Oscillation of Third Order Nonlinear Neutral Type Difference equations

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Abstract: This paper is concerned with the oscillatory behavior of solutions of third order difference equations with neutral term. Sufficient conditions guarantee that every solution of

\[ \Delta[a_1(n)\Delta[a_2(n)\Delta(z(n))]] + q(n)f(x(\sigma(n))) = 0, \quad n \geq n_0 > 0 \]

is oscillatory.

Index Terms: Oscillation, Third order, Difference Equation, Neutral type.

I. INTRODUCTION

Last few decades have seen rapid growth in the study of the qualitative theory of difference equations [1]. In the recent years, researchers have paid great attention to the oscillation theory of third-order differential and difference equations [2, 5, 3, 4, 6]. This paper considers the following form of third order neutral difference equations

\[ \Delta[a_1(n)\Delta[a_2(n)\Delta(z(n))]] + q(n)f(x(\sigma(n))) = 0, \quad n \geq n_0 > 0 \]

(1)

Where \( z(n) = [x(n) + p(n)x^{\alpha}(\tau(n))] \) for the oscillation of its solutions. Assume that the following Conditions holds,

(\( H_1 \)) \( 0 < \alpha \leq 1 \) is the ratio of odd positive integers.

(\( H_2 \)) \( a_1(n), a_2(n), p(n), q(n) \) are positive sequences.

(\( H_3 \)) \( \tau(n), \sigma(n) \) are all positive and \( \tau(n) \geq n, \sigma(n) \geq n \).

(\( H_4 \)) \( f \) is non decreasing such that \( uf(u) \geq k > 0 \) for \( u \neq 0 \) and \( \lim_{n \rightarrow +\infty} \sigma(n) = \lim_{n \rightarrow +\infty} \delta(n) = \infty \).

This paper considers the following two cases:

(A) \( \sum_{n=n_0}^{\infty} \frac{1}{a_1(n)} = \infty; \sum_{n=n_0}^{\infty} \frac{1}{a_2(n)} = \infty \).

(B) \( \sum_{n=n_0}^{\infty} \frac{1}{a_1(n)} < \infty; \sum_{n=n_0}^{\infty} \frac{1}{a_2(n)} = \infty \).

A Solution \( x(n) \) of (1) is said to be oscillatory if the terms \( x(n) \) of the solution are not eventually positive or eventually negative. Otherwise the solution is called non oscillatory.

II. MAIN RESULT

Theorem 2.1. Let \( 0 \leq p(n) \leq p_1 \leq 1 \). Assume \( (H_1) \sim (H_4) \) and (A) holds if there exists a positive function \( \phi \), such that for all sufficiently large \( n_3 > n_2 > n_1 \geq n_0 \), we have

\[ \lim_{n \rightarrow +\infty} \sup_{m=n_0}^{n-1} \sum_{m=n_0}^{n-1} \left[ kq(m)\phi(m)\rho_*(m)\Theta(m) - \frac{a_1(m)\Delta^2\phi(m)}{4\phi(m)} \right] = \infty \]

(2)

hold for all constants \( M > 0 \). Then any solution \( x(n) \) of (1) is oscillatory.

Proof. Suppose that \( x(n) \) is solution of (1). By condition (A), there exists a possible case

\( z(n) > 0, \Delta(z(n)) > 0, \Delta[a_2(n)\Delta(z(n))] > 0, \Delta[a_1(n)\Delta[a_1(n)\Delta(z(n))] < 0 \), for \( n > n_1, n_1 \)

is large enough. Also,

\[ \Delta[a_1(n)\Delta[a_2(n)\Delta(z(n))] = -q(n)f(x(\sigma(n))) \]
Assume that $z(n)$ satisfying the case, there exists $n \geq n_1$ such that $z(n) > 0$; $z(\sigma(n)) > 0$ $\Delta z(n) > 0$. then $z(n)$ is monotonically increasing and there exists a constant $M > 0$ such that $z(n) \geq M$. Now by definition of $z(n)$ we have

$$x(n) = z(n) - p(n)x^\alpha(\tau(n)) \geq z(n) - p(n)z^\alpha(\sigma(n)) \geq \left[1 - \frac{p(\sigma(n))}{M^{1-\alpha}}\right]z(n) \quad (3)$$

Where $p_*(n) = \left[1 - \frac{p(n)}{M^{1-\alpha}}\right]$. Since, $\Delta a_1(n)\Delta a_2(n)\Delta z(n) < 0$, then $a_1(n)\Delta a_2(n)\Delta z(n)$ is decreasing, so

$$a_2(n)\Delta z(n) \geq \sum_{m=n_1}^{n-1} \frac{a_1(m)\Delta[a_2(m)\Delta z(m)]}{a_1(m)} \quad (4)$$

We have that

$$\Delta \left[\sum_{m=n_1}^{n-1} \frac{1}{a_1(m)} a_2(n)\Delta z(n)\right] \leq 0 \quad (5)$$

Now, from $\sum_{m=n_2}^{n-1} \Delta z(m) = z(n) - z(n_2)$, we get

$$z(n) = z(n_2) + \sum_{m=n_2}^{n-1} \Delta z(s) = z(n_2) + \sum_{m=n_2}^{n-1} a_2(m)\Delta z(m) \sum_{i=n_1}^{m-1} \frac{1}{a_1(i)}$$

IV.

$$\geq a_2(n)\Delta z(n) \sum_{i=n_1}^{n-1} \frac{1}{a_1(i)} \left[\sum_{m=n_1}^{m-1} \frac{1}{a_2(m)} \sum_{i=n_1}^{m-1} \frac{1}{a_1(i)}\right]$$

$$\frac{z(n)}{a_2(n)\Delta z(n)} \geq \frac{1}{\sum_{i=n_1}^{n-1} \frac{1}{a_1(i)}} \left[\sum_{m=n_1}^{m-1} \frac{1}{a_2(m)} \sum_{i=n_1}^{m-1} \frac{1}{a_1(i)}\right] \quad (6)$$

Let us define $\omega(n)$ as follows

$$\omega(n) = \phi(n) \frac{a_1(n)\Delta[a_2(n)\Delta z(n)]}{a_2(n)\Delta z(n)} , n \geq n_1 , \quad (7)$$

Notice that $\omega(n) > 0$ for $n \geq n_1$. Now

$$\Delta(\omega(n)) = \phi(n) \frac{\Delta[a_1(n)\Delta[a_2(n)\Delta z(n)]]}{a_2(n)\Delta z(n)} + \Delta\phi(n) \frac{a_1(n+1)\Delta[a_2(n+1)\Delta z(n+1)]}{a_2(n+1)\Delta z(n+1)}$$

$$- \phi(n) \frac{a_1(n+1)\Delta[a_2(n+1)\Delta z(n+1)]\Delta[a_2(n)\Delta z(n)]}{a_2(n)\Delta z(n) a_2(n+1)\Delta z(n+1)}$$

It follows from (1) and (7)
\[ \Delta a(n) = - q(n) f(x(\sigma(n))) + \Delta \phi(n) \frac{a(n+1)}{\phi(n+1)} - \frac{\phi(n)}{a_n} \Delta a(n) \Delta z(n) \]

\[ a(n) \Delta [a_2(n) \Delta z(n)] \] is a non increasing sequence and \( a_2(n) \Delta z(n) \) is an increasing sequence. From (3)

\[ \Delta(\omega(n)) \leq \Delta \phi(n) \frac{\omega(n+1)}{\phi(n+1)} - \frac{\phi(n)}{a_1(n)} \Delta \phi(n) \frac{\Delta z(n)}{\phi(n+1)} - k q(n) \phi(n) p\ast(n) \frac{z(\sigma(n))}{a_2(n) \Delta z(n)} \]

Summing from \( n_3 > n_2 \) to \( n-1 \), we get

\[ \omega(n) - \omega(n_3) \leq \sum_{m=n_3}^{n-1} \left[ \frac{\Delta^2 \phi(m) a_1(m)}{4 \phi(m)} - k q(m) \phi(m) p\ast(m) \Theta(m) \right] \]

Letting \( n \to \infty \),

\[ \lim_{n \to \infty} \sup \sum_{m=n_3}^{n-1} [k q(m) \phi(m) p\ast(m) \Theta(m) - \frac{\Delta \phi(m) a_1(m)}{4 \phi(m)}] \leq \omega(n_3) \]

Which contradicts (2) which completes the proof.

**Theorem 2.2.** Assume \((H_1) - (H_4)\) and \((A)\) holds and we have

\[ \sum_{j=n_1}^{n} \frac{1}{a_2(j)} \sum_{i=1}^{j} \sum_{m=1}^{i} q(m) = \infty \]  

(8)

Then any solution \( x(n) \) of (1) is either oscillatory or \( \lim_{n \to \infty} x(n) = 0 \)

Proof. Suppose \( x(n) \) is the positive solution of (1). By condition \((A)\), there exists a possible case such that

\[ z(n) > 0, \Delta z(n) < 0, \Delta [a_2(n) \Delta z(n)] > 0, \Delta [a_1(n) \Delta [a_2(n) \Delta z(n)]] < 0 \]

Since \( z(n) > 0 \) and \( \Delta z(n) < 0 \), there exists a finite limit, \( \lim_{n \to \infty} z(n) = l \). We shall prove that \( l = 0 \);

assume that \( l > 0 \), then for any \( \varepsilon > 0 \), we have \( l + \varepsilon > z(n) > l \); choose \( 0 < \varepsilon < \frac{l(l - p)}{p} \),

\[ z(n) = x(n) + p(n) \lambda_a(\tau(n)) \]

\[ x(n) = z(n) - p(n) \lambda_a(\tau(n)) > l - p z(\sigma(n)) > l - p(l + \varepsilon) = k(l + \varepsilon) \]

\[ > k z(n) \] where, \( k = \frac{l - p(l + \varepsilon)}{l + \varepsilon} > 0 \)

From (1),

\[ \Delta [a_1(n) \Delta [a_2(n) \Delta z(n)]] = - q(n) f(x(\sigma(n))) \]

\[ \leq - k q(n) z(\sigma(n)) \]

Summing from \( n \) to \( \infty \), we get
\[
\sum_{m=1}^{\infty} \Delta[a_1(m)] \Delta[a_2(m)] \Delta(z(m)) \leq -k \sum_{m=1}^{\infty} q(m) z(\sigma(m))
\]

\[
\frac{kl}{a_1(n)} \sum_{m=1}^{\infty} q(m) \leq \Delta[a_2(n)] \Delta(z(n))
\]

Summing again from \(n_1\) to \(\infty\), we get

\[
\sum_{i=n_1}^{\infty} \left[ \frac{kl}{a_1(i)} \sum_{m=1}^{\infty} q(m) \right] \leq \sum_{i=n_1}^{\infty} \Delta[a_2(i)] \Delta(z(i))
\]

\[
\frac{kl}{a_2(n_1)} \sum_{i=n_1}^{\infty} \frac{1}{a_1(i)} \sum_{m=1}^{\infty} q(m) \leq -\Delta(z(n_1))
\]

Again taking summing from \(n_0\) to \(\infty\)

\[
kl \left[ \sum_{j=n_0}^{\infty} \frac{1}{a_2(j)} \sum_{m=1}^{\infty} a_1(i) \sum_{m=1}^{\infty} q(m) \right] \leq -\sum_{j=n_0}^{\infty} \Delta(z(j))
\]

\[
\sum_{j=n_0}^{\infty} \frac{1}{a_2(j)} \sum_{i=n_0}^{\infty} a_1(i) \sum_{m=1}^{\infty} q(m) \leq \frac{z(n_0)}{kl}
\]

Which contradicts (8). Hence due to condition (8), we get the required result.

**Theorem 2.3.** Let \(0 \leq p(n) \leq p_i \leq 1\). Assume \((H_1) - (H_4)\) and \((B)\) holds. If there exists a positive function \(\delta\), such that for all sufficiently large \(n_2 > n_1 \geq n_0\), we have

\[
\lim_{n \to \infty} \text{Sup} \sum_{m=1}^{n-1} kq(m) \delta(m) p_\sigma(m) \sum_{i=1}^{\sigma(m)-1} \frac{1}{a_2(i)} \sum_{m=1}^{\infty} q(m) \leq \infty
\]

(9)

Where \(p_\sigma(n) = 1 - \frac{p(n)}{M^{1-\alpha}}\) and \(\delta(n) = \sum_{m=1}^{\infty} \frac{1}{a_1(m)}\) hold for all constants \(M > 0\). Then any solution \(x(n)\) of (1) is oscillatory.

Proof. Suppose \(x(n)\) is positive solution of (1). By condition \((B)\), there exists a possible case such that

\[
z(n) > 0, \Delta(z(n)) > 0, \Delta[a_2(n)] \Delta(z(n)) < 0, \Delta[a_1(n)] \Delta[a_2(n)] \Delta(z(n)) < 0.
\]

Hence \(a_1(n) \Delta[a_2(n)] \Delta(z(n))\) is non increasing.

\[a_1(m) \Delta[a_2(m)] \Delta(z(m)) \leq a_1(n) \Delta[a_2(n)] \Delta(z(n)), \quad m > n \geq n_1\]

Dividing by \(a_1(s)\) and summing \(n\) to \(l - 1\), we get

\[
\sum_{m=n}^{l-1} \Delta[a_2(m)] \Delta(z(m)) \leq a_1(n) \Delta[a_2(n)] \Delta(z(n)) \sum_{m=n}^{l-1} \frac{1}{a_1(m)}
\]

\[a_2(l) \Delta(z(l)) \leq a_2(n) \Delta(z(n)) + a_1(n) \Delta[a_2(n) \Delta(z(n))] \sum_{m=n}^{l-1} \frac{1}{a_1(m)}
\]

Letting \(l \to \infty\)

\[
\lim_{l \to \infty} a_2(l) \Delta(z(l)) \leq a_2(n) \Delta(z(n)) + a_1(n) \Delta[a_2(n) \Delta(z(n))] \sum_{m=n}^{\infty} \frac{1}{a_1(m)}
\]

V. \(0 \leq a_2(n) \Delta(z(n)) + a_1(n) \Delta[a_2(n) \Delta(z(n))] \sum_{m=n}^{\infty} \frac{1}{a_1(m)}\)
\[
\frac{a_1(n)\Delta[a_2(n)\Delta z(n)]}{a_2(n)\Delta z(n)} - \frac{1}{\sum_{m=n}^{\infty} a_i(m)} \leq 1
\]

Now define \(\phi(n)\) as follows:

\[
\phi(n) = \frac{a_1(n)\Delta[a_2(n)\Delta z(n)]}{a_2(n)\Delta z(n)}, n \geq n_1.
\]  

(10)

Notice that \(\phi(n) < 0\) for \(n \geq n_1\). Thus

\[
-\phi(n)\sum_{m=n}^{\infty} \frac{1}{a_i(m)} \leq 1
\]

\[
-\phi(n)\delta(n) \leq 1 \quad \text{where} \quad \delta(n) = \sum_{m=n}^{\infty} \frac{1}{a_i(m)}
\]

We get

\[
\Delta \phi(n) = \Delta \left[ \frac{a_1(n)\Delta[a_2(n)\Delta z(n)]}{a_2(n)\Delta z(n)} \right]
\]

\[
= \frac{a_2(n)\Delta z(n)}{[a_2(n)\Delta z(n)]^2} - \frac{a_1(n)\Delta^2[a_2(n)\Delta z(n)]}{a_2(n)\Delta z(n)} \quad \text{since} \quad \Delta[a_2(n)\Delta z(n)] < 0
\]

\[
\Delta \phi(n) = \frac{-kq(n)x(\sigma(n)}{a_2(n)\Delta z(n)} - \frac{a_1(n)\Delta^2[a_2(n)\Delta z(n)]}{a_2(n)\Delta z(n)}
\]

In view of case:

\[
z(n) \geq a_2(n)\sum_{m=n}^{\frac{n-1}{2}} \frac{1}{a_2(m)} \Delta z(n)
\]

(11)

\[
\frac{z(n)}{a_2(n)\Delta z(n)} \geq \frac{\sum_{m=n}^{\frac{n-1}{2}} \frac{1}{a_2(m)}}{a_2(n)\Delta z(n)}
\]

Hence, \(\Delta \left[ \frac{z(n)}{\sum_{m=n}^{\frac{n-1}{2}} a_2(m)} \right] \leq 0\), which implies that

\[
\frac{z(\sigma(n))}{z(n)} \geq \frac{\sum_{m=n}^{\frac{\sigma(n)-1}{2}} \frac{1}{a_2(m)}}{\sum_{m=n}^{\frac{n-1}{2}} \frac{1}{a_2(m)}}
\]

Now use (3) and (11) in \(\Delta \phi(n)\), we have

\[
\Delta \phi(n) \leq \frac{-kq(n)p_\ast(n)z(\sigma(n))}{a_2(n)\Delta z(n)} - \frac{\phi^2(n)}{a_1(n)}
\]

\[
\leq -kq(n)p_\ast(n) \sum_{m=n}^{\frac{\sigma(n)-1}{2}} \frac{1}{a_2(m)} - \frac{\phi^2(n)}{a_1(n)}
\]

Thus multiply by \(\delta(n)\) and summing from \(n_2 > n_1\) to \(n-1\)

\[
\sum_{m=n_1}^{n-1} \Delta \phi(m)\delta(m) \leq \sum_{m=n_1}^{n-1} -kq(m)p_\ast(m)\delta(m) \sum_{m=n_1}^{\frac{\sigma(m)-1}{2}} \frac{1}{a_2(m)} - \frac{\phi^2(m)}{a_1(m)}
\]

\[
- \sum_{m=n_1}^{n-1} \frac{\phi^2(m)}{a_1(m)} \delta(m) + \frac{\delta(m)}{a_1(m)}
\]

\[
\frac{1}{a_2(i)}
\]

\[
\frac{1}{a_2(i)}
\]
\[
\sum_{m=n_2}^{n_1} kq(m)p_*(m)\delta(s) \sum_{i=n_1}^{\sigma(m)-1} \frac{1}{a_z(i)} + \sum_{m=n_2}^{n_1} \left[ \frac{\phi^2(m)}{a_1(m)} \delta(m) + \frac{\phi(m)}{a_1(m)} \right] \\
\leq -\phi(n)\delta(n) + \phi(n_2)\delta(n_2)
\]

Since \(-\phi(n)\delta(n) \leq 1\)

\[
\sum_{m=n_2}^{n_1} kq(m)p_*(m)\delta(m) \sum_{i=n_1}^{\sigma(m)-1} \frac{1}{a_z(i)} + \sum_{m=n_2}^{n_1} \left[ \frac{\delta(m)}{a_1(m)} + \frac{1}{2a_1(m)\delta(m)} \right] \\
- \sum_{m=n_2}^{n_1} \frac{1}{4a_1(m)\delta(m)} \leq 1 + \phi(n_2)\delta(n_2)
\]

\[
\sum_{m=n_2}^{n_1} kq(m)p_*(m)\delta(m) \sum_{i=n_1}^{\sigma(m)-1} \frac{1}{a_z(i)} - \frac{1}{4a_1(m)\delta(m)} \leq 1 + \phi(n_2)\delta(n_2)
\]

Letting \(n \to \infty\)

\[
\lim \sup_{n \to \infty} \sum_{m=n_2}^{n_1} \left[ kq(m)p_*(m)\delta(m) \sum_{i=n_1}^{\sigma(m)-1} \frac{1}{a_z(i)} - \frac{1}{4a_1(m)\delta(m)} \right] \leq 1 + \phi(n_2)\delta(n_2)
\]

Which contradicts (9). This completes the proof.

VI. REFERENCES