Generated Monotone Class

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Abstract

In this paper we study generated monotone class. We also discuss generated sigma ring and the relation between generated sigma ring and generated monotone class.

Monotone class

Let \( \Gamma \) be a non-empty set and \( D \) be a class of subsets of \( \Gamma \). We say \( D \) is a "monotone class" if

\[
\text{a) } \bigcup_{n=1}^{\infty} H_n \in D, \text{ whenever } H_n \in D \text{ and } H_n \subseteq H_{n+1} \text{ for } n=1,2,3,\ldots \text{, and } \lim_{n \to \infty} H_n = \bigcup_{n=1}^{\infty} H_n,
\]

\[
\text{b) } \bigcap_{n=1}^{\infty} H_n \in D \text{ and } H_n \supseteq H_{n+1} \text{ for } n=1,2,3,\ldots \text{, and } \lim_{n \to \infty} H_n = \bigcap_{n=1}^{\infty} H_n [2,3].
\]

Generated monotone class

Let \( \Gamma \) be any non-empty set and \( \emptyset \) be any collection of subsets of \( \Gamma \). Let \( D(\emptyset) = I \) where the intersection is over all those monotone classes \( D \) of subsets of \( \Gamma \) such that \( \emptyset \subseteq D \).

Clearly, \( D(\emptyset) \) is itself a monotone class and if \( D \) is any monotone class such that \( \emptyset \subseteq D \) then \( D(\emptyset) \subseteq D \). Thus \( D(\emptyset) \) is the smallest monotone class of subsets of \( \Gamma \) such that \( \emptyset \subseteq D(\emptyset) \).

The class \( D(\emptyset) \) is called as the "generated monotone class" generated by \( \emptyset \) [2, 3].

Sigma Ring (\( \sigma-\)ring)

A non-empty class \( \mathcal{S} \) of subsets of \( \Gamma \) is said to be a "sigma ring" denoted as "\( \sigma\)-ring" if

\[
\text{a) } P, Q \in \mathcal{S} \Rightarrow P-Q \in \mathcal{S},
\]

\[
\text{b) } \{P_n\}_{n \in \mathbb{N}} \in \mathcal{S} \Rightarrow \bigcup_{n=1}^{\infty} P_n \in \mathcal{S} [1, 3].
\]

Generated sigma ring (\( \sigma-\)ring)

We define \( \mathcal{S}(K) \) to be the smallest "\( \sigma\)-ring" containing a class of sets \( K \) which is called as "generated \( \sigma\)-ring" generated by \( K \) [1, 3].

"Let \( \mathcal{R} \) be a ring of subsets of a set \( \Gamma \). Then \( \mathcal{S}(\mathcal{R}) = D(\mathcal{R}) \), where \( \mathcal{S}(\mathcal{R}) \) is the \( \sigma \)-ring generated by the ring \( \mathcal{R} \) and \( D(\mathcal{R}) \) is the generated monotone class generated by the ring \( \mathcal{R} \)."

Since every \( \sigma\)-ring is a monotone class and \( \mathcal{R} \subseteq \mathcal{S}(\mathcal{R}) \) which implies \( D(\mathcal{R}) \subseteq \mathcal{S}(\mathcal{R}) \).

Now we will show that \( \mathcal{S}(\mathcal{R}) \subseteq D(\mathcal{R}) \). Since \( D(\mathcal{R}) \) already contains \( \mathcal{R} \) and \( D(\mathcal{R}) \) is a monotone class.
If we will show $D(\mathcal{R})$ is a ring then $D(\mathcal{R})$ becomes a monotone ring and consequently becomes a $\sigma$-ring containing $\mathcal{R}$, since $\mathcal{F}$ is a $\sigma$-ring if and only if it is a monotone ring. As $\mathcal{F}(\mathcal{R})$ is the smallest $\sigma$-ring containing $\mathcal{R}$ which implies $\mathcal{F}(\mathcal{R}) \subseteq D(\mathcal{R})$. So we need to show that $D(\mathcal{R})$ is a ring.

For $F \in \Gamma$, let us define $\mathcal{N}(F)$ as $\mathcal{N}(F) = \{E: E \cup F, E-F, F-E \in D(\mathcal{R})\}$. Here we note that $E \in \mathcal{N}(F)$ is equivalent of saying $F \in \mathcal{N}(E)$.

We now show that $\mathcal{N}(F)$ is a monotone class when it is non-empty. Let $\{E_n\}_{n=1}^\infty$ be a monotonically increasing sequence of sets of $\mathcal{N}(F)$.

Now, $\left(\lim_n E_n\right) \cup F = \left(\bigcup_{n} E_n\right) \cup F = \bigcup_{n} (E_n \cup F) = \lim_n (E_n \cup F) \in D(\mathcal{R})$, since $\{E_n \cup F\}$ is again monotonically increasing.

$\left(\lim_n E_n\right) - F = \left(\bigcup_{n} E_n\right) - F = \bigcup_{n} (E_n - F) = \lim_n (E_n - F) \in D(\mathcal{R})$, since $\{E_n - F\}$ is again monotonically increasing and,

$F - \left(\lim_n E_n\right) = F - \left(\bigcup_{n} E_n\right) = \bigcup_{n} (F - E_n) = \lim_n (F - E_n) \in D(\mathcal{R})$, since $\{F - E_n\}$ is monotonically decreasing.

Thus $\lim_n E_n \in \mathcal{N}(F)$.

Similarly for monotonically decreasing sequence from $\mathcal{N}(F)$, we can show that limit set belongs to $\mathcal{N}(F)$. Hence $\mathcal{N}(F)$ is a monotone class.

Now let $E, F \in \mathcal{R}$. Since $\mathcal{R}$ is a ring so $E-F, F-E, E \cup F \in \mathcal{R}$.

Also $\mathcal{R} \subseteq D(\mathcal{R})$ which implies $E-F, F-E, E \cup F \in D(\mathcal{R}) \Rightarrow E \in \mathcal{N}(F)$.

Since this is true for any $E \in \mathcal{R}$, therefore we have $\mathcal{R} \subseteq \mathcal{N}(F)$.

Thus $\mathcal{N}(F)$ is a monotone class containing ring $\mathcal{R}$ which shows $D(\mathcal{R}) \subseteq \mathcal{N}(F)$ since $D(\mathcal{R})$ is the smallest monotone class containing $\mathcal{R}$.

Finally, let $E \in D(\mathcal{R})$ then $E \in \mathcal{N}(F)$, $F \in \mathcal{R}$ since $D(\mathcal{R}) \subseteq \mathcal{N}(F)$

$\Rightarrow F \in \mathcal{N}(E)$.

Since this is true for any $F \in \mathcal{R}$ so $\mathcal{R} \subseteq \mathcal{N}(E)$.

$\Rightarrow D(\mathcal{R}) \subseteq \mathcal{N}(E)$, since $D(\mathcal{R})$ is the smallest monotone class containing $\mathcal{R}$.

Thus $D(\mathcal{R})$ is a ring since $D(\mathcal{R}) \subseteq \mathcal{N}(F)$ and $D(\mathcal{R}) \subseteq \mathcal{N}(E)$. 


References

[1]. Measure Theory and Integration, G. de Barra, New age International Publication.
