality constraints along with the other constraints like budget constraints and non-negativity constraints. They also impose a ceiling constraint for a single asset. The problem is solved as single objective optimization problem using Genetic Algorithm, Tabu Search and Simulated Annealing. However, they do not use multi-objective formulation. Also, some other constraints are left out. Lin et al [17] extend the MV model by incorporation of fixed transaction costs and minimum transaction lots. Streichert et al. [58] use fixed cardinality constraints, Buy-in threshold constraints and round lot constraints along with the classical budget constraints as well as no short sales constraints. Schaerf [52] applies cardinality constraint and constraints on individual equities on mean-variance model along with the original constraints of MV model. He utilizes one of the local search techniques i.e. Tabu Search. In addition, the problem is solved as single objective optimization problem considering expected return as a constraint instead of considering the problem as a multi-objective problem. Crama and Schyns [53] use floor and ceiling constraints, budget constraints, turnover constraints (purchase and sale), trading constraints and cardinality constraints. The problem is solved as single objective optimization function by considering expected return as one of the constraints, instead of considering the problem as a multi-objective problem. They use Simulated Annealing to solve the formulated problem. Fieldsend et al. [72] use cardinality constraints as additional objective. Skolpadungket et al. [54] use additional constraints like Cardinality constraints (range), Floor& ceiling constraints as well as round lot constraints. Chiam et al. [55] also use the similar set of constraints except that they use fixed cardinality constraints. Krink et al [77]use Buy in Threshold constraint, Asset class constraint, % change compare to the previous allocation as well as maximum turnover constraints. Golmakani&Frazel [19] extended the MV by adding the bound on cardinality, holdings, minimum transaction lots and sector capitalization constraints. They formulated the problem as single objective optimization problem instead of a multi-objective problem. The problem thus constructed is solved utilizing Particle Swarm Optimization technique. Soleimani et al. [18]use three constraints like minimum transaction lots, cardinality constraints

and market capitalization constraint and solve this using Genetic Algorithm. Fernández and Gómez [61] extend the mean-variance problem by adding cardinality constraints and bound constraints (i.e. Floor and Ceiling constraints). They formulate the problem as a single optimization problem by using weights and use Neural Network to trace the efficient frontier. Mishra et al. [60]as well as Ma et al. [65] uses cardinality constraints (fixed), buy-in threshold constraints. Chen [59] uses same set of constraints but added Transaction cost as additional constraints. However, no researcher has considered all the constraints in one problem and attempted to solve the same as a multi-objective problem. Maximum number of constraints is incorporated by Lwin et al [74]and Meghwani et al [69]. They use Cardinality Constraints (fixed), floor and ceiling constraints, Pre-assignment constraints, round lot constraints along with the classical constraints. But there are few more constraints which are also important as sectoral constraints, liquidity constraints. Also, transaction cost is not included.

A considerable amount of research is conducted on Portfolio optimization using evolutionary Algorithms in the previous decade. However, it is evident from the review that most of the researchers used only a few i.e. 2-3 in additions to the constraints used by Markowitz. Cardinality Constraints and Floor & ceiling constraints are most popular among the researchers.

Hence there is a scope to use all the constraints, at least most of the common ones faced by Investors, together. This will make the model more realistic.

We are proposing the model considering the following constraints:

- **Guideline Constraints**
  - ✓ Budget Constraints
  - ✓ No short sale Constraints
  - ✓ Cardinality Constraints (range)

- **Trading Constraints**
  - ✓ Round lot Constraints
  - ✓ Liquidity Constraints

- **Risk Management Constraints**
  - ✓ Floor/Buy in Threshold and Ceiling Constraints
  - ✓ Sectoral Constraints

- **With Transaction Cost**

While classifying the constraints, in different categories, we observe the process of maximum likelihood concept. If a constraint can be part of two buckets, we check where the constraint is more relevant. For example, the floor and ceiling constraints can be part of Guideline Constraints as well as Risk Management Constraints. In that case, we put these in Risk Management category. Reason being its relevance to managing risks is considerable. Admittedly, this is subjective.

Proposed Model with Transaction cost

Objective Functions

$MinimizeVariance_{portfolio} = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}$ (1)

$MaximizeExpectedReturn_{Portfolio} = \sum_{i=1}^{n} w_i \mu_i - \frac{\sum_{i=1}^{n} z_{i*} C_{ti}}{(\sum_{i=1}^{n} wi)*C}$ (2)

Subject to

$\sum_{i=1}^{n} w_i \leq 1$, i =1,2,…..n(Budget Constraints) (A)

$0 \leq w_i \leq 1$ (Non-negativity Constraints) (B)

$C_l \leq \sum_{k=1}^{n} z_k \leq C_u$,$z = \{0,1\}$ (Cardinality Constraints) (C)

$0 \leq C_l \leq C_u \leq n$

$a_i \leq w_i \leq b_i$ (Floor and Ceiling constraints) (D)

$0 \leq a_i \leq b_i \leq 1$

$a_i \geq \max(userdefinedlowerlimit, \frac{100XC_f}{w_i C C_p})$

Ct=max $\{Cf, p.ni.S0i\}$ $(Proportionaltransactioncost)$ (E)

wiC = LiPi(Round Lot Constraints) (F)

$S_l \leq w_i C \leq S_u$ (Sectoral Constraints) (G)

$w_i C \leq AverageTradingvolume$(Liquidity Constraints) (H)

Where

wi = weight of the asset i.e. percentage of fund allocated to a particular asset

σij = covariance between the two assets i and j (i =1,2,….,n; j = 1,2,…..,n)

μ_i= Expected /Estimated Return of assets i (i = 1, 2… n) and

n = maximum number of assets

Cl = Lower limit i.e. minimum number of assets to be part of the Portfolio

Cu = Upper limit i.e. maximum number of assets to be part of the Portfolio

ai = lower limit or floor constraint on weights,

bi = upper limit or ceiling constraint on weights and

C = total investable capital

Li = tradable lot size of the assets
Pi = Price of the Lot
Z = Assets or individual equity
S_l= Lower limit in a sector
S_u= Upper limit in a sector
n_i =Natural, positive numbers of assets i={$1, \dots \dots .., N$}
S_{0i} = Present price of the asset
C_f = Fixed cost
c_p = Proportional cost

Possible methods to solve the problem

With the inclusion of the other factors or relaxation of the assumptions of the MV, the problem of portfolio optimization becomes intractable, too complex to be solved by classical methods. Given that the classical methods like quadratic programming, mixed linear Integer programming model, etc. are inappropriate, researchers attempt to apply metaheuristic algorithm. Chang et. al. [4]use Cardinality Constraints and floor-ceiling constraints. They solve using Simulated Annealing, Tabu Search and Genetic Algorithm. They find that all the algorithms' performances are comparable. Skolpadungketet. al. [54]use different forms of multi-objective like VEGA (Vector Evaluated Genetic Algorithm), MOGA (Multi Objective Genetic Algorithm), SPEA (Strength Pareto Genetic Algorithm) and Fuzzy VEGA. The model they use includes Cardinality Constraints, Floor-Ceiling constraints and Roundlot constraints in addition to the ones used by MV model. Liagkouraset. al. [66] use modified NSGA-II and SPEA II. Their model includes Cardinality Constraints, Floor-Ceiling constraints and Roundlot constraints in addition to the ones in MV. Mishra et. al [60] use self-regulated PSO (particle Swarm Optimization) to solve an extended MV model. Saboridoet. al. [95] use NSGA-II with other GA along with newly introduced crossover, mutation and reparation operators. Meghwaniet. al. [69]use NSGA-II, SPEA 2 as well as PESA 2. Other heuristics are also used by researchers like Simulated Annealing [4, 53], Particle Swarm optimization [2, 3], Tabu Search [4], Fireworks Algorithm [96] Based on our review of the literature we found that the most popular two multi-objective algorithms among researchers are NSGA-II and SPEA 2. Reason being, evolutionary algorithms do not assume anything related to the shape of the function [97]. Second, Genetic Algorithms have the ability to explore enormous space, as well as inbuilt flexibility [26]. Considering that, we propose to use NSGA II for solving the problem.

CONCLUSION

Researchers criticize the MV model for the assumptions. This paper is an attempt to address some of the assumptions by incorporation of constraints like round lot constraint, transaction cost, floor-ceiling constraints etc. We propose an enhancement to the MV with more number of realistic constraints for portfolio optimization to bring this closure to reality. We also suggest way to solve this model. The problem can be solved using evolutionary algorithm specifically Genetic Algorithm.

The study can further be extended by

- Solving the model using heuristics and plotting the efficient frontier.
- Incorporation of Investors' Preference and generate the relevant portion of the efficient frontier.
- Include the Tax implication.
- Addition of Pre-assignment constraints.

REFERENCES

[1] H. Markowitz, "Portfolio Selection," *The Journal of Finance*, vol. 7, no. 1, pp. 77-91, 1952.

[2] F. R. Busetti, *Metaheuristic approaches to realistic portfolio optimisation*, MS Degree Dissertation, University of South Africa., 2000.

[3] J. Zheng, Q. Wu and W. Song, "An improved particle swarm algorithm for solving nonlinear constrained optimization problems," In Third International Conference on Natural Computation, 2007.

[4] T. J. Chang, N. B. J. E. Meade and Y. M. Sharaiha, "Heuristics for cardinality constrained portfolio optimisation," *Computers and Operations Research*, vol. 27, no. 2000, p. 1271–1302, 2000.

[5] D. Maringer and P. Parpas, "Global Optimization of Higher order moments in portfolio selection," *Journal of Global Optimization*, pp. 219-230, 2009.

[6] T. Mills, "Stylized facts on the temporal and distributional properties of daily FT-SE returns," *Applied Financial Economics*, vol. 7, pp. 599-604, 1997.

[7] R. Oberuc, Dynamic Portfolio Theory and Management, New York: McGraw-Hill, 2004.

[8] E. W. Saad, D. V. Prokhorov and D. C. Wunsch, "Comparative study of stock trend prediction using time delay, recurrent and probabilistic neural networks," *IEEE Transactions on Neural Networks*, vol. 9, no. 6, pp. 1456-1470, 1998.

[9] L. C. F. E. Tay, "Application of support vector machines in financial time series forecasting,," *Omega*, vol. 29, no. 4, pp. 309-317, 2001.

[10] K.-j. Kim, "Financial time series forecasting using support vector machines," *Neurocomputing*, vol. 55, no. 1, pp. 307-319,

2003.

[11] Y. N. S.-Y. W. W. Huang, "Forecasting stock market movement direction with support vector machine," *Computers & Operations Research,* vol. 32, no. 10, pp. 2513-2522, 2005.

[12] R. Michaud, Efficient Asset Allocation : A practical guide to stock portfolio optmization and asset allocation, Boston: Harvard Business School Press, 1998.

[13] J. Jobson and B. Korkie, "Putting markowitz theory to work," *Journal of Portfolio Management,* vol. 7, pp. 70-74, 1981.

[14] P. Jorion, "International portfolio diversification with estimation risk," *Journal of Business,* vol. 58, pp. 259-278, 1985.

[15] V. DeMiguel, L. Garlappi and R. Uppal, "Optimal versus naive diversification:How inefficient is the 1/N portfolio strategy?," *Review of Financial Studies,* vol. 22, pp. 1915-1953, 2009.

[16] A. Perold, "Large-scale portfolio optimization," *Management Science 30,* pp. 1143-1160, 1984.

[17] D. Lin, S. Wang and H. Yan, "A multiobjective genetic algorithm for portfolio selection," *Working Paper. Beijing China: Institute of Systems Science, Academy of Mathematics and Systems Science Chinese Academy of science;2001 ,* 2001.

[18] H. Soleimani, H. R. Golmakani and M. H. Salimi, "Markowitz-based portfolio selection with minimum transaction lots, cardinality constraints and regarding sector capitalization using genetic algorithm," *Expert Systmes with Application,* vol. 36, no. 3, pp. 5058-5063, 2009.

[19] H. R. Golmakani and M. Frazel, "Constrained portfolio selection using particle swarm optimization," *Expert Systems with Applications,* vol. 38, no. 7, pp. 8327-8335, 2011.

[20] H. Markowitz, Portfolio Selection: Efficient Diversification of Investments, New York: John Wiley & Sons, 1959.

[21] V. S. Bawa, "Optimal Rules For Ordering Uncertain Prospects," *Journal of Financial Economics,* vol. v2, no. 1, pp. 95-121, 1975.

[22] P. C. Fishburn, "Mean-Risk Analysis with Risk Associated with Below-Target Returns," *American Economic Review,* vol. 67, pp. 116-126, 1977.

[23] H. Konno and H. Yamazaki, "Mean-Absolute Deviation Portfolio Optimization Model and Its Appplications to Tokyo Stock Market," *Management Science,* vol. 37, no. 5 (May, 1991), pp. 519-531, 1991.

[24] J. Estrada, "Downside Risk in Practice," *Journal of Applied Corporate Finance,* vol. 18, no. No. 1, Winter, pp. 117-125, 2006.

[25] A. Ponsich, A. L. Jaimes and C. A. C. Coello, "A survey on Multiobjecyive Evolutionary Algorithm for the solution of the Portfolio Optimization problem and other Finance and Economics Applications," *IEEE Transactions on Evolutionary Computation,* vol. 17, no. 3, 2013.

[26] S. Arone, A. Loraschi and A. Tettamanzi, "A genetic approach to Portfolio selection," *Journal on Neural and MAss-parallel Computing and Information Systems,* vol. 3, pp. 597-604, 1993.

[27] A. Loraschi, A. Tettamanzi, M. Tomassini and P. Verda, "Distributed Genetic Algorithms with an Application to Portfolio Selection," in *Artificial Neural Networks and Geentic Algorithms (ICANNGA'95),* E. N. Steele and R. Albrecht, Ed., Wien:Springer, 1995, pp. 384-387.

[28] L. C. C. D. G. G. Vedarajan, "Investment portfolio optimization using Genetic Algorithms," *Proc. Late Break. Papers Genet. Program. Conf.,* pp. 255-263, Jul 1997.

[29] M. Ehrgott, K. Klamroth and C. Schwehm, "An MCDM approach to portfolio optimization," *Eur. J. Oper. Res,* vol. 155, no. 3, pp. 752-770, 2004.

[30] R. Subbu, P. Bonissone, N. Eklund and K. Bollapragada, "Multi Objective financial portfolio design: A hybrid evolutionary approach," *Proc. IEEE Congr. Evol. Comput. (CEC),* vol. 2, pp. 1722-1729, Sep. 2005.

[31] C. Ong, J. Huang and G. Tzeng, "A novel hybrid model for portfolio selection," *Appl. Math. Computat,* vol. 169, no. 2, pp. 1195-1210, 2005.

[32] R. M. S. L. W. Yan, "Multi-period semi-variance portfolio selection : Model and munerical solution," *Appl. Math. Computat.,* vol. 194 , pp. 128-134, 2007.

[33] C. Aranha and H. Iba, "A tree based GA representation for the portfolio optimization problem," *Proc. 10th Annu. Gen. Evol. Comput. Conf. (GECCO),,* pp. 873-880, 2008.

[34] C. Aranha and H. Iba, "Application of a memetic Algorithm to the Portfolio Optimization problem," *Australasian Joint Conf. AI,* pp. 512-521, 2008.

[35] X. Huang, "Mean semi-variance model for fuzzy portfolio selection," *J. Computat. Appl. Math.,* vol. 217, no. 1, pp. 1-8, 2008.

[36] I. Radziukyniene and A. Zilinskas, "Approximation of Pareto set in multi objective portfolio optimization," 2008.

[37] P. Lin and P. Ko, "Portfolio value-at-risk forecasting with GA based extreme value theory," *Expert Syst. Applicat.,* vol. 36, no. 2, pp. 2503-2512, 2009.

[38] N. Bradshaw, C. Walshaw, C. Lerotheou and A. Parrot, "A multiobjective evolutionary algorithm for portfolio optimization," *Proc. Symp. Evol. Syst. (AISB),*, pp. 27-32, 2009.

[39] F. Duran, C. Cotta and A. Fernandez, "On the use of Sharp's index in evolutionary portfolio optimization under Markowitz's model," *Proc. Convent. Adapt. Emergent Behavior Complex Syst. (AISB),* pp. 9-14, Apr. 2009.

[40] X. Li, Z. Qin and S. Kar, "Mean-variance-skewness model for portfolio selection with fuzzy returns," *Eur. J. Oper. Res,* vol. 202, no. 1, pp. 239-247, 2010.

[41] L. L. Macedo, P. Godinho and M. J. Alves, "Mean-semivariance portfolio optimization with multiobjective evolutionary algorithms and technical analysis rules," *Expert Systems with Applications,* vol. 79, pp. 33-43, 2017.

[42] F. Black, M. C. Jensen and M. Scholes, "The Capital Asset Pricing Model: Some Empirical Tests," in *Studies of the Theory of Capital Market*, M. C. Jensen, Ed., New York, Praeger Publishers, 1972.

[43] P. N. Kolm, R. Tutuncu and F. J. Fabozzi, "60 year of Portfolio Optimization: Practical Challenges and current trends," *European Journal of Operation Research,* vol. 234, pp. 356-371, 2014.

[44] I. Bajeux-Besnainou, R. Belhaj, D. Maillard and R. Portait, "Portfolio Optimization under Tracking Error and Weights Constraints," *Working Paper, George Washington University,* 2007.

[45] InvG., "Investmentgesetz," 2005. [Online]. Available: http://www.bafin.de/gesetze/invg.htm . [Accessed 14 4 2015].

[46] alfi, "How UCITS funds protect investors," 3 May 2015. [Online]. Available: http://www.alfi.lu/investor-centre/investor-protection/how-ucits-funds-protect-investors. [Accessed 3 May 2015].

[47] U. Derigs and N.-H. Nickel, "On a metaheuristic-based DSS for portfolio optimization and managing investment guideleines," in *Metaheuristics International Conference,* 2001.

[48] B. S. S. K. D. H. S. J. Branke, "Portfolio optimization with an envelop based multiobjective evolutionalry algorithm," *??,* 2009.

[49] G.-F. Deng, W.-T. Lin and C.-C. Lo, "Markowitz-based portfolio selection with cardinality constraints using improved particle swarm optimization," *Expert Systems with Applications,* vol. 39, pp. 4558-4566, 2012.

[50] G. V. Pai and T. Michel, "Evolutionary optimization of constrained k-means clustered assets for diversification in small portfolios," *IEEE Trans. Evol. Computat.,* vol. 13, no. 5, pp. 1030-1053, 2009.

[51] J. L. M. M. B. a. X. Y. Campbell, "Have individual stocks become more volatile? An empirical exploration of idiosyncratic risk Working Paper 7590," *National Bureau of Economic Research,* 2000.

[52] A. Schaerf, "Local search techniques for constrained portfolio selection problems," *Computational Economics,* vol. 20, pp. 177-190, 2002.

[53] Y. Crama and M. Schyns, "Simulated annealing for complex portfolio selection problems," *European Journal of Operational Research,* vol. 150, pp. 546-571, 2003.

[54] P. D. K. &. H. N. Skolpadungket, "Portfolio optimization using multi-objective genetic algorithms," 2007.

[55] S. C. Chiam, K. C. Tan and A. A. Mamum, "Evolutionary Multi-Objective Portfolio Optimization in Practical Context," *International Journal of Automation and Computing,* vol. 5, no. 1, pp. 67-80, 2008.

[56] K. Anagnostopoulos and G. Mamanis, "A portfolio optimization model with three objectives and discrete variables," *Computers & Operations Research,* no. 37, pp. 1285-1297, 2010.

[57] F. Streichert, H. Ulmer and A. Zell, "Evaluating a Hybrid Encoding and Three Crossover Operators on the Constrained Portfolio Selection Problem," *Congress on Evolutionary Computation,* vol. 1, pp. 932-939, 2004.

[58] F. Streichert, H. Ulmer and A. Zell, "Comparing Discrete and Continuous Genotypes on the Constrained Portfolio Selection Problem," *Genetic and Evolutionary Computation,* vol. 3103, pp. 1239-1250, 2004a.

[59] W. Chen, "Artificial bee colony algorithm for constrained possibilistic portfolio optimization problem," *Physica A,* vol. 429, pp. 125-139, 2015.

[60] S. K. Mishra, G. Panda and B. Majhi, "Prediction based mean-variance model for constrained portfolio assets selection using multiobjective evolutionary algorithms," *Swarm and Evolutionary Computation,* vol. 28, pp. 117-130, 2016.

[61] A. Fernández and S. Gómez, "Portfolio selection using neural networks," *Computers & Operations Research,* vol. 34, pp. 1177-1191, 2007.

[62] J. Li and J. Xu, "Multi-objective portfolio selection model with fuzzy random returns and a compromise approach-based genetic algorithm," *Information Sciences,* vol. 220, pp. 507-521, 2013.

[63] A. Silva, R. Neves and N. Horta, "A hybrid approach to portfolio composition based on fundamental and technical indicators," *Expert Systems with Applications,* vol. 42, no. 4, pp. 2036-2048, 2015.

[64] M. Gomez, C. Flores and M. Osorio, "Hybrid search for cardinality constrained portfolio optimization," *Proc. 8th Annu. Gen. Evol Comput. Conf. (GECCO),* pp. 1865-1866, 2006.

[65] X. Ma, Y. Gao and B. Wang, "Portfolio Optimization with Cardinality Constraints Based on Hybrid Differential Evolution," *AASRI Procedia,* pp. 311-317, 2012.

[66] K. Liagkouras and K. Metaxiotis, "A new Probe Guided Mutation operator and its application for solving the cardinality constrained portfolio optimization problem," *Expert Systems with Applications*, vol. 41, no. 14, pp. 6274-6290, 2014.

[67] R. Qi and G. G. Yen, "Hybrid bi-objective portfolio optimization with pre-selection strategy," *Information Sciences*, vol. 417, pp. 401-419, 2017.

[68] T. Chang, S. Yang and K. Chang, "Portfolio Optimization problems in different risk Measures using Genetic Algorithm," *Expert Systems with Applications*, vol. 36, no. 7, pp. 10529-10537, 2009.

[69] S. S. Meghwani and M. Thakur, "Multi-criteria algorithms for portfolio optimization under practical constraints," *Swarm and Evolutionary Computation*, vol. 37, pp. 104-125, 2017.

[70] C. B. Kalayci, O. Ertenlice, H. Akyer and H. Aygoren, "An artificial bee colony algorithm with feasibility enforcement and infeasibility toleration procedures for cardinality constrained portfolio optimization," *Expert Systems With Applications*, vol. 85, pp. 61-75, 2017.

[71] A. Baykasoğlu, M. G. Yunusoglu and F. B. Özsoydan, "A GRASP based solution approach to solve cardinality constrained portfolio optimization problems," *Computers & Industrial Engineering*, vol. 90, pp. 339-351, 2015.

[72] J. E. Fieldsend, J. Matatko and M. Peng, "Cardinality constrained portfolio optimisation," in *Processdings of the Fifth International Conference on Intelligent Data Engineering and Automated Learning (IDEAL'04)*, R. E. H. Y. Z. Yang, Ed., Springer-verlag. Lecture Notes in Computer Sceience Vol 3177, 2004, pp. 788-793.

[73] L. Di Gaspero, G. Di Tollo, A. Roli and A. Schaerf, "Hybrid metaheuristics for constrained portfolio selection problems," *Quantitative Finance*, vol. 11, no. 10, p. 1473–1487, 2011.

[74] K. Lwin, R. Qu and G. Kendall, "A learning-guided multi-objective evolutionary algorithm for constrained portfolio optimization," *Applied Soft Computing*, 2014.

[75] C. Lin and Y. Liu, "Genetic algorithms for portfolio selection problems with minimum transaction lots," *European Journal of Operational Research*, vol. 185, no. 1, pp. 393-404, 2008.

[76] H. Dastkhan, N. S. Gharneh and H. Golmakani, "A linguistic based portfolio Selection model using weighted max-min operator and hybrid genetic algorithm," *Expert Systems with Applications*, vol. 38, no. 9, pp. 11735-11743, 2011.

[77] T. Krink and S. Paterlini, "Differential evolution for multiobjective portfolio optimization," *Center Economic Res. (RECent), Dept. Economics, Univ. Modena Reggio E., Working Paper*, 2008.

[78] R. Hochreiter, "An Evolutionary computation approach to scenario based risk-return portfolio optimization for general risk measures," in *Applications of Evolutionary Computing. EvoWorkshops 2007. Lecture Notes in Computer Science*, vol. 4448, Berlin, Heidelberg, Springer, 2007.

[79] R. Hochreiter, "Evolutionary Stochastic Portfolio Optimization," in *Natural Computing in Computational Finance. Studies in Computational Intelligence*, vol. 100, Berlin, Heidelberg, Springer, 2008, pp. 67-87.

[80] K. R. F. Eugene F. Fama, "Business conditions and expected returns on stocks and Bonds," *Journal of Financial Economics*, vol. 25, pp. 23-49, 1989.

[81] e. a. John Y. Campbell, The econometrics of financial markets, Princeton, NJ: Princeton University Press, 1997.

[82] C. H. J. Cox, "A variational problem arising in financial economics,," *Journal of Mathematical Economy*, vol. 20, pp. 465-487, 1991.

[83] L. M. V. John Y. Campbell, "Consumption and portfolio decisions when expected returns are time varying,," *The Quarterly Journal of Economics*, vol. 114, pp. 433-495, 1999.

[84] D. Maringer, "Portfolio Management with Heuristic Optimization," *Advanced in Computational Management Science Series*, vol. 8, 2005.

[85] R. Ruiz-Torrubiano and A. Suárez, "A memetic algorithm for cardinality-constrained portfolio optimization with transaction costs," *Applied Soft Computing*, vol. 36, pp. 125-142, 2015.

[86] A. A. Najafi and S. Mushakhian, "Multi-stage stochastic mean–semivariance–CVaR portfolio optimization under transaction costs," *Applied Mathematics and Computation*, vol. 256, pp. 445-458, 2015.

[87] Y.-J. Liu, W.-G. Zhang and W.-J. Xu, "Fuzzy multi-period portfolio selection optimization models using multiple criteria," *Automatica*, vol. 48, pp. 3042-3053, 2012.

[88] S. Guo, L. Yu, X. Li and S. Kar, "Fuzzy multi-period portfolio selection with different investment horizons," *European Journal of Operational Research*, vol. 254, no. 3, pp. 1026-1035, 2016.

[89] S. S. Meghwani and M. Thakur, "Multi-objective heuristic algorithms for practical portfolio optimization and rebalancing with transaction cost," *Applied Soft Computing*, vol. 67, pp. 865-894, 2018.

[90] F. Gupta and D. Eichhorn, "Mean-variance optimization for practitioners of asset allocation.," in *Handbook of portfolio management*, F. J. F. Associates, Ed., New Hope, Pennsylvania, 1998.

[91] P. Behr, A. Guettler and F. Miebs, "On portfolio optimization: Imposing the right constraints," *Journal of Banking & Finance*, vol. 37, pp. 1232-1242, 2013.

[92] K. J. Oh, T. Y. Kim, S.-H. Min and H. Y. Lee, "Portfolio algorithm based on portfolio beta using genetic algorithm," *Expert Systems with Applications*, vol. 30, pp. 527-534, 2006.

[93] K. T. Lwin, R. Qu and B. L. MacCarthy, "Mean-VaR portfolio optimization: A nonparametric approach," *European Journal of Operational Research*, vol. 260, no. 2, pp. 751-766, 2017.

[94] B. Graham, The Intelligent Investor, First Collins Business Essentials edition ed., New York: HarperCollins, 2006.

[95] R. Saborido, A. B. Ruiz, J. D. Bermúdez, E. Vercher and M. Luque, "Evolutionary multi-objective optimization algorithms for fuzzy portfolio selection," *Applied Soft Computing*, vol. 39, pp. 48-63, 2016.

[96] T. Zhang and Z. Liu, "Fireworks algorithm for mean-VaR/CVaR models," *Physica A*, vol. 483, pp. 1-8, 2017.

[97] C. A. V. V. D. A. &. L. G. B. Coello Coello, Evolutionary algorithms for solving multi-objective problems, New York: Kluwer Academic Publishers, 2002.