The study of some partial differential equation with fuzzy trapezoidal, slipped and Sine-fuzzy initial conditions using fuzzy Laplace transforms.

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Abstract
In this paper, different partial differential equations are formulated under some fuzzy initial conditions and the fuzzy Laplace transform is applied to get fuzzy solutions. Three different initial conditions with trapezoidal, slipped and Sine-fuzzy, numbers are imposed and systems of fuzzy differential equations are prepared. The fuzzy Laplace transform is established for all defined trapezoidal, Sine-fuzzy numbers. The lower central and upper solutions according the applied fuzzy initial conditions are presented. The solution in terms of membership grade has also illustrated.

1. Introduction
The simulation of physical and engineering problems such as dynamic processes in solid and fluid dynamics, viscoelastics and anatomy leads to the Mathematical modelling through differential equations. The parameters and main conditions of a model are generally considered to be described accurately. In fact owing to calculation, analysis or investigation errors, the data on factors and conditions can be ambiguous or inelegant for some realistic modelling. In order to counter uncertainty or imprecision, a floated system can be utilized by transforming general differential equations into fuzzy differential equations. Several analyses were pursued for the last few decades on the various types of fuzzy derivatives and differential equations. The derivatives of fuzzy functions were examined in [1] then a theorem that provides a solution to a fuzzy differential equation was established [2]. The general characteristics of the Sumudu transformation and the Sumudu transformation of the first order equation were discussed in [3]. Abbasbandy and Viranloo [4] analysed the Taylor scheme numerical computations for solving “fuzzy ordinary differential equations. New set of results for fuzzy differential equations are presented in [5] using stacking theorem .Regan et al. [6] did the nonlinear analysis of fuzzy differential equitation with initial and boundary conditions.

The generalized Hukuahra difference and fuzzy interval partition was numerically explored to approximate the fuzzy initial value problem [7]. While a few more research to solve the fuzzy system of differential equation have been carried out in [9], [10] and [12-14]. Basic characteristics of Sumudhu and Laplace transform were described in [11] and [15] respectively. Allahviranloo and Salahshour presented Eulers technique and new
approaches like fuzzy Laplace to solve the fuzzy hybrid, integro and fractional differential equations in [16-19]. F.V.I. method was employed to obtain the approximate results of nonlinear fuzzy differential equation [20], [25]. Mondal and Roy [21] considered the coefficient and initial condition of differential equation as triangular fuzzy number and fuzzy Laplace method is implemented to solve the first order equation and then fuzzy nth order derivatives were evaluated in [22]. Several Intuitionist differential equations solutions were carried out in [23]. Nayak and Chakraverty [24] numerically approximated the fuzzy stochastic differential equation. Some solutions of resolving a complex and linear fuzzy differential equation were derived by You and Zhang [26] and then stability analysis of credibility for FDEs was also done in [28]. Gholami et al.[27] provided the approach to solve the 2-point boundary-value problem by fuzzy kernel method.

2. Preliminaries

2.1: Fuzzy Set and its Components

Definition 2.1.1: Let $X^*$ be the universal space and a fuzzy set $\tilde{q}$, is a set in which each element of the set $X^*$ is associated with a membership grade defined as:

$$\tilde{q} = \{(x^*, \tilde{q}(x^*)): x^* \epsilon X^*, \tilde{q}(x^*) \rightarrow 0\}$$

Definition 2.1.2: Let a fuzzy set $\tilde{q}$ defined on the universal space $X^*$ with $\tilde{q}(x^*) \rightarrow [0, 1]$ for $\alpha^* \in \tilde{q}(x^*)$, then $\alpha^*$ cut of $\tilde{q}$ is defined as $\tilde{q}^{\alpha^*} = \{x^* | \tilde{q}(x^*) \geq \alpha^*, x^* \epsilon X^*\}$

Definition 2.1.3: Let a fuzzy set $\tilde{q}$ defined on the universal space $X^*$ with $\tilde{q}(x^*) \rightarrow [0, 1]$ for $\alpha^* \in \tilde{q}(x^*)$, then strong $\alpha^*$ cut of $\tilde{q}$ is defined as $\tilde{q}^{\alpha^*} = \{x^* | \tilde{q}(x^*) > \alpha^*, x^* \epsilon X^*\}$

Definition 2.1.4: Let a fuzzy set $\tilde{q}$ defined on the universal space $X^*$ with $\tilde{q}(x^*) \rightarrow [0, 1]$ for $\alpha^* \in \tilde{q}(x^*)$, then the support of $\tilde{q}$ is defined as

$$\tilde{A}^* = \{x^* | \tilde{q}(x^*) > 0, x^* \epsilon X^*\}$$

Definition 2.1.5: Let a fuzzy set $\tilde{q}$ defined on the universal space $X^*$ with $\tilde{q}(x^*) \rightarrow [0, 1]$ for $\alpha^* \in \tilde{q}(x^*)$, then the height of $\tilde{q}$ is defined as $H(\tilde{q}) = \max(\tilde{q}(x_i^*))$ and $\tilde{q}$ is said to be normal if $H(\tilde{q}) = 1$.

Definition 2.1.6: Let a fuzzy set $\tilde{q}$ defined on the universal space $X^*$ with $\tilde{q}(x^*) \rightarrow [0, 1]$ then it is called convex, if for any two $x_i^*, x_j^* \epsilon X^*$

$$\tilde{q} \{ \lambda x_i^* + (1- \lambda)x_j^* \} \geq \min\{\tilde{q}(x_i^*), \tilde{q}(x_j^*)\}, \text{where } 0 \leq \lambda \leq 1.$$

Definition 2.1.7: Let a set of fuzzy number $\varphi^{**} = \{\varphi^*: \Re \rightarrow [0, 1]\}$ if

1) $\varphi^*$ is normal
2) $\varphi^*$ is convex $\forall x_i^*, x_j^* \epsilon X^*$
3) upper semi-continuous on $\Re \epsilon X^*$
4) $\tilde{F}^S$ and its closure are compact
Definition 2.1.8: The r-level of the fuzzy number \( \varphi^* \) is defined as

\[ \varphi^{*r} = \{ x^* | \mu_F(x^*) \geq r, 0 \leq x^* \leq X^* \} \]

Definition 2.1.9: Let \( \varphi_1^* \) and \( \varphi_2^* \) be \( \varphi^* \), \( \exists \varphi_3 \in \varphi^* \) such that \( \varphi_1^* = \varphi_2^* + \varphi_3^* \) then \( \varphi_3^* \) is known as of Hukuhara, \( H \) difference of \( \varphi_1^* \) and \( \varphi_2^* \) defined by \( \varphi_1^* \ominus \varphi_2^* \) its r-level if

\[ (\varphi_1^* \ominus \varphi_2^*)^r = [\varphi_1^* - \varphi_2^*]^r \]

3. Methodology

Definition 3.1: Fuzzy differentiation: A mapping \( \psi^*(\alpha_1, \alpha_2) \rightarrow \varphi^* \) is strongly generalized differentiable at \( \vartheta^*_0 \in (\alpha_1, \alpha_2) \) if \( \exists \) an element \( \chi^*(\vartheta^*_0) \in \varphi^* \) such that

(i) For a small \( p > 0 \) \( \exists \chi^*(\vartheta^*_0 + p) \ominus \chi^*(\vartheta^*_0), \chi^*(\vartheta^*_0) \ominus \chi^*(\vartheta^*_0 - p) \)

\[ \text{and } \lim_{p \to 0^+} \frac{\chi^*(\vartheta^*_0 + p) \ominus \chi^*(\vartheta^*_0)}{p} = \lim_{p \to 0^+} \frac{\chi^*(\vartheta^*_0) \ominus \chi^*(\vartheta^*_0 - p)}{p} = \chi^*(\vartheta^*_0) \]

(ii) For a small \( p > 0 \) \( \exists \chi^*(\vartheta^*_0 - p) \ominus \chi^*(\vartheta^*_0), \chi^*(\vartheta^*_0) \ominus \chi^*(\vartheta^*_0 + p) \)

\[ \text{and } \lim_{p \to 0^+} \frac{\chi^*(\vartheta^*_0) \ominus \chi^*(\vartheta^*_0 + p)}{-p} = \lim_{p \to 0^+} \frac{\chi^*(\vartheta^*_0) \ominus \chi^*(\vartheta^*_0 + p)}{-p} = \chi^*(\vartheta^*_0) \]

(iii) For a small \( p > 0 \) \( \exists \chi^*(\vartheta^*_0 + p) \ominus \chi^*(\vartheta^*_0), \chi^*(\vartheta^*_0) \ominus \chi^*(\vartheta^*_0 - p) \)

\[ \text{and } \lim_{p \to 0^+} \frac{\chi^*(\vartheta^*_0 + p) \ominus \chi^*(\vartheta^*_0)}{p} = \lim_{p \to 0^+} \frac{\chi^*(\vartheta^*_0 - p) \ominus \chi^*(\vartheta^*_0)}{-p} = \chi^*(\vartheta^*_0) \]

(iv) For a small \( p > 0 \) \( \exists \chi^*(\vartheta^*_0) \ominus \chi^*(\vartheta^*_0 + p), \chi^*(\vartheta^*_0) \ominus \chi^*(\vartheta^*_0 - p) \)

\[ \text{and } \lim_{p \to 0^+} \frac{\chi^*(\vartheta^*_0) \ominus \chi^*(\vartheta^*_0 + p)}{-p} = \lim_{p \to 0^+} \frac{\chi^*(\vartheta^*_0) \ominus \chi^*(\vartheta^*_0 - p)}{p} = \chi^*(\vartheta^*_0) \]

Definition 3.2: Fuzzy Laplace Transform: Let \( \chi^*(\vartheta) \) is a continuous fuzzy membership grade function and \( e^{-\beta \vartheta} \chi^*(\vartheta) \) is improper fuzzy Riemann integrable on \( [0, \infty) \) then the fuzzy Laplace transform of \( \chi^*(\vartheta) \) is defined as \( \tilde{\mathcal{L}} (\chi^*(\vartheta)) = \int_0^\infty e^{-\beta \vartheta} \chi^*(\vartheta) d\vartheta \beta > 0 \)

Theorem 3.3: Let \( \chi''^*(\vartheta) \) is an integrable fuzzy membership grade function on \( [0, \infty) \) then

(a) if \( \chi^*(\vartheta) \) is (i)-differentiable: \( \tilde{\mathcal{L}} [\chi''^*(\vartheta)] = \beta \chi^*(\vartheta) \ominus \chi^*(0) \)

(b) if \( \chi^*(\vartheta) \) is (ii)-differentiable: \( \tilde{\mathcal{L}} [\chi''^*(\vartheta)] = (-\chi^*(0)) \ominus (-\beta \ominus \chi^*(\vartheta)) \)

3.4: Triangular fuzzy number: \( \tau_{b,d,c} \)

Let \( X \) be the universal space of real number and \( a, b \) and \( c \in X \) such that \( a < b < c \) then a triangular fuzzy number \( \tau_{b,d,c} \) with membership grade \( \tau_{b,d,c}(x) \) is defined as
\[ \tau_{b,d,c}(x) = \begin{cases} 
0 & x < a \\
\frac{x-a}{b-a} & a < x < b \\
\frac{c-x}{c-b} & b < x < c \\
0 & x > c 
\end{cases} \]

where \( d_c = b - a, \ i_c = c - b \).

And graph 1 represents the membership grade of \( \tau_{b,d,i} \)

And the fuzzy Laplace transformation of \( \tau_{b,d,i} \) is defined as

\[ \tilde{\mathcal{L}}(\tau_{b,d,c,l}) = \left( \tilde{\mathcal{L}}(\tau_{b,d,c,l}), \tilde{\mathcal{L}}(\tau_{b,d,c,l}) \right) \]

\[ \tilde{\mathcal{L}}(\tau_{b,d,c,l}) = \begin{cases} 
0 & x < a \\
\tilde{\mathcal{L}}(\tau_{b,d,c,l}) & a < x < b \\
\tilde{\mathcal{L}}(\tau_{b,d,c,l}) & b < x < c \\
0 & x > c 
\end{cases} \]

Where

\[ \tilde{\mathcal{L}}(\tau_{b,d,c,l}) = \frac{e^{-a\beta} - ((b-a)\beta+1)e^{-b\beta}}{\beta^2(b-a)} \]

\[ \tilde{\mathcal{L}}(\tau_{b,d,c,l}) = \frac{e^{-c\beta} - ((c-b)\beta-1)e^{-b\beta}}{\beta^2(c-b)} \]

3.5: Triangular fuzzy number: \( \tau_{b,i} \)

Let \( X \) be the universal space of real number and \( a, b \ and \ c \in X \) such that \( a < b < c \) then a triangular fuzzy number \( \tau_{b,i} \) with membership grade \( \tau_{b,i}(x) \) is defined as

\[ \tau_{b,i}(x) = \begin{cases} 
\frac{b-x}{b-a} & x < a \\
1 & a \leq x \leq b \\
0 & x > b 
\end{cases} \]

where \( i = b - a \).

And graph 2 represents the membership grade of \( \tau_{b,i} \)
Graph 2: Triangular fuzzy number $\tau_{b,i}$ with its membership grade

And the fuzzy Laplace transformation of $\tau_{b,i}$ is defined as $\mathcal{L} \left[ \tau_{b,i} \right] = \left[ \mathcal{L} (\tau_{b,i})^*, \mathcal{L} (\tau_{b,i}) \right]$

$$\mathcal{L} (\tau_{b,i}) = \begin{cases} \mathcal{L} (\tau_{b,i})^* & a < x \\ \mathcal{L} (\tau_{b,i}) & a < x < b \\ 0 & x > b \end{cases}$$

Where

$$\mathcal{L} (\tau_{b,i})^* = \int_0^a e^{-\beta x} \cdot 1 \, dx = \frac{1-e^{-\alpha \beta}}{\beta}, \quad \beta > 0$$

Similarly

$$\mathcal{L} (\tau_{b,i}) = e^{-b\beta} - (b-a)\beta - 1 \cdot e^{-a\beta}$$

3.6: Trapezoidal fuzzy number: $\tau_{b \sim c, d_c, i_c}$

Let $X$ be the universal space of real number and $a, b, c, d \in X$ such that $a < b < c < d$.

Then a trapezoidal fuzzy number $\tau_{b \sim c, d, i}$ with membership grade $\tau_{b \sim c, d_c, i_c}(x)$ is defined as

$$\tau_{b \sim c, d_c, i_c}(x) = \begin{cases} 0 & x < a \text{ or } x > d \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & b \leq x < c \\ \frac{d-x}{d-c} & c \leq x \leq d \end{cases}$$

where $d_c = b - a, i_c = d - c$

And graph 3 represents the membership grade of $(\tau_{b \sim c, d_c, i_c})$.

Graph 3: Trapezoidal fuzzy number $\tau_{b \sim c, d_c, i_c}$ with its membership grade
And the fuzzy Laplace transformation of $\tau_{b\sim c,d,i}$ is defined as

$$\mathcal{L}(\tau_{b\sim c,d,i}) = \begin{cases} 0 & x < a \& x > d \\ \mathcal{L}(\tau_{b\sim c,d,i}) & a \leq x < b \\ \mathcal{L}(\tau_{b\sim c,d,i})^* & b \leq x < c \\ \mathcal{L}(\tau_{b\sim c,d,i}) & c < x \leq d \end{cases}$$

Where

$$\mathcal{L}(\tau_{b\sim c,d,i}) = \frac{e^{-a\beta} - ((b - a)\beta + 1)e^{-b\beta}}{\beta^2(b - a)}$$

$$\mathcal{L}(\tau_{b\sim c,d,i})^* = \frac{e^{-b\beta} - e^{-a\beta}}{\beta}$$

$$\mathcal{L}(\tau_{b\sim c,d,i}) = \frac{e^{-d\beta} - ((d - c)\beta - 1)e^{-c\beta}}{\beta^2(d - c)}$$

### 3. Problem formulation and solution

Consider the following partial differential equation with Trapezoidal Fuzzy initial condition

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = x$$

$$U(0, x) = \left(U(0, x), U^*(0, x), \bar{U}(0, x)\right) = (3\alpha + 1, k, 8 - 3\alpha) \quad k \in [4, 5]$$

And $U(0, t) = 0$. Where $\alpha$ is the alpha cut of fuzzy set

Apply fuzzy Laplace Transformation both side

$$\frac{d\bar{L}\left(U(x,s)\right)}{dx} + s \bar{L}\left(U(x,s)\right) - (3\alpha + 1) = \frac{x}{s}$$

$$\frac{d\bar{L}\left(U(x,s)\right)}{dx} + s \bar{L}\left(U(x,s)\right) - (k) = \frac{x}{s}$$

$$\frac{d\bar{L}\left(U(x,s)\right)}{dx} + s \bar{L}\left(U(x,s)\right) - (8 - 3\alpha) = \frac{x}{s}$$

Let $\bar{L}\left(U(x,s)\right) = V$

$$\frac{dv}{dx} + sV - (3\alpha + 1) = \frac{x}{s}$$

$$\frac{dv}{dx} + sV - (k) = \frac{x}{s}$$

$$\frac{dv}{dx} + sV - (8 - 3\alpha) = \frac{x}{s}$$

Integrating Factor: $e^{\int s dx} = e^{sx}$

Solution is given by:

$$Ve^{sx} = \int_{\frac{x}{s}}^{x} (3\alpha + 1) e^{sx} dx + C$$

$$Ve^{sx} = \int_{\frac{x}{s}}^{x} (k) e^{sx} dx + C$$
\[ Ve^{sx} = \int \left[ \frac{x}{s} + (8 - 3\alpha) \right] e^{sx} \, dx + C \] (3.11)

Then
\[ V = \frac{x}{s^2} - \frac{1}{s^3} + \frac{(3\alpha + 1)}{s} + Ce^{-sx} \] (3.12)
\[ V = \frac{x}{s^2} - \frac{1}{s^3} + \frac{(k)}{s} + Ce^{-sx} \] (3.13)
\[ V = \frac{x}{s^2} - \frac{1}{s^3} + \frac{(8 - 3\alpha)}{s} + Ce^{-sx} \] (3.14)

Using initial conditions
\[ C = \frac{1}{s^3} - \frac{U(x,0)}{s} \]
\[ V = \frac{x}{s^2} - \frac{1}{s^3} + \frac{(3\alpha + 1)}{s} + \left[ \frac{1}{s^3} - \frac{(3\alpha + 1)}{s} \right] e^{-sx} \] (3.15)
\[ V = \frac{x}{s^2} - \frac{1}{s^3} + \frac{(k)}{s} + \left[ \frac{1}{s^3} - \frac{(k)}{s} \right] e^{-sx} \] (3.16)
\[ V = \frac{x}{s^2} - \frac{1}{s^3} + \frac{(8 - 3\alpha)}{s} + \left[ \frac{1}{s^3} - \frac{(8 - 3\alpha)}{s} \right] e^{-sx} \] (3.17)

Inverse Laplace Transformation

\[ U(x,t) = xt - \frac{t^2}{2} + (3\alpha + 1) + \frac{H(t-x)(t-x)^2}{2} - (3\alpha + 1)H(t-x)(t-x) \] (3.18)
\[ U(x,t) = xt - \frac{t^2}{2} + (k) + \frac{H(t-x)(t-x)^2}{2} - (k)H(t-x)(t-x) \] (3.19)
\[ U(x,t) = xt - \frac{t^2}{2} + (8 - 3\alpha) + \frac{H(t-x)(t-x)^2}{2} - (8 - 3\alpha)H(t-x)(t-x) \] (3.20)

Where \( H \) is Unit-Step Function defined as \[ H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \] (3.21)

The equation (3.18-3.21) provides the solutions of fuzzy initial value problem equation (3.1-3.2).

Now we consider another partial differential equation with following condition
\[ \frac{\partial U(x,t)}{\partial t} = \frac{\partial^2 U}{\partial x^2} \] (3.21)
\[ U(0,x) = \left( U(0,x), U^*(0,x), U(0,x) \right) = (( 3\alpha + 1)sin\pi x, ks\sin\pi x, (8 - 3\alpha)sin\pi x) k \in [4, 5] \] (3.22)

Where \( \alpha \) is the alpha cut of fuzzy set

\[ L \left( \frac{\partial U(x,t)}{\partial t} \right) = L \left( \frac{\partial^2 U}{\partial x^2} \right) \] (3.23)
\[ sU(x,s) - U(x,0) = \frac{d^2 U(x,s)}{dx^2} \] (3.24)

Then the following system is going to be obtained.
\[ \frac{d^2 U(x,s)}{dx^2} - sU(x, s) - U(0, x) = 0 \] (3.25)
\[ \frac{d^2 U(x,s)}{dx^2} - sU(x, s) - U^*(0, x) = 0 \] (3.26)
\[ \frac{d^2 U(x,s)}{dx^2} - sU(x, s) - U(0, x) = 0 \] (3.27)

The initial fuzzy trapezoidal condition changes the equations into following system
\[ \frac{d^2 U(x,s)}{dx^2} - sU(x,s) - (3\alpha + 1)\sin\pi x = 0 \]  \hspace{1cm} (3.28)

\[ \frac{d^2 U(x,s)}{dx^2} - sU(x,s) - (k)\sin\pi x = 0 \]  \hspace{1cm} (3.29)

\[ \frac{d^2 U(x,s)}{dx^2} - sU(x,s) - (8 - 3\alpha)\sin\pi x = 0 \]  \hspace{1cm} (3.30)

Complete solution = General Solution + Particular Solution

C.F. = \( c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} \)

P.I = \( \frac{U(x,0)}{-\pi^2 - s} \)

U(x,s) = \( c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} + \frac{U(x,0)}{-\pi^2 - s} \)  \hspace{1cm} (3.31)

Using initial condition, \( U(0,1) = 0 ; U(1,1) = 0 ; c_1 = c_2 = 0 \)  \hspace{1cm} (3.32)

Then the solution is \( U(x,s) = \frac{U(x,0)}{-\pi^2 - s} \)  \hspace{1cm} (3.33)

Applying the inverse Laplace the above coupled equations deduced into

\[ U(x,s) = \sin\pi x(3\alpha + 1)L^{-1}\left(\frac{1}{-\pi^2 - s}\right) \]  \hspace{1cm} (3.34)

\[ U'(x,s) = k\sin\pi xL^{-1}\left(\frac{1}{-\pi^2 - s}\right) \]  \hspace{1cm} (3.35)

\[ U(x,s) = \sin\pi x(8 - 3\alpha)L^{-1}\left(\frac{1}{-\pi^2 - s}\right) \]  \hspace{1cm} (3.36)

Hence

\[ U(x,t) = \begin{cases} 
-\sin\pi x(3\alpha + 1)e^{-\pi^2 t} & \text{ae[0 1] and (3\alpha + 1) \in [1 3]} \\
-k\sin\pi xe^{-\pi^2 t} & \text{k \in [4 5]} \\
-\sin\pi x(8 - 3\alpha)e^{-\pi^2 t} & \text{ae[0 1] and (8 - 3\alpha) \in [5 8]} 
\end{cases} \]  \hspace{1cm} (3.37)

The equation (3.37) provides the solutions of fuzzy initial value problem equation (3.21-3.22).

We consider one more partial differential equation with Sine-fuzzy condition

\[ \frac{\partial^2 U(x,t)}{\partial t^2} = \frac{\partial^2 U}{\partial x^2} + \sin\pi x \]  \hspace{1cm} (3.38)

\[ U(0,x) = \sin\pi x \hspace{1cm} \text{Where} \ x \in [0 \frac{1}{2}] \]  \hspace{1cm} (3.39)

Apply Laplace Transformation

\[ L\left(\frac{\partial^2 U(x,t)}{\partial t^2}\right) = L\left(\frac{\partial^2 U}{\partial x^2} + \sin\pi x\right) \]  \hspace{1cm} (3.40)

\[ s^2 U(x,s) - sU(x,0) - \sin\pi x = \frac{d^2 U(x,s)}{dx^2} \]  \hspace{1cm} (3.41)

Then the following system is going to be obtained.

\[ \frac{d^2 U(x,s)}{dx^2} - s^2 U(x,s) - sU(0,x) - \frac{\sin\pi x}{s} = 0 \]  \hspace{1cm} (3.42)

The initial fuzzy trapezoidal condition changes the equations into following system

\[ \frac{d^2 U(x,s)}{dx^2} - s^2 U(x,s) - s\frac{\sin\pi x}{s} - \frac{\sin\pi x}{s} = 0 \]  \hspace{1cm} (3.43)

\[ \frac{d^2 U(x,s)}{dx^2} - s^2 U(x,s) - \sin\pi x - \frac{\sin\pi x}{s} = 0 \]  \hspace{1cm} (3.44)

Complete solution = General Solution + Particular Solution

C.F. = \( c_1 e^{sx} + c_2 e^{-sx} \)
P.I = \left( - \frac{\sin \pi x}{s^2 + \pi^2} \right) + \frac{\sin \pi x}{s(s^2 + \pi^2)}

U(x,s) = c_1 e^{sx} + c_2 e^{-sx} + \frac{-\sin \pi x}{s^2 + \pi^2} + \frac{\sin \pi x}{s(s^2 + \pi^2)} \quad (3.45)

Using initial condition, \( U(0,t) = 0 \); \( U(1,t) = 0; \ c_1 = c_2 = 0 \) \quad (3.46)

Then the solution is \( U(x,s) = \frac{-\sin \pi x}{s^2 + \pi^2} + \frac{\sin \pi x}{s(s^2 + \pi^2)} \) \quad (3.47)

Applying the inverse Laplace the above coupled equations deduced into

\[ U(x,t) = \left( -\frac{\sin \pi x + \sin \pi t}{\pi} \right) + \frac{1}{\pi^2} (1 - \cos \pi t) \sin \pi x \]

The equation (3.48) provides the solution of fuzzy initial value problem equation (3.38-3.39).

4. Result and analysis

Using the described methodology the solution of partial differential equations (3.1, 3.21 and 3.38) under all three imposed fuzzy initial condition is obtained. Figure 1 shows the exact solution of partial differential equation (3.1) under the trapezoidal initial conditions and the three decaying curves represents the lower, central and upper solutions. Figure 2 shows the exact solution of partial differential equation (3.21) under the slipped initial conditions and the curves represents the lower, central and upper solutions initially they shows the fluctuation in nature then become steady. Figure 3 shows the exact solution of partial differential equation (3.38) under the Sine-fuzzy initial conditions and the curve represents the decaying nature of the solution.
Figure 1: Solution of the partial differential equation 3.1 by using trapezoidal \((τ_{b~c,d_{c},l_{c}})\) initial conditions 3.2

Figure 2: Solution of the partial differential equation 3.21 by using slipped initial conditions 3.22

Figure 3: Solution of the partial differential equation 3.38 by using Sine-fuzzy initial conditions 3.39.

5. Conclusion

In this paper, three partial differential equations are considered under the \textit{trapezoidal}(\(τ_{b~c,d_{c},l_{c}}\)) fuzzy, slipped and Sine-fuzzy initial conditions. Different systems of partial differential equations are formulated and solved by fuzzy Laplace transform. A new slipped and Sine-fuzzy fuzzy number as initial condition is introduced first time to
explore the FDEs. The lower central and upper solutions according the applied fuzzy initial conditions are presented. The respective fuzzy solutions under all three imposed conditions have presented.

References


