BOUNDARY-LAYER FLOW OF A POWER - LAW FLUID

Sanjay Kumar Shrivastava
Lecturer,
Department of Mathematics,
D.A.V Govt. Inter College, Siwan,
Bihar(India)-841226

Abstract : In this paper, we have discussed the boundary layer flow of a power-Law fluid. We have considered the laminar boundary layer flow of a non-Newtonian fluid that is modelled by power-law constitutive relation along a semi-infinite horizontal flat plate. The plate is permeable and permits the same non-Newtonian fluid to be injected into the boundary layer. We have formulated the boundary-layer equations for a power-law fluid with an arbitrary external flow. A numerical scheme is developed and used to obtain solutions to the governing equations.

Key words : Boundary layer, Power-law fluid

Introduction: Self-similar solutions for the boundary-layer equations of non-Newtonian, and Newtonian, fluids provide valuable insights into the behaviour of the fluid flow. However, the conditions under which self-similar solutions are obtained may be either too restrictive, or only applicable over a limited range of relevant parameters. To obtain a fuller understanding of the fluid flow it is necessary to treat the boundary-layer equations as a system of parabolic partial differential equations. Furthermore, as the boundary-layer equations are nonlinear in character they must in general be solved numerically.

Various techniques for the numerical solution of the boundary-layer equations for Newtonian fluids have been developed over many years. An early method for the solution of the boundary-layer equations was described by Hartree and Womersley (1937), in which the x derivatives are replaced by finite-differences so as to approximate the partial differential equation by an ordinary differential equation. This ordinary differential equation is then solved using a technique based on finite-differences. This method has been employed by others such as Leigh (1955) and Smith and Clutter (1963) to produce quite satisfactory solutions to the boundary-layer equations for a number of flow regimes. Blottner (1975) has compared the use of other finite-difference methods such as the Crank-Nicolson scheme and the Keller box scheme for the numerical solution of the boundary-layer equations.

Any of these techniques should, in principle, be suitable for finding a numerical solution to the boundary-layer equations governing the base flow of a non-Newtonian fluid modelled by a power-law relationship. Andersson and Toften (l989) has described the use of the Keller-box scheme to obtain solutions to the Falkner-Skan-type equation for a power-law fluid. Though the numerical scheme was applied to a linearised form of the boundary-layer equations, rather than the complete nonlinear version of the boundary-layer equations, the results presented indicate that modern finite-difference techniques can be successfully applied to find solutions to non-Newtonian fluid flows.

In this paper, we look at the effect that mass transfer through the surface has on the boundary-layer flow of a power-law fluid. The corresponding problem for Newtonian fluids, though with zero pressure gradient in the external flow, has been considered by Catherall et al. (1965). They found that fluid injection reduces the skin friction which subsequently approaches zero and the boundary layer separates from the surface.

We formulate the boundary-layer equations for a power-law fluid with an arbitrary external flow. A numerical scheme, based on that used by Catherall et al. (1965), is developed and used to obtain solutions to the governing equations. This numerical scheme facilitates an investigation into the effect that the fluid index, n, has on the location of the separation point subject to a uniform rate of fluid injection through the flat plate.

Equations of Motion: In this section, we consider the laminar boundary-layer flow of a non-Newtonian fluid that is modelled by a power-law constitutive relation along a semi-infinite horizontal flat plate. The plate is permeable and permits the same non-Newtonian fluid to be injected into the boundary layer; Figure 1.1 shows the flow geometry along with the co-ordinate system that is used. All variables are in non-dimensional form and have been rescaled to a suitable boundary-layer thickness.

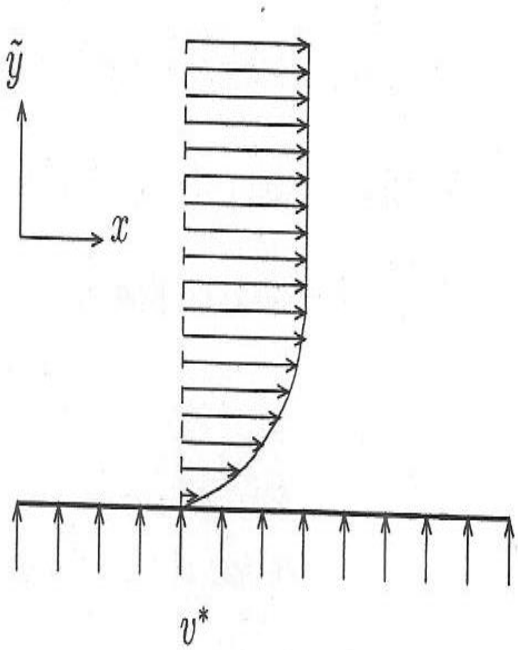

Figure 1.1: Representation of flow geometry and co-ordinate system.

For the flow depicted in Figure 1.1 the origin for the co-ordinate system used is assumed to be at the leading edge of the flat plate. Although the fluid injection rate is shown as being constant along the entire length of the semi-infinite plate, the injection rate may vary with distance along the plate. The boundary-layer approximation is known to have limited applicability at or near the leading edge, and the effect of this limitation on the proposed numerical scheme will be considered briefly in due course.

The equations governing the flow are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \qquad (1.1a)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{dp}{dx}+n\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial^2 u}{\partial y^2} \qquad (1.1b)$$

To close the system we impose the boundary conditions

$$u=0, \quad v=V(x) \quad \text{on} \quad y=0, \qquad (1.1c)$$

$$u\to U_e(x) \qquad \text{as} \quad y\to\infty, \qquad (1.1b)$$

These boundary conditions reflect the prescribed physical requirement that the fluid flow satisfies full viscous no-slip at the surface and normal flow through the surface. The stream-wise velocity within the boundary layer is required to match smoothly onto the free-stream,$U_e(x)$, at a large distance from the surface. The mass transfer, denoted by $V(x)$, may be either suction of fluid from or injection of fluid into the boundary layer. While the mass transfer may depend on the stream-wise location x, we will be concerned mainly with injection of fluid through the surface at a constant rate.

In the free-stream $u(x,y)\to U_e(x)$, where $U_e(x)$ is the external free-stream velocity. Using this far-field behaviour of the streamwise velocity, the x-momentum equation (1.1b) allows the pressure gradient to be expressed as

$$-\frac{dp}{dx}=U_e(x)\frac{dU_e}{dx}$$

The form of the free-stream velocity, $U_e(x)$, can be used to set up various flow types of interest, such as the flow under a zero, adverse or favourable pressure gradient.

For most types of boundary-layer flow the velocity component aligned with the main flow direction,u, shows very rapid changes across the boundary layer. Additionally, a boundary-layer flow involving an adverse pressure gradient shows rapid growth of the boundary-layer thickness with streamwise distance. Hence it is common practice to employ new independent variables that are less sensitive to these effects.

We introduce new independent variables (ξ,η) for the streamwise and wall-normal directions respectively. The new variables are given by

$$\xi = A_1 x^{\alpha_1} \text{ and } \quad \eta = y\frac{A_2 U_e^{\alpha 2}}{x^{\frac{1}{n+1}}}$$

where the scaling constants A_1, A_2 and the exponents α_1, α_2 are to be determined. The partial derivatives with respect to the original co-ordinates can be expressed as

$$\frac{\partial}{\partial x} = \alpha_1 A_1 x^{\alpha_{1-1}} \frac{\partial}{\partial \xi} + \eta\left(\alpha_2 \frac{\frac{dU_e}{dx}}{U_e} - \frac{1}{(n+1)x}\right)\frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial x} = \frac{A_2 U_e^{\alpha 2}}{x^{\frac{1}{n+1}}}\frac{\partial}{\partial \eta}$$

Any occurrences of the variable x are replaced accordingly by ξ to give

$$\frac{\partial}{\partial x} = \alpha_1 A_1 \left(\frac{\xi}{A_1}\right)^{\frac{\alpha_{1-1}}{\alpha_1}} \frac{\partial}{\partial \xi} + \eta\left(\alpha_1\alpha_2 A_1\left(\frac{\xi}{A_1}\right)^{\frac{\alpha_{1-1}}{\alpha_1}} \frac{\frac{dU_e}{d\xi}}{U_e} - \frac{1}{(n+1)}\left(\frac{\xi}{A_1}\right)^{-\frac{1}{\alpha_1}}\right)\frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial y} = A_2 U_e^{\alpha_2}\left(\frac{\xi}{A_1}\right)^{-\frac{1}{\alpha_1(n+1)}}\frac{\partial}{\partial \eta}$$

We note that U_e is now a function of ξ.

Under these transformations the x-momentum equation (1.1b) becomes

$$n\left|A_2 U_e^{\alpha_2}\left(\frac{\xi}{A_1}\right)^{-\frac{1}{\alpha_1(n+1)}}\frac{\partial u}{\partial \eta}\right|^{n-1}\left[A_2 U_e^{\alpha_2}\left(\frac{\xi}{A_1}\right)^{-\frac{1}{\alpha_1(n+1)}}\right]^2\frac{\partial^2 u}{\partial \eta^2} + \alpha_1 A_1\left(\frac{\xi}{A_1}\right)^{\frac{\alpha_{1-1}}{\alpha_1}} U_e \frac{dU_e}{d\xi} =$$

$$u\left[\alpha_1 A_1\left(\frac{\xi}{A_1}\right)^{\frac{\alpha_{1-1}}{\alpha_1}}\frac{\partial u}{\partial \xi} + \eta\left(\alpha_1\alpha_2 A_1\left(\frac{\xi}{A_1}\right)^{\frac{\alpha_{1-1}}{\alpha_1}}\frac{\frac{dU_e}{d\xi}}{U_e} - \frac{1}{(n+1)}\left(\frac{\xi}{A_1}\right)^{-\frac{1}{\alpha_1}}\right)\frac{\partial u}{\partial \eta}\right]$$

$$+vA_2 U_e^{\alpha_2}\left(\frac{\xi}{A_1}\right)^{-\frac{1}{\alpha_1(n+1)}}\frac{\partial u}{\partial \eta}$$

which after some simplification gives

$$nA_2^{n+1}U_e^{\alpha_2(n+1)}\frac{\partial^2 u}{\partial \eta^2}\left|\frac{\partial u}{\partial \eta}\right|^{n-1} + \alpha_1 \xi U_e \frac{dU_e}{d\xi} =$$

$$u\left[\alpha_1 \xi \frac{\partial u}{\partial \xi} + \eta\left(\alpha_1\alpha_2\xi\frac{\frac{dU_e}{d\xi}}{U_e} - \frac{1}{(n+1)}\right)\frac{\partial u}{\partial \eta}\right] + vA_2 U_e^{\alpha_2}\left(\frac{\xi}{A_1}\right)^{\frac{n}{\alpha_1(n+1)}}\frac{\partial u}{\partial \eta} \qquad (1.2)$$

Equation (1.2) allows us to determine the form of α_1, A_1 and A_2 in the following manner. Requiring that the exponent of the streamwise variable, ξ, be equal to unity wherever it occurs in the right-hand side of equation (1.2) results in $\alpha_1 = \frac{n}{n+1}$. Similarly, requiring that the coefficient of the highest derivative term in the left-hand side of equation(1.2) is equal to unity results in $A_2 = \left(\frac{1}{n}\right)^{\frac{1}{n+1}}$. Lastly, by setting $\frac{A_2}{A_1} = \frac{1}{n}$ we obtain $A_1 = n^{\frac{n}{n+1}}$.

The appropriate form of α_2 is still to be determined.

Making use of these values for α_1, A_1 and A_2, the x-momentum equation (1.2) becomes

$$U_e^{\alpha_{2(n+1)}}\frac{\partial^2 u}{\partial \eta^2}\left|\frac{\partial u}{\partial \eta}\right|^{n-1} + \frac{n}{(n+1)}\xi U_e\frac{dU_e}{d\xi} = \frac{u}{(n+1)}\left[n\xi\frac{\partial u}{\partial \xi} + \eta\left(\alpha_2 n\xi\frac{\frac{dU_e}{d\xi}}{U_e} - 1\right)\frac{\partial u}{\partial \eta}\right]$$

$$+\frac{1}{n}U_e^{\alpha_2}\xi v\frac{\partial u}{\partial \eta} \qquad (1.3)$$

and the continuity equation (1.1a) becomes

$$\frac{1}{n}U_e^{\alpha_2}\xi\frac{\partial v}{\partial \eta}=\frac{\eta}{(n+1)}\left(1-\alpha_2 n\xi\frac{\frac{dU_e}{d\xi}}{U_e}\right)\frac{\partial u}{\partial \eta}-\frac{\eta}{n+1}\xi\frac{\partial u}{\partial \xi}$$

Integrating the continuity equation with respect to η gives

$$\frac{1}{n}U_e^{\alpha_2}\xi v=\frac{\eta}{(n+1)}\int_0^\eta\left[\left(1-\alpha_2 n\xi\frac{\frac{dU_e}{d\xi}}{U_e}\right)\frac{\partial u}{\partial \eta}-n\xi\frac{\partial u}{\partial \xi}\right]d\eta+G(\xi).$$

Making use of the boundary condition on $\eta=0, i.e,\ \ v=V(\xi)$, gives

$$G(\xi)=\frac{U_e^{\alpha_2}\xi V}{n}$$

so that the integrated form of the continuity equation is given by

$$\frac{1}{n}U_e^{\alpha_2}\xi v=\frac{1}{n+1}\left(1-\alpha_2 n\xi\frac{\frac{dU_e}{d\xi}}{U_e}\right)\left[\eta u-\int_0^\eta u d\eta\right]-\frac{n}{n+1}\int_0^\eta\xi\frac{\partial u}{\partial \xi}d\eta+\frac{U_e^{\alpha_2}\xi V}{n} \tag{1.4}$$

By substituting (1.4) into (1.3) and rearranging terms we obtain the following integro differential equation

$$U_e^{\alpha_2(n+1)}\frac{\partial^2 u}{\partial \eta^2}\left|\frac{\partial u}{\partial \eta}\right|^{n-1}+\left\{\frac{1}{n+1}\int_0^\eta\left[\left(1-\alpha_2 n\xi\frac{\frac{dU_e}{d\xi}}{U_e}\right)u+n\xi\frac{\partial u}{\partial \xi}\right]d\eta-\frac{U_e^{\alpha_2}\xi V}{n}\right\}\frac{\partial u}{\partial \eta}$$

$$+\frac{n}{n+1}\xi U_e\frac{dU_e}{d\xi}-\frac{n}{n+1}\xi u\frac{\partial u}{\partial \xi}=0. \tag{1.5a}$$

The corresponding boundary conditions expressed in the new variables are

$$u=0,\quad v=V(\xi)\quad \text{on}\quad \eta=0, \tag{1.5b}$$

$$u\to U_e(\xi)\quad \text{as}\ \eta\to\infty \tag{1.5c}$$

We note that equation (1.5a) along with the boundary conditions (1.5b,c)provides a concise description of the fluid flow being considered.

By defining $u=U_e q$ where $q=\frac{\partial f}{\partial \eta}$ and f is the dimensionless stream function, equation (1.5a) becomes

$$U_e^{\alpha_2(n+1)+n}\frac{\partial^2 q}{\partial \eta^2}\left|\frac{\partial q}{\partial \eta}\right|^{n-1}$$

$$+U_e^2\left\{\frac{1}{n+1}\int_0^\eta\left[\left(1-\alpha_2 n\xi\frac{\frac{dU_e}{d\xi}}{U_e}\right)q+n\xi\frac{\frac{dU_e}{d\xi}}{U_e}+n\xi\frac{\partial q}{\partial \xi}\right]d\eta-\frac{U_e^{\alpha_2-1}\xi V}{n}\right\}\frac{\partial q}{\partial \eta}$$

$$+\frac{n}{n+1}\xi U_e\frac{dU_e}{d\xi}(1-q^2)-\frac{n}{n+1}\xi U_e^2 q\frac{\partial q}{\partial \xi}=0.$$

We can now determine the form of α_2 by requiring that $\alpha_2(n+1)+n=2$ to give $\alpha_2=\frac{2-n}{n+1}$

Additionally, we define the pressure gradient function

$$\beta(\xi)=\frac{n}{n+1}\xi\frac{\frac{dU_e}{d\xi}}{U_e}$$

to give

$$\frac{\partial^2 q}{\partial \eta^2}\left|\frac{\partial q}{\partial \eta}\right|^{n-1}+\left\{\frac{1}{n+1}\int_0^\eta\left[\left((2n-1)\beta+1\right)q+n\xi\frac{\partial q}{\partial \xi}\right]d\eta-\frac{\xi V}{n\mathrm{U}_e^{\frac{2n-1}{n+1}}}\right\}\frac{\partial q}{\partial \eta}$$

$$+\beta(1-q^2)-\frac{n}{n+1}\xi q\frac{\partial q}{\partial \xi}=0 \tag{1.6a}$$

The boundary conditions now take the form

$$q=0,\quad v=V(\xi)\quad \text{on}\quad \eta=0, \tag{1.6b}$$

$$q\to 0\ \ \text{an}\quad \eta\to\infty, \tag{1.6c}$$

The solution of equation (1.6a) subject to the boundary conditions (1.6 b,c) can be used to determine u at any streamwise location using $u = U_e(\xi)q$. Note that in the case of a power-of- x free-stream, i.e. $U_e \propto x^m$, the pressure gradient function $\beta(\xi)$ becomes a constant.

The final form for the transformation of independent variables is

$$\xi = (nx)^{\frac{n}{n+1}} \qquad (1.7a)$$

$$\eta = y\left(\frac{U_e^{2-n}}{nx}\right)^{\frac{1}{n+1}} \qquad (1.7b)$$

and we note that η has the form of a Falkner-Skan-like similarity variable that has the effect of compensating for the growth of the boundary layer. The solution grid in the transformed domain can use a uniform step-size for ξ and η so that errors introduced during discretisation will be smaller than if a non-uniform grid had been used.

Numerical Method: The existence of a closed-form solution to equation (1.6 a) is very unlikely due to the non-linear terms appearing in the equation. We can gain some understanding about the nature of the solutions to this problem through the use of numerical simulations. A number of numerical methods that are available for solving partial differential equations, such as finite elements and spectral methods, may also be applicable to this class of problem. The numerical method that was chosen for solving equation (1.6 a) is based on finite differences.

The boundary-layer equations are a system of parabolic partial differential equations and are commonly solved by numerical schemes that march along in the streamwise direction. The numerical scheme developed to solve equation (1.6 a) is very similar to that used by Catherall et al. (1965). This technique was described by Hartree and Womersley (1937) in the context of solving the classical boundary-layer equations and is conceptually related to the Method of Lines.

We begin by replacing the variables q, ξ and β by averages, and ξ-derivatives are replaced by finite-differences. Let

$$\hat{q} = \frac{q_1+q_2}{2}, \quad \hat{\xi} = \frac{\xi_1+\xi_2}{2}, \quad \frac{\partial q}{\partial \xi} = \frac{q_2-q_1}{\Delta\xi} \qquad \text{where } \Delta\xi = \xi_2 - \xi_1.$$

The subscripts denote two closely spaced locations along the streamwise direction. The integro-differential equation (1.6a) was derived from a set of partial differential equations with two independent variables, however, the use of these discretised variables changes the governing equation to an ordinary differential equation that is dependent on only one independent variable, η. The integro-differential equation (1.6a) now becomes

$$\frac{d^2\hat{q}}{d\eta^2}\left|\frac{d\hat{q}}{d\eta}\right|^{n-1} + \left\{\int_0^\eta \left[\frac{\left((2n-1)\beta_1+1\right)q_1 + \left((2n-1)\beta_2+1\right)q_2}{2\,(n+1)} + \frac{n\hat{\xi}(q_2-q_1)}{(n+1)\Delta\xi}\right]d\eta - \frac{\hat{\xi}V}{n\widehat{U_e^{\frac{2n-1}{n+1}}}}\right\}\frac{d\hat{q}}{d\eta}$$
$$+\frac{\beta_1(1-q_1^2)+\beta_2(1-q_2^2)}{2} - \frac{n}{n+1}\hat{\xi}\hat{q}\frac{(q_2-q_1)}{\Delta\xi}$$
$$= 0 \qquad (1.8)$$

Note that the term $\widehat{U_e^{\frac{2n-1}{n+1}}}$ indicates the average after the exponentiation of U_e has been carried out.

The solution to equation (1.6a) is the streamwise velocity for any value of the fluid index n in the range $0 < n < 2$. The far-field boundary condition, requiring that the streamwise velocity matches onto the free-stream velocity, manifests itself as $\frac{d\hat{q}}{d\eta}$ becoming vanishingly small. For shear-thickening fluids, with the fluid index in the range $1<n<2$, the numerical scheme that is suggested by the partly discretised equation (1.8) should work well. However, for shear-thinning fluids with the fluid index in the range $0<n<1$, this far-field matching requirement results in vanishingly small numbers being raised to a negative exponent along with the associated computer arithmetic problems that follow. Hence, we choose to develop and describe a numerical scheme that will solve equation (1.6a) specifically for shear-thinning fluids. A variant of this numerical scheme that is appropriate for dealing with the boundary-layer flow of a shear-thickening power-law fluid will be discussed later.

We proceed by multiplying equation (1.8) throughout by $\left(\frac{d\hat{q}}{d\eta}\right)$, then replacing all appearances of q_2 by $2\hat{q} - q_1$, or equivalently with $q_2 - q_1 = 2(\hat{q} - q_1)$, to give

$$\frac{d^2\hat{q}}{d\eta^2} + \left\{\int_0^{\eta}\left[\frac{(2n-1)(\beta_1-\beta_2)\,q_1+2((2n-1)\beta_2+1)\hat{q}}{2\,(n+1)} + \frac{n\theta(\hat{q}-q_1)}{n+1}\right]d\eta - K_{in}\right\}\left(\frac{d\hat{q}}{d\eta}\right)^{2-n}$$

$$+\left\{\hat{\beta}(1-q_1^2) - 2\beta_2\hat{q}(\hat{q}-q_1) - \frac{n\theta}{n+1}\hat{q}(\hat{q}-q_1)\right\}\left(\frac{d\hat{q}}{d\eta}\right)^{1-n} = 0,$$

where $K_{in} = \frac{\hat{\xi}V}{nU_e^{\frac{2n-1}{n+1}}}$, $\theta = \frac{\xi_1+\xi_2}{\Delta\xi}$, and $\hat{\beta} = \frac{\beta_1+\beta_2}{2}$. Further regrouping of the terms in the above equation gives

$$\frac{d^2\hat{q}}{d\eta^2} + \left\{\int_0^{\eta}\left[\left(\frac{(2n-1)(\beta_1-\beta_2)-2n\theta}{2\,(n+1)}\right)q_1 + \left(\frac{(2n-1)\beta_2+n\theta+1}{n+1}\right)\hat{q}\right]d\eta - K_{in}\right\}\left(\frac{d\hat{q}}{d\eta}\right)^{2-n}$$

$$+\left\{\hat{\beta}(1-q_1^2) - \left(2\beta_2+\frac{n\theta}{n+1}\right)\hat{q}(\hat{q}-q_1)\right\}\left(\frac{d\hat{q}}{d\eta}\right)^{1-n} = 0. \tag{1.9}$$

Next the derivatives with respect to η are discretised using second-order accurate finite-difference approximations:

$$\left.\frac{d^2\hat{q}}{d\eta^2}\right]_j = \frac{\hat{q}_{j+1}-2\hat{q}_j+\hat{q}_{j-1}}{h^2},$$

$$\left.\frac{d\hat{q}}{d\eta}\right]_j = \frac{\hat{q}_{j+1}-\hat{q}_{j-1}}{2h}.$$

The suffix j is the index to the mesh points and h is the step-size in the η direction. After making these substitutions and multiplying through by h^2, equation (1.9) takes the following discretised form

$$(\hat{q}_{j+1}-2\hat{q}_j+\hat{q}_{j-1}) + \left\{h^2\int_0^{\eta_j}\left[\left(\frac{(2n-1)(\beta_1-\beta_2)-2n\theta}{2\,(n+1)}\right)q_1 + \left(\frac{(2n-1)\beta_2+n\theta+1}{n+1}\right)\hat{q}\right]d\eta - K_{in}^*\right\}\left(\frac{\hat{q}_{j+1}-\hat{q}_{j-1}}{2h}\right)^{2-n}$$

$$+h^2\left\{\hat{\beta}(1-q_1^2)_j - \left(2\beta_2+\frac{n\theta}{n+1}\right)\hat{q}_j(\hat{q}-q_1)_j\right\}\left(\frac{\hat{q}_{j+1}-\hat{q}_{j-1}}{2h}\right)^{1-n} = 0,$$

Where $K_{in}^* = h^2K_{in}$. The definite integral in the above expression is evaluated by the trapezoidal rule, which is denoted by $\sum''$ with first and last terms halved. Hence, the above equation takes the following form

$$(\hat{q}_{j+1}-2\hat{q}_j+\hat{q}_{j-1}) + \left\{h^3\sum_{r=0}^{j}\left[\left(\frac{(2n-1)(\beta_1-\beta_2)-2n\theta}{2\,(n+1)}\right)q_{1_r} + \left(\frac{(2n-1)\beta_2+n\theta+1}{n+1}\right)\hat{q}_r\right] - K_{in}^*\right\}\left(\frac{\hat{q}_{j+1}-\hat{q}_{j-1}}{2h}\right)^{2-n} + h^2\left\{\hat{\beta}(1-q_1^2)_j - \left(2\beta_2+\frac{n\theta}{n+1}\right)\hat{q}_j(\hat{q}-q_1)_j\right\}\left(\frac{\hat{q}_{j+1}-\hat{q}_{j-1}}{2h}\right)^{1-n} = 0. \tag{1.10}$$

Equation (1.10) needs to be solved at the uniformly spaced mesh points $j = 1,\ldots,J$, with $j = 0$ corresponding to the flat plate and $j = J+1$ to the free-stream. The solution of this equation at the j^{th} mesh point involves the unknowns $\hat{q}_1,\ldots.\hat{q}_{j+1}$. At the J^{th} mesh point use is made of the far-field boundary condition, viz. the given stream wise velocity, by setting the $J+1^{th}$ mesh point to the given value. The velocity in the far-field may be normalised, hence allowing the velocity at the final mesh point to be given by $\hat{q}_{J+1} = 1$. This system of non-linear algebraic equations requires a solution to be found at each streamwise location of the marching scheme being used to solve the governing parabolic partial differential equation.

Let the vector $\hat{q} = (\hat{q}_1,\hat{q}_2,\ldots,\hat{q}_J)^{\mathrm{T}}$ and define a function F by

$$\mathrm{F}(\hat{q}) = \left(f_1(\hat{q}),\ldots\ldots,f_j(\hat{q}),\ldots f_J(\hat{q})\right),$$

where $f_j(\hat{q})$ is given by the left hand side of equation (1.10). We note that at mesh point j the unknowns $\hat{q}_{j+2},\ldots.\hat{q}_J$ are understood to have coefficients identically equal to zero in the function $f_j(\hat{q})$. Using vector notation the system of non-linear equations assumes the form

$$F(\hat{q}) = 0. \tag{1.11}$$

This system of non-linear algebraic equations will need to be solved by an iterative process. An iteration scheme similar to that used by Terrill (1960) that results in a set of simultaneous linear equations expressed in matrix form as $\mathrm{A}\hat{q} = \mathrm{b}$ could be implemented. However, equation (1.11) lends itself to solution more directly using Newton's iterative method for non-linear systems. The iteration procedure is based upon the ansatz

$$\hat{\mathrm{q}}^{(\mathrm{k})} = \hat{\mathrm{q}}^{(\mathrm{k}-1)} - J\left(\hat{\mathrm{q}}^{(\mathrm{k}-1)}\right)^{-1}\mathrm{F}\left(\hat{\mathrm{q}}^{(\mathrm{k}-1)}\right),$$

where $J(\hat{q})$ is the Jacobian matrix and k is the iteration index. Expanding the trapezoidal sum and applying the no-slip condition at $j = 0$ gives the following form for the function $f_i(\hat{q})$ at mesh point j

$$\begin{aligned} f_j(\hat{q}^{(k)}) = {} & (\hat{q}_{j+1}^{(k)} - 2\hat{q}_j^{(k)} + \hat{q}_{j-1}^{(k)}) \\ & + \left\{ h^3 \left[\left(\frac{(2n-1)(\beta_1 - \beta_2) - 2n\theta}{2\,(n+1)} \right) \left(q_{1,1} + \cdots + q_{1,j-1} + \frac{1}{2} q_{1,j} \right) \right. \right. \\ & + \left. \left. \left(\frac{(2n-1)\beta_2 + n\theta + 1}{n+1} \right) \left(\hat{q}_1^{(k)} + \cdots + \hat{q}_{j-1}^{(k)} + \frac{1}{2} \hat{q}_j^{(k)} \right) \right] - K_{in}^* \right\} \left(\frac{\hat{q}_{j+1}^{(k)} - \hat{q}_{j-1}^{(k)}}{2h} \right)^{2-n} \\ & + h^2 \left\{ \hat{\beta}(1 - q_1^2)_j - \left(2\beta_2 + \frac{n\theta}{n+1} \right) \hat{q}_j^{(k)} \left(\hat{q}^{(k)} - q_1 \right)_j \right\} \left(\frac{\hat{q}_{j+1}^{(k-1)} - \hat{q}_{j-1}^{(k-1)}}{2h} \right)^{1-n} \qquad (1.12) \end{aligned}$$

In equation (1.12) the superscript k is used to indicate both the iteration step and which variables are being solved for at the current step of the iterative procedure. We also note that the finite-difference approximation for $\left(\frac{d\hat{q}}{d\eta}\right)^{1-n}$ is calculated using values of $\hat{q}$ from the previous iteration step. The Jacobian matrix that is required to solve this system of equations using a Newton iteration procedure is sparse with all elements above the super-diagonal being zero.

Falkner-Skan flows are defined by a free-stream potential flow with a streamwise velocity given by $U_e(\xi) \propto \xi^m$. The pressure gradient parameter β for Falkner-Skan flows is constant, i.e. $\beta = m$. Hence, setting $\beta_1 = \beta_2 = \beta$ allows equation (1.12) to take the following simplified form

$$\begin{aligned} f_j(\hat{q}^{(k)}) = {} & (\hat{q}_{j+1}^{(k)} - 2\hat{q}_j^{(k)} + \hat{q}_{j-1}^{(k)}) \\ & + \left\{ h^3 \left[\left(\frac{(2n-1)\beta + n\theta + 1}{n+1} \right) \left(\hat{q}_1^{(k)} + \cdots + \hat{q}_{j-1}^{(k)} + \frac{1}{2} \hat{q}_j^{(k)} \right) \right. \right. \\ & - \left. \left. \frac{n\theta}{n+1} \left(q_{1,1} + \cdots + q_{1,j-1} + \frac{1}{2} q_{1,j} \right) \right] - K_{in}^* \right\} \left(\frac{\hat{q}_{j+1}^{(k)} - \hat{q}_{j-1}^{(k)}}{2h} \right)^{2-n} \\ & + h^2 \left\{ \beta(1 - q_1^2)_j - \left(2\beta_2 + \frac{n\theta}{n+1} \right) \hat{q}_j^{(k)} \left(\hat{q}^{(k)} - q_1 \right)_j \right\} \left(\frac{\hat{q}_{j+1}^{(k-1)} - \hat{q}_{j-1}^{(k-1)}}{2h} \right)^{1-n} \qquad (1.13) \end{aligned}$$

Equation (1.13) generates a system of non-linear algebraic equations that are specific for a Falkner-Skan flow. The Jacobian matrix needed to solve this non-linear system is slightly simpler than was needed for solving equation (1.12). The solution of this non-linear system via a Newton iteration procedure yields the streamwise velocity profile at a given down-stream location. The numerical marching scheme is simply the application of this sequence of calculations along the length of the flat plate.

The set of equations (1.1) governing the boundary-layer flow of a power-law fluid are parabolic partial differential equations. The accompanying boundary conditions have been accounted for in the design of the numerical marching scheme described in this section. However, the numerical marching scheme will require an essential additional condition to work as expected. That extra condition is a prescribed velocity profile at an initial station ξ_0.

Initial Velocity Profile: The marching based numerical scheme described in Section first needs an initial condition to start the iteration process at the first streamwise location. Such an initial condition is given by a streamwise velocity profile appropriate for the class of flow being considered.

The velocity profile at the leading edge $\xi = 0$ is given by the solution of the Falkner-Skan-like equation for power-law fluids. The appropriate form of this equation is obtained by substituting $\xi = 0$ into equation (1.6a) to give

$$\frac{\partial^2 q}{\partial \eta^2} \left| \frac{\partial q}{\partial \eta} \right|^{n-1} + \left\{ \frac{1}{n+1} \int_0^\eta \left[\left((2n-1)\beta + 1 \right) q \right] d\eta \right\} \frac{\partial q}{\partial \eta} + \beta(1 - q^2) = 0. \qquad (1.14)$$

Equation (1.14) can be expressed in a more familiar form by letting $q = \frac{df}{d\eta}$, where f is the normalised stream function and ordinary derivatives are used to indicate that q and f are independent of ξ, viz.

$$\frac{\partial^3 f}{\partial \eta^3} \left| \frac{\partial^2 f}{\partial \eta^2} \right|^{n-1} + \left\{ \frac{1}{n+1} \int_0^\eta \left[\left((2n-1)\beta + 1 \right) \frac{df}{d\eta} \right] d\eta \right\} \frac{\partial^2 f}{\partial \eta^2} + \beta \left(1 - \left(\frac{df}{d\eta} \right)^2 \right) = 0.$$

We recall that β is constant when the free-stream potential flow is of Falkner-Skan-type and is referred to as the pressure gradient parameter.

Re-arranging and simplifying some of the terms finally gives

$$f''' + \frac{(2n-1)\beta + 1}{n+1}(f'')^{2-n} f + \beta(1-(f')^2)(f'')^{1-n} = 0. \qquad (1.15a)$$

The boundary conditions for this third-order non-linear ODE are

$$f = 0, \quad f' = 0, \ \text{ on } \eta = 0, \qquad (1.15b)$$

$$f' \to 1 \ \text{ an } \ \eta \to \infty. \qquad (1.15c)$$

Equation (1.15a) is a generalised version of the classical Falkner-Skan equation for power-law fluids and it can readily be seen that setting $n = 1$ results in equation (1.15a) taking the form of the standard Falkner-Skan equation.

Equation (1.15a) and the boundary conditions (1.15b,c) form a two-point boundary-value problem that can be solved numerically using a shooting method based around a fourth order Runge-Kutta quadrature scheme. The solution obtained is then used as the initial guess for the iteration procedure that begins at the first streamwise location and marches along the length of the plate until a suitable termination condition is satisfied.

Conclusion: In this paper, the set of partial differential equations governing the boundary-layer flow of a generalised Newtonian fluid were introduced. Then a co-ordinate transformation was applied to yield a corresponding integro-differential equation. A general numerical marching scheme was developed to solve this integro-differential equation which was subsequently adapted to deal with shear-thinning power-law fluids.

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