Comprehensive Study on Completeness Properties of Function Space

S.Mishra

Department of Mathematics

School of Chemical Engineering and Physical Sciences

Lovely Professional University, Phagwara-144411, Punjab, India.

Abstract We are going to start a scheme for the comprehensive study and critical analysis about properties of several topologies (from classical to recent) on function space. In the first plan of this systematic study we focus on completeness property of the metric space and also examine the relation of completeness with compactness for function space. In addition, we also discuss some interesting applications of completeness property.

Introduction

The completeness property of metric space is the most fundamental concept which give several results that not only help in understanding about many other properties of metric space but also help other aspects of analysis. As completeness is a metric property but many results which involve complete metric space have topological character. Many researcher investigated completeness property of function space in different prospective in [1-7]. In this paper we are going to study and critically analyze the completeness property of function space and further to move investigation on influence of the completeness on other property like compactness. In the first section we discuss about the complete m-space. In the second section we describe the relation of compactness with other properties of m-space. In the third section we present some interesting application of completeness property mainly Peano space-filling curve. In the last section we see how compactness can be formulated with the help of completeness especially in the case of function space. The letters $\boldsymbol{R}$ and $\boldsymbol{Z}$ are notation of set real number and set of integers respectively. We use following abbreviation: complete metric space as CMS, metric space as m-space top-space as top-space.

Complete Metric Space and Its Properties

In section we are going to explore concepts about CMS and its properties. In 1935, Neumann in [8] discussed about the Cauchy sequence in m-space and studies its fundamental properties. Any sequence (x_n) of points of m-space (X, d) is **Cauchy sequence** in X when for given positive real number ϵ, there exist positive interger N such $d(x_n,\ x_m) < \epsilon$, whenever $n\ , m \geq N$. A m-space (X, d) is **complete** when every Cauchy sequence in X is converges. There is also another way by which we can verify the completeness of the m-space which is more convenient than the definition as follows: "A m-space X is complete when every Cauchy sequence in X has a convergent subsequence." Since convergence of sequence is relative w.r.t metric on X so the completness property of X is also relative w.r.t. metric that defined on X. The completeness of m-space require both things that every Cauchy sequence in X converge and any convergent sequence in X is necessarily Cauchy sequence.

A new CMS can be obtained by given m-space in following two ways: (**i**) A closed subset A of CMS (X, d) is complete under the restricted metric d_A of metric d on X. In this case (A, d_A) is complete subspace of m-space X. (**ii**) If (X, d) is complete m-space, then $\left(X,\ \bar{d}\right)$ is new CMS w.r.t. standard bounded metric $\bar{d}$, which is defined as $\bar{d}(x, y) = \min\{d(x, y), 1\}$.

Now we are going to discuss interesting examples of complete and non-complete m-spaces. (**i**) The Euclidean space $\boldsymbol{R}$ with usual metric $d(x, y) = |x - y|$ is complete. (ii) The n-Euclidian space $\boldsymbol{R}^n$ is complete w.r.t. usual metric d and square metric ρ. (**iii**) The m-space $(\boldsymbol{R}^\omega, D)$ is complete, where the metric D is defined by $D(\bar{x}, \bar{y}) = \sup\left\{\frac{\bar{d}(x,\ y)}{i}\right\}$, and $\bar{d}(a, b) = \min\{|a - b|, 1\}$ is standard bounded on $\boldsymbol{R}$. (**iv**) The space of rational numbers $\boldsymbol{Q}$ with usual metric d is not CMS. (**v**) The open interval $(-1, 1)$ as subset of $\boldsymbol{R}$ is not CMS w.r.t. usual metric.

To study about the detail theory of completeness property of m-space first here we would like to investigate the topological character of this property. Is it completeness is topological property? In other words, Is the completeness is preserve during the homeomorphism? We can answer this with an example. The Euclidean m-space $\boldsymbol{R}$ is complete w.r.t. usual metric but the subspace $(-1, 1)$ of $\boldsymbol{R}$ w.r.t. same usual metric is not complete but $\boldsymbol{R}$ and $(-1, 1)$ both are homeomorphic. Therefore, it is clear from this example that completeness does not preserve during the homeomorphism i.e. it is not a topological property.

As we know that the spaces $\boldsymbol{R}^n$ and $\boldsymbol{R}^\omega$ both are CMS w.r.t. certain metric. So, Is it possible to describe the completeness of much more richer space $\boldsymbol{R}^J$ in same style. Unfortunately, it is not possible as like $\boldsymbol{R}^J$ is not metrizable if J is uncountable whereas both $\boldsymbol{R}^n$ and $\boldsymbol{R}^\omega$ are metrizable. Now in this time a specific metric as uniform metric play a major role to describe the completeness properties of $\boldsymbol{R}^J$ and we can say that $\boldsymbol{R}^J$ is complete w.r.t. uniform metric $\bar{\rho}$ corresponding to usual metric d on $\boldsymbol{R}$.

The completeness of Euclidean m-space $\boldsymbol{R}^J$ w.r.t. uniform metric give an idea to investigate the completeness of arbitrary m-space as Y^J w.r.t. uniform metric $\bar{\rho}$. The set Y^J is cartesion products of m-space Y and it is defined by $Y^J = \left\{ \bar{x} \colon \bar{x} = (x_\alpha)_{\alpha \in J}, x_\alpha \in Y, \forall\, \alpha \in J \right\}$. First we define uniform metric for Y^J, further describing its completeness property. Let Y be a m-space w.r.t usual metric d and standard bounded metric $\bar{d}\,(a, b) = \min\{d(a, b), 1\}$, then **uniform metric** $\bar{\rho}$ on Y^J is defined by $\bar{\rho}\,(\bar{x}\,, \bar{y}) = \sup\{\bar{d}(x_\alpha, y_\alpha) \colon \alpha \in J\}$ corresponding to metric d on Y. The topology induced by the uniform metric $\bar{\rho}$ is **uniform topology** on Y^J and it is denoted by $\tau_{\bar{\rho}}$. Thus we can denote new top-space as $(Y^J, \tau_{\bar{\rho}}\,)$ and describe as that Y^J is top-space with topology $\tau_{\bar{\rho}}$ induced by the uniform metric $\bar{\rho}$ corresponding to metric d on Y.

In this paper, we will use functional notation for the elements of space Y^J instead of tuple notation as discussed earlier. Therefore, in functional notation we can define uniform metric on Y^J as follows: the elements of Y^J are simply functions from J to Y so let $f,\ g : J \to Y$ are elements of Y^J, then uniform metric $\bar{\rho}$ is defined by $\bar{\rho}\,(f, g) = \sup\{\bar{d}\left(f(\alpha), g(\alpha)\right) \colon \alpha \in J\}$.

After defining the uniform metric for Y^J now we describe most important result on completeness of m-space. If Y is CMS with metric d, then the m-space Y^J is also complete with uniform metric $\bar{\rho}$ corresponding to metric d. This result tell us condition of completeness of Y^J in term of completeness of m-space Y. In particularly, since $\boldsymbol{R}$ is complete with usual metric d, then we can say that $\boldsymbol{R}^J$ is also complete with uniform metric $\bar{\rho}$ corresponding to usual metric d.

Once again we summaries all the results about completeness of several m-spaces and after that we move to discuss more general discussion of completeness of m-space. The m-space is complete under the condition of convergent of every Cauchy sequence or existence of convergent subsequence of every Cauchy sequence in the m-space. For example $\boldsymbol{R}$ is complete with usual and square metric, $\boldsymbol{R}^n$ and $\boldsymbol{R}^\omega$ are complete with specific metric, $\boldsymbol{R}^J$ is complete with uniform metric. Finally, in general case Y^J is complete with uniform metric $\bar{\rho}$ corresponding to metric d on Y, where J is any index set. In the prospective of more general form of completeness of m-space one question arise on the index set J in m-space Y^J. What happened about the completeness properties of space when we replace index set J by a top-space X. Since the space Y^J is nothing is a set of sequence of functions from set J to m-space Y so if we replace J by X, then new set comes in picture and that is denoted by Y^X and it is set of sequence of functions from top-space X to a m-space Y. In symbolically, $Y^J = \{f : f : J \to Y\}$, where Y is m-space and $Y^X = \{f: f: X \to Y\}$, where X is top-space and Y is m-space. So, now our aim is discuss about the completeness of new object as Y^X. There are two types of results we found regarding the completeness of Y^X as follows: (**i**) The completeness properties Y^X is irrelevant with the top-space X when we consider only a functions between X and Y. (**ii**) But if we consider the specific subset $C(X,Y)$ of Y^X which is defined as $C(X,Y) = \{f : f : X \to Y, f \text{ is continous}\}$, then completeness of $C(X,Y)$ is also depend on the nature of top-space X due to continuity between X and Y. (**iii**) Similarly, if we consider the another subset $B(X,Y)$ of Y^X which is defined as $B(X,Y) = \{f : f : X \to Y, f \text{ is bounded}\}$, then completeness of $B(X,Y)$ is also depend on the nature of space X. So finally we can say that in first case completeness of space Y^X is same as Y^J but in case of spaces $C(X,Y)$ and $B(X,Y)$ the completeness of both spaces depend upon the top-space X also. So our next objective is to find out the condition on X which make both spaces either $C(X,Y)$ or $B(X,Y)$ to be complete. In this direction we have following results which tell about require conditions on X: If the m-space Y is complete and the set $C(X,Y)$ is closed in Y^X under uniform metric (i.e. if $f \in Y^X$ that is limit point of $C(X,Y)$, then there is a sequence (f_n) of elements of $C(X,Y)$ converginning to f in the uniform metric $\bar{\rho}$), then space $C(X,Y)$ is complete in the uniform metric. Similarly if the m-space Y is complete and the set $B(X,Y)$ is closed in Y^X, then space $B(X,Y)$ is complete in uniform metric.

In the discussion of completeness of spaces Y^X, $C(X,Y)$ and $B(X,Y)$ one thing is common and that is selection of uniform metric on these spaces. Now question arises that except this uniform metric it is possible to find out the condition of completeness of above space in term of other metric. Fortunately, its answer is affirmative and that metric is sup metric. So, now first we define this new sup metric and also see how it helps to describe the completeness of above spaces.

Let Y be a m-space with usual metric d, then the metric ρ is **sup metric** on set $B(X, Y)$ defined by $\rho(f, g) = \sup\{d(f(x), g(x)) : x \in X\}$. The sup metric ρ and uniform metric $\bar{\rho}$ on the set $B(X, Y)$ are related to each other by the equation $\bar{\rho}(f, g) = \min\{\rho(f, g), 1\}$. This relation show that on $B(X, Y)$ the uniform metric $\bar{\rho}$ is the standard bounded metric derived from the sup metric ρ. As we know that if X is compact, then every continuous function $f : X \rightarrow Y$ is bounded, hence sup metric is also possible to defined on set $C(X, Y)$. If Y is complete under metric d, then $C(X, Y)$ is complete under the corresponding uniform metric $\bar{\rho}$ so, it is also complete under sup metric ρ.

Now we have an important result about the isometric imbedding of m-space. If X is any m-space, then there is an isometric imbedding of X into complete m-space. Let $h : X \rightarrow Y$ is isometric imbedding of X into a CMS Y, then the subspace $\overline{h(X)}$ of Y is CMS and it is called the completion of X. This process gives us one another tool to find out the new CMS from old one with the help of isometric imbedding.

Relation of Completeness of Metric Space with other Properties

Once again here we are going to discuss about the completeness property of arbitrary m-space and see how other properties of m-space influence the completeness. Let us see that how the ϵ −ball of any m-space X play a role in the discussion of completeness properties of X. If for given some $\epsilon > 0$, every ϵ −ball in X has compact closure then X must be complete. This result give us another tool to investigate about the completness of m-space under the condition of existence of compact closure of every ϵ −ball of X. But instead of each ϵ −ball in X if for each element x of X there is an $\epsilon > 0$ s.t. the ϵ −ball has compact closure, then it not give the guarantee for completeness of m-space X. Let us consider two metric d and d' on a space X and both metrices are metrically equivalent i.e. identity map as $i : (X, d) \rightarrow (X, d')$ and its inverse are both uniform continuous, X is complete under d iff X is complete under d'. This result gives us the tool to check the completeness of any m-space X w.r.t. two different metrically equivalent metrices. For example any metric d is metrically equivalenct to standard bounded metric $\bar{d}$ so we can say that space X is complete under metric d iff X is also complete under $\bar{d}$.

As we find so many necessary conditions for completeness of m-space in various terms, but still this is not enough discussion on the completeness of space until we have found a sufficient condition for the completeness. So our first necessary and sufficient condition of completeness of m-space as follows: a space X is complete iff for every nested sequence $A_1 \supset A_2 \supset \cdots$ of non-empty closed sets in X s.t. the dim $A_n \to 0$, the intersection of the sets $A_n \neq \phi$.

As we seen that completeness is not a topological property i.e. it is not preserve during the homeomorphism but now here we are going to see that how this property interact with topological concept. A space X is a **topologically complete** when there exists a metric for the topology on X relative to which X is complete. Now we discuss the fundamental properties of topologically complete space. (**i**) The topologically completeness is not a hereditary property but is it weakly hereditary because closed subspace as well as open subspace of topologically complete space is topologically complete. (**ii**) Topologically completeness is closed w.r.t. countable product under the product topology. (**iii**) The G_δ subset of topologically complete space is topologically complete.

Application Completeness of Metric Space

There are many applications of completeness properties of m-space in analysis and other fields of mathematics. Let us discuss some application of this property as follows: (**i**) The evaluation map is continuous under certain condition on the m-space $C(X,Y)$ w.r t. uniform metric. The evaluation map $e : X \times C(X,Y) \to Y$ defined by $e(x,f) = f(x)$, where $f : X \to Y$ is any function, then the evaluation map is continuous under the uniform metric on $C(X,Y)$ corresponding to metric d on Y. (**ii**) Another interesting application of this property which help to construct famous Peano space filling curve. In 1890, Peano in [9] given first example of space-filling curve which is a surjective, continuous function form unit interval onto the unit square. This can be state as “There exists a continuous map $f : I \to I^2$ whose image fill up the entire square I^2, where $I = [0,1]$”. This type of construction is possible due to the completeness of the m-space $C(X,Y)$ in the uniform metric when Y is complete. (**iii**) Let us see application of completeness of m-space by which we can find the fixed point in m-space X. But first, recall an interesting map called **contraction map** on X defined as if there is a number $\alpha < 1$ s.t. $d\big(f(x),f(y)\big) \leq \alpha d(x,y), \forall\, x,y \in X$. Therefore, if map f is a contraction of the complete m-space, then $\exists x \in X$ s.t. $f(x) = x$ is a fixed point of space X.

Relation between Compactness and Completeness

Now is the time to discuss about the relation of completeness property of m-space with other properties and in this direction first we choose most important properties that is compactness. As we know that for m-space compactness, limit point compactness and sequential compactness are equivalent to each other. In addition, to this we have another formulation of the compactness of m-space which includes completeness. Now the question is, how does compactness relate to completeness? The answer to this question comes from the result that each compact space is complete. But unfortunately it is not the complete answer because the converse of this result is not true. So our discussion is now focusing under what extra condition the converse of above is true. The extra condition for compactness of m-space except completeness is totally boundedness. Therefore, now we describe this new property as totally boundedness of m-space in details. A space (X, d) is **totally bounded** when for every positive real number ϵ there exists a finite covering of X by ϵ −balls. Let us see some examples for better understanding (**i**) All the bounded m-space are totally bounded but remembers its converse is not true for example $\boldsymbol{R}$ is bounded but it is not totally bounded because it have not finite covering by ϵ −balls. (**ii**) The relation between completeness and totally boundedness can be understand with some examples as $\boldsymbol{R}$ is complete under usual metric but not totally bounded, the subspace $(-1,1)$ of $\boldsymbol{R}$ is totally bounded but not complete while $[-1, 1]$ is both complete as well as totally bounded.

Now our main objective is to describe the necessary and sufficient condition for compactness of m-space in term of completeness and that is "A m-space is complete iff it is complete and totally bounded." This is the most useful result in the theory of complete m-space. The major application of this result is in classical version of Ascoli's Theorem, where we can find out the compact subspaces of function space $C(X, R^n)$ w.r.t. uniform topology. Later, we will discuss the details. By the way the idea of finding the compact subspace of space $C(X, R^n)$ comes from the space R^n. As we know that a subspace of R^n is compact iff it is closed and bounded. Therefore, in same way we can find out compact subspace of $C(X, R^n)$ i.e. subspace must be closed and bounded. But unfortunately this procedure not works for our function space (X, R^n) even X is compact. In this case, an additional condition called equicontinuity for subspace is required for compactness. Therefore, first we discuss about this new property and with it an important result which helps in the discussion of Ascoli's theorem. Let (Y, d) is m-space and $\boldsymbol{F}$ is a subset of function space $C(X, Y)$. The set $\boldsymbol{F}$ is called **equicontinuous at $\boldsymbol{x_0} \in \boldsymbol{X}$** when for given positive real number ϵ, there exists a neighbourhood U of x_0 s.t.

$\forall x \in U$ and all $f \in F$, $d\big(f(x), f(x_0)\big) < \epsilon$. If the set $\boldsymbol{F}$ is equicontinous at each point x_0 of X, then $\boldsymbol{F}$ is called **equicontinuous**. For better understanding of equicontinuous we will compare this with continuity at point of set. Any function f on set X is continuous at a point $x_0 \in X$ if for given positive real number numberϵ, there exists a neighbourhood U of x_0 s.t. $\mathrm{d}\big(f(x), f(x_0)\big) < \epsilon$ for $x \in U$. The equicontinuity of set $\boldsymbol{F}$ tell us that a single neighbourhood U can be choose that will work for all the functions f in the collection $\boldsymbol{F}$. Actually the set $\boldsymbol{F}$ is a set of continuous functions from X to $Y$and if it equicaontinuous at particular at point $x_0 \in X$, if we are able to choose single neighbourhood U of x_0 s.t. for other points x of U and for all functions f of F we have $d\big(f(x), f(x_0)\big) < \epsilon$. What is condition under which any subset $\boldsymbol{F}$ of functions space $C(X, Y)$ is equicontinuous under the metric d? The conditions is $\boldsymbol{F}$ must be totally bounded under the uniform metric $\bar{\rho}$ corresponding to metric d on Y . So, we can say that set subset $\boldsymbol{F}$ of function space $C(X, Y)$ is equicontinuous under the metric d if $\boldsymbol{F}$ is totally bounded under the uniform metric $\bar{\rho}$ corresponding to metric d on Y. Is it conversely true? i.e. if $\boldsymbol{F}$ is equiconntinuous under metric d, then $\boldsymbol{F}$ is totally bounded. No, but fortunately it is partial true as follows: Let X is a space and (Y, d) is a m-space and both are compact. If subset $\boldsymbol{F}$ of $C(X, Y)$ is equiconntinuous under d, then $\boldsymbol{F}$ is totally bounded under uniform metric as well as sup metric corresponding to metric d. Here we can see that converse of the above result is true only in case of compactness of X and Y. The subset $\boldsymbol{F}$ of $C(X, Y)$ is **pointwise bounded** under the metric d when $\forall x \in X$, the subset $F_a = \{f(a) : f \in \boldsymbol{F}\}$ of Y is bounded under d, where (Y, d) is m-space.

Finally, we can state the classical version of Ascoli's Theorems as follows: Let X is compact space, (R^n, d) is m-space with square metric or Euclidean metric and $C(X, R^n)$ is function space with uniform topology induced by uniform metric corresponding to either square metric or Euclidean metric on R^n. Then a subspace $\boldsymbol{F}$ of $C(X, R^n)$ has a compact closure if an only if $\boldsymbol{F}$ is equicontinuous and pointwise bounded under metric d. We can draw one important result from classical version of Ascoli's Theorems which tell us about directly compactness of subset $\boldsymbol{F}$ of $C(X, R^n)$. Let X is compact space, (R^n, d) is m-space with either square or Euclidean metric and $C(X, R^n)$ is function space with uniform topology induced by the uniform metric corresponding to metric on R^n. Then a subspace $\boldsymbol{F}$ of $C(X, R^n)$ is compact iff $\boldsymbol{F}$ is closed, bounded under the sup metric and equicontinuous under d.

The classical version of Ascoli's Theorems tells us that **(i)** It is necessary and sufficient condition for compactness of any subspace of function space in terms of equicontinuity and pointwise boundedness of that subspace. (**ii**) It is generalization of the result that m-space is compact iff space is complete and totally bounded. (**iii**) A particular topology as uniform topology plays significant role in all discussion of compactness. Although, unfortunately this theorem does not give the actual general form for the compactness of subspace of function space. This gives condition for compactness of subspace of only a particular continuous functions space as $C(X, R^n)$. Now the question arise that what is condition for compactness of subspace of general continuous function space $C(X, Y)$. We will discuss such condition in general version of Ascoli's Theorem, but before discussing this theorem we have to understand some more concepts about several topologies on the functions space.

In continuation of this study, our next objective is to describe several topologies (from classical to recent) on functions spaces, along with a discussion of the relationship of topologies with completeness property.

Reference

[1] D.J. Lutzer and R.A. McCoy, *Category in function spaces* I, Pacific J. Math. 90 145-168 (1980).

[2] R.A. McCoy, *Complete function spaces*, Internat. J. Math. Math. Sci. 6, 271-278 (1983).

[3] L. Holá, L. Zsilinszky, Completeness and related properties of the graph topology on function spaces. Topol. Proc. **46**, 1–14 (2015).

[4] Jindal, A., Jindal, V., Kundu, S. *et al. Completeness properties of the open-point and bi-point-open topologies on C(X).* Acta Math. Hungar. **153,** 109–119 (2017).

[5] D. Buhagiar and I. Yoshioka, *Sieves and completeness properties*, Questions Answers Gen. Topology, 18, 143–162(2000).

[6] Alexander V. Osipov, *On the Completeness Properties of the C-Compact-Open Topology on* $C(X)$, Ural Mathematical Journal, **1**(1), 61- 67 (2015).

[7] S. Kundu and P. Garg, *Completeness Properties of the Pseudocompact-Open Topology on* $C(X)$, Ural Mathematical Journal, Math. Slovaca, **58** (3), 325–338 (2008).

[8] J. V. Neumann, *On Complete Topological Spaces*, Transactions of the American Math-ematical Society **37**(1), 1–20 (1935).

[9] G. Peano, *Sur une courbe, qui remplit toute une aire plane*. *Math. Ann.* **36,** 157–160 (1890).