Review of Lagrange-Euler Mechanics applied for Dynamic Study of Robotic Arm

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ABSTRACT

A robotic manipulator's dynamic analysis based on various parameters such as velocity, acceleration and forces etc., is the examination of robotic arm motions in relation to a coordinate system known as a fixed frame. There exists always a challenge while solving the problem of inverse kinematics because many solutions are available for the same problem. This work focused on defined trajectory tracking in a given workspace.

1. INTRODUCTION

The origin of the robot word is come from Robota, which means “work” or “forced labor” in Czech. A robot is a multi-function control unit with reprogrammable abilities, to move materials, parts, tools or specialized devices, by means of varying programmed instructions.

Robotics is a multidisciplinary science and engineering branch that integrate mechanical, electrical, software and other fields of study and implementation. It manages the purpose, expansion, and operation of robots, as well as IOT control, sensory feedback, and information handling systems.

- Robot Kinematics

The kinematics is mechanical division which explores the action of things without taking account of the forces that cause action. Robot arm kinematics conduct an analytical study of robot arm geometry as a time function, regardless of force / moment, against the fixed reference coordinating system.

2. LITERATURE SURVEY

- Forward Kinematics Problems

The problems of the forward kinematics are explained by the transformation matrices between an end-effector coordinate frame and base coordinate frame. The position vector is signified by the homogeneous transformation matrix in a 3-D space along with the rotation matrix of the body. The entire homogeneous transformation matrix is generated simply by multiplying specific frame transformations fixed in contiguous chain links. Denavit and Hartenberg [1] were the first to make this judgment for the manipulator's spatial geometry, and its benefit is in the universal algorithm to solve the kinematics of a manipulator.

- Inverse Kinematics Problems

Indirect kinematics problems can be resolved by using two types of approaches first one is closed form solutions and second is numerical approach. Closed form solutions are robot dependent and faster than the numerical approach. This approach classified into two type of methods Analytical method [2] and Geometric method [3]. Analytical method it is also called as algebraic method, analytically invert the direct kinematics equations. By solving an algebraic equations scheme, the problem of inverse kinematics can be summarized. If Cosine and Sine values are evaded by fixed substitutes, robot kinematics gives an arithmetical system or a sequence of equations. Then algebraic method is used to solve these equations that are obtained. The numerical approach method can be applied to any kinematic arrangement because it is not
robot dependent. It is slower and, in some case, it is not possible to compute the solution. In this approach, there are different methods like symbolic elimination method \[4\], continuation method \[5\] and iterative method. Symbolic elimination method gives a set of nonlinear equations by eliminating variables to shorten it into a smaller set of equations. Different numbers of iterative methods are using now a day to resolve the inverse kinematics problems. Like Newton-Raphson method \[6\], Optimization approach \[7,8\], Cyclic coordinate descent method \[9\], Pseudoinverse method \[10\].

3. DYNAMIC MODELING OF MANIPULATOR

The set of equations related to the dynamics of a robot arm are its motion equations. A manipulator move at consistent speed, must accelerate, and decelerate amid the work cycle. In order to adjust internal forces and external forces, the time varies from one torque to the other with the help of actuators. In this section, numerical model and properties of dynamic motion equations for the robot arm dynamics have been developed Lagrange-Euler approach.

Lagrange-Euler Mechanics

In physics, Lagrangian mechanics is widely used to solve mechanical problems. In optimisation problems of dynamic systems Lagrange’s equations are applied. The Lagrange function \( L \) which is a scaler function is given by

\[
L = \text{Kinetic Energy} - \text{Potential Energy}
\]  

To define the manipulator variables based on a set of generalized coordinates dynamic model formulation utilising Lagrange-Euler approach is used. In the generalized coordinates the joint variables are described as displacement ‘\( q \)’. For prismatic joint which is defined as linear displacement ‘\( d \)’ and while for rotary joint defined as angular displacement ‘\( \theta \)’. The velocity for prismatic joint ‘\( \dot{q} \)’ describes linear velocity ‘\( \dot{d} \)’ and velocity for rotary joint ‘\( \dot{q} \)’ describes angular velocity ‘\( \dot{\theta} \)’.

To obtain the dynamic model for robot arm based on Lagrange-Euler approach is given by the Lagrangian, as a set of equations,

\[
\frac{d}{dt} \left( \frac{dL}{dq_i} \right) - \frac{dL}{dq_i} = \tau_i
\]  

Here,

- \( L \) = Lagrangian function.
- \( q_i \) = Generalized coordinates of the manipulator.
- \( \dot{q}_i \) = Generalized coordinates of the joint velocity.
- \( \tau_i \) = Generalized force at joint \( i \).

• Lagrange-Euler Formulation

The dynamic model of a \( n \) degree of freedom robotic arm can be systematically developed using L-E model. The \( n \) degree of freedom open kinematic chain serial link manipulator has \( n \) joint position,

\[
q = [q_1, \ldots, q_n]^T.
\]  

The derivation of equation of motion utilizing Lagrange-Euler formulation is done in the accompanying subtopics.

Joint Velocity of a Point on the Manipulator

For computing the K.E., link velocity is required.

Let’s take a point \( p \) on the link \( i \) of as revealed in figure (2). The vector \( r_i \) represents the location of \( p \) on the link w.r.t. frame \( \{i\} \).

\[
r_i = [x_i, y_i, z_i]^T
\]  

w.r.t. frame \( \{0\} \), the position of point \( p \) is given by

\[
r_i^0 = T_i^0 r_i
\]
If joint \( i' \) is prismatic, transformation matrix is

\[
T_{i}^{i+1} = \begin{bmatrix}
\cos \theta_i & -\sin \theta_i & \cos \alpha_i & \sin \alpha_i & 0 \\
\sin \theta_i & \cos \theta_i & -\sin \alpha_i & \cos \alpha_i & 0 \\
0 & 0 & \sin \alpha_i & \cos \alpha_i & d_i \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]  

(3.7)

If joint \( i' \) is rotary, the transformation matrix is

\[
T_{i}^{i+1} = \begin{bmatrix}
\cos \theta_i & -\sin \theta_i & \cos \alpha_i & \sin \alpha_i & a_i \cos \theta_i \\
\sin \theta_i & \cos \theta_i & -\sin \alpha_i & \cos \alpha_i & a_i \sin \theta_i \\
0 & 0 & \sin \alpha_i & \cos \alpha_i & d_i \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]  

(3.8)

Considering, \( a_i, \alpha_i \) are link parameters of robot arm and \( d_i, \theta_i \) are joint parameters of joint \( i \). The partial derivates of above rotary transformation matrix with respect to \( \theta_i \) gives,

\[
\frac{\partial T_{i}^{i+1}}{\partial \theta_i} = \begin{bmatrix}
-\sin \theta_i & -\cos \theta_i & \sin \alpha_i & \cos \alpha_i & -a_i \sin \theta_i \\
\cos \theta_i & -\sin \theta_i & \sin \alpha_i & \cos \alpha_i & a_i \cos \theta_i \\
0 & 0 & \sin \alpha_i & \cos \alpha_i & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]  

(3.9)

The velocity of \( r_i \) can be expressed as

\[
v_i^0 = \frac{d}{dt} (r_i^0) = \frac{d}{dt} (T_i^0 r_i^i) 
\]

(3.10)

\[
\frac{d}{dt} (r_i^0) = \left( \sum_{j=1}^{i} \frac{\partial T_i^0}{\partial q_j} \ddot{q}_j \right) r_i^i
\]

(3.11)

When equation number (3.8) and (3.9) are compared they gives a pattern, that equation (3.9) can be obtained from equation (3.8) by using some matrix operations,

Hence, the partial derivation of homogeneous transformation matrix \( T_{i}^{i+1} \) with respect to \( \theta_i \) can be obtained using a \( 4 \times 4 \) matrix \( Q_i \). For revolute joint \( Q_i \) is defined as
For prismatic joint $Q_i$ is defined as

$$Q_i = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ (3.12)

and by premultiplying $T_{i-1}^{-1}$ with $Q_i$,

$$Q_i T_{i-1}^{-1} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos \theta_i & -\sin \theta_i \cos \alpha_i & \sin \theta_i \sin \alpha_i & a_i \cos \theta_i \\ \sin \theta_i & \cos \theta_i \cos \alpha_i & -\sin \theta_i \sin \alpha_i & a_i \sin \theta_i \\ \sin \alpha_i & \cos \alpha_i & 0 & b_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ (3.14)

$$Q_i T_{i-1}^{-1} = \begin{bmatrix} -\sin \theta_i & -\cos \theta_i \cos \alpha_i & \sin \theta_i \sin \alpha_i & a_i \sin \theta_i \\ \cos \theta_i & -\sin \theta_i \cos \alpha_i & \sin \alpha_i \cos \theta_i & -a_i \sin \theta_i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ (3.15)

It is observed that equation (3.9) and (3.13) are same,

$$\frac{\partial T_{i-1}^{-1}}{\partial \theta_i} = Q_i T_{i-1}^{-1}$$ (3.16)

Since $T_i^0 = T_1^0 T_2^1 ... T_i^{i-1}$, therefore the partial derivative $T_i^0$ with respect to $q_j$,

$$\frac{\partial T_i^0}{\partial q_j} = T_1^0 T_2^1 ... T_{j-1}^{j-1} \frac{\partial T_{j-1}^{-1}}{\partial q_j} T_j^1 ... T_{i-1}^{i-1}$$ (3.17)

After simplifying equation (3.17), the result is valid for $j \leq i$. Therefore,

$$\frac{\partial T_i^0}{\partial q_j} = \begin{cases} T_{j-1}^0 Q_j T_{i-1}^{-1} & \text{for } j \leq i \\ 0 & \text{for } j > i \end{cases}$$ (3.18)

The link velocity $v_i$ as given in equation (3.11), is simplified using equation (3.18),

$$v_i = \left( \sum_{j=1}^{i} T_{j-1}^0 Q_j T_{i-1}^{-1} \dot{q}_j \right) r_i^i$$ (3.19)

**The Inertia Tensor of the Manipulator**

During the motion of links, the mass of links contributes inertia forces. The symmetric matrix is a $4 \times 4$ matrix characterizing the mass distributions of a inflexible thing. The moment of inertia tensor is
\[ I_i = \begin{bmatrix} \int x_i^2 \, dm_i & \int x_i y_i \, dm_i & \int x_i z_i \, dm_i & \int x_i \, dm_i \\ \int x_i y_i \, dm_i & \int y_i^2 \, dm_i & \int y_i z_i \, dm_i & \int y_i \, dm_i \\ \int x_i z_i \, dm_i & \int y_i z_i \, dm_i & \int z_i^2 \, dm_i & \int z_i \, dm_i \\ \int x_i \, dm_i & \int y_i \, dm_i & \int z_i \, dm_i & \int dm_i \end{bmatrix} \]  

(3.20)

Where \( dm_i \) is the mass of the element on link \( i \) located at \( r_i' = [x_i \ y_i \ z_i \ 1]^T \).

\[ I_i = \begin{bmatrix} 0.5(-I_{xx} + I_{yy} + I_{zz}) & I_{xy} & I_{xz} & m_i x_i \\ I_{xy} & 0.5(I_{xx} - I_{yy} + I_{zz}) & I_{yz} & m_i y_i \\ I_{xz} & I_{yz} & 0.5(I_{xx} + I_{yy} - I_{zz}) & m_i z_i \\ m_i x_i & m_i y_i & m_i z_i & m_i \end{bmatrix} \]  

(3.21)

Taking, \( i^{th} \) link mass = \( m_i \) and center of mass = \( \vec{r}_i' = [x_i \ y_i \ z_i \ 1] \).

**Manipulator’s Kinetic Energy**

K.E. of the small mass \( dm_i \) on link \( i \), positioned at \( r_i^0 \) and moving with velocity \( \dot{v}_i^0 \) w.r.t base frame is,

\[ dk_i = \frac{1}{2} dm_i (\dot{v}_i^0)^2 \]  

(3.22)

\[ (\dot{v}_i^0)^2 = v_i \cdot \dot{v}_i = r_i^0 \cdot \dot{r}_i^0 = Trace \left( r_i^0 \dot{r}_i^0^T \right) = Trace(\dot{v}_i \cdot v_i^T) \]  

(3.23)

By substituting equation (3.19) in equation (3.22),

\[ dk_i = \frac{1}{2} Trace \left[ \left( \sum_{j=1}^{i} T_{j-1}^0 Q_j T_i^{j-1} \dot{q}_j \right) \left( \sum_{k=1}^{i} T_{k-1}^0 Q_k T_i^{k-1} \dot{q}_k \right) \right] dm_i \]  

(3.24)

The total kinetic energy can be given by integration of equation (3.24),

\[ k_i = \int dk_i \]  

(3.25)

\[ k_i = \frac{1}{2} Trace \left[ \sum_{j=1}^{i} \sum_{k=1}^{i} \left( T_{j-1}^0 Q_j T_i^{j-1} \right) I_i \left( T_{k-1}^0 Q_k T_i^{k-1} \right)^T \dot{q}_j \dot{q}_k \right] \]  

(3.26)

Form equation (3.21), the term \( \int r_i^T r_i \, dm_i \) of equation (3.26) is the moment of inertia tensor \( I_i \), therefore the above equation (3.26) is,

\[ k_i = \frac{1}{2} Trace \left[ \sum_{j=1}^{i} \sum_{k=1}^{i} \left( T_{j-1}^0 Q_j T_i^{j-1} \right) I_i \left( T_{k-1}^0 Q_k T_i^{k-1} \right)^T \dot{q}_j \dot{q}_k \right] \]  

(3.27)

The final K.E. of manipulator is,

\[ K = \sum_{i=1}^{n} k_i \]

\[ = \frac{1}{2} \sum_{i=1}^{n} Trace \left[ \sum_{j=1}^{i} \sum_{k=1}^{i} \left( T_{j-1}^0 Q_j T_i^{j-1} \right) I_i \left( T_{k-1}^0 Q_k T_i^{k-1} \right)^T \dot{q}_j \dot{q}_k \right] \]  

(3.28)

By simplifying equation (3.28),

\[ K = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{i} Trace \left[ \left( T_{j-1}^0 Q_j T_i^{j-1} \right) I_i \left( T_{k-1}^0 Q_k T_i^{k-1} \right)^T \dot{q}_j \dot{q}_k \right] \]  

(3.29)
\[ K = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{i} Trace \left[ (U_{ij}) I_i (U_{ik})^T \right] q_j q_k \] 

(3.30)

### Manipulator’s Potential Energy

P.E for \( i^{th} \) link is

\[ p_i = -m_i g \left( \frac{\bar{r}_i}{\bar{r}_i^0} \right) = -m_i g T_i^0 (\frac{\bar{r}_i}{\bar{r}_i^0}) \] 

(3.31)

Considering, \(-\)ve sign represents the work is done on the system to raise link \( i \) against gravity. \( \bar{r}_i^0 \) represents \( i^{th} \) link COM w.r.t. frame \( \{ l \} \), and \( \bar{r}_i^0 \) represents \( i^{th} \) link COM w.r.t. base frame. The gravity vector w.r.t. bottom frame is \( g = [g_x \quad g_y \quad g_z \quad 0]^T \).

Total potential energy is given as,

\[ p = \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} -m_i g T_i^0 (\frac{\bar{r}_i}{\bar{r}_i^0}) \] 

(3.32)

### Manipulator’s Motion Equation

Motion equation of manipulator is formulated as following. After substituting kinetic energy equation (3.30) and potential energy equation (3.32) in Lagrange-Euler equation (3.1), \( L = K - P \),

\[ L = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{i} Trace \left[ (U_{ij}) I_i (U_{ik})^T \right] q_j q_k - \left( \sum_{i=1}^{n} -m_i g T_i^0 (\frac{\bar{r}_i}{\bar{r}_i^0}) \right) \] 

(3.33)

The generalized torque \( \tau_i \) of actuator at joint \( i \), as described in equation (3.2), by substituting above Lagrange-Euler equation (3.33) in equation (3.2), the final equation of motion is,

\[ \tau_i = \sum_{j=1}^{n} M_{ij} (q) d_j + \sum_{j=1}^{n} \sum_{k=1}^{n} h_{ijk} q_j q_k + G_i \] 

(3.34)

Where,

\[ M_{ij} = \sum_{p = \max(i,j)}^{n} Trace \left[ (U_{pj}) I_i (U_{pk})^T \right] \] 

(3.35)

\[ h_{ijk} = \sum_{p = \max(i,j,k)}^{n} Trace \left[ \frac{\partial (d_{pk})}{\partial q_p} I_p U_p^T \right] \] 

(3.36)

\[ G_i = -\sum_{p = i}^{n} m_p g U_{pi} (\frac{\bar{r}_p^0}{\bar{r}_p^0}) \] 

(3.37)

\[ U_{ij} = \begin{cases} T_{j-1}^0 Q_i T_{j-1}^0 & \text{for } j \leq i \\ 0 & \text{for } j > i \end{cases} \] 

(3.38)

\[ \frac{\partial U_{ij}}{\partial q_k} = \begin{cases} T_{j-1}^0 Q_i T_{j-1}^k Q_k T_{i}^{k-1} & \text{for } i \geq k \geq j \\ T_{k-1}^0 Q_k T_{j}^{k-1} & \text{for } i \geq j \geq k \\ 0 & \text{for } i < j \text{ or } i < k \end{cases} \] 

(3.39)

The dynamic model for a manipulator is given in equation (3.34)

**DEVELOPMENT OF DYNAMIC MODEL FOR 5-DOF MANIPULATOR**

The following is the example to develop dynamic model using Lagrange-Euler equation for 5- link manipulator which tracks square trajectory.

Let’s consider a 5-link manipulator which is having all revolute joint. The physical dimensions of the links of the manipulator and the link parameter are given in table (3.1).
Table no 3.1: Physical dimensions and parameters of 5-link manipulator

<table>
<thead>
<tr>
<th>No. of links (i)</th>
<th>Link Masses (m) (kg)</th>
<th>Link Parameters</th>
<th>Link Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Initial Joint Angle (θ_i) (deg)</td>
<td>Link Length (a_i) (m)</td>
</tr>
<tr>
<td>1</td>
<td>40</td>
<td>70</td>
<td>0.8</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>-30</td>
<td>0.7</td>
</tr>
<tr>
<td>3</td>
<td>30</td>
<td>15</td>
<td>0.7</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
<td>-40</td>
<td>0.52</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td>2.5</td>
<td>0.3</td>
</tr>
</tbody>
</table>

Dynamic model for 5 link manipulators as discussed in above section can be formulated using MATLAB. The final homogeneous transformation matrices with respect to base frame {0} is,

\[
T_5^0 = \begin{bmatrix}
0.9537 & -0.3007 & 0 & 1.9997 \\
0.3007 & 0.9537 & 0 & 1.9999 \\
0 & 0 & 1.0000 & 0 \\
0 & 0 & 0 & 1.0000 \\
\end{bmatrix}
\]  

\[
M = \begin{bmatrix}
389.2631 & 245.9620 & 131.7287 & 41.4152 & 7.2257 \\
245.9620 & 168.7943 & 93.3589 & 32.7960 & 5.7647 \\
131.7287 & 93.3589 & 55.4902 & 20.9589 & 3.8246 \\
41.4152 & 32.7960 & 20.9589 & 10.9277 & 2.1585 \\
7.2257 & 5.7647 & 3.8246 & 2.1585 & 0.6000 \\
\end{bmatrix}
\]  

4. NUMERICAL SIMULATIONS

In this present section numerical simulations are carry out for a manipulator having five degrees of freedom, all the joint is rotary joint. The link lengths, masses, joint angles, max joint limits are specified in following table (5.1).

Table 5.1: Parameters of 5-DOF manipulator

<table>
<thead>
<tr>
<th>No. of Links (i)</th>
<th>Link Lengths (l_i) (m)</th>
<th>Link Masses (m_i) (kg)</th>
<th>Joint Angle (θ_i) (deg)</th>
<th>Max, Joint Limit (θ_max) (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.8</td>
<td>40</td>
<td>70</td>
<td>180</td>
</tr>
<tr>
<td>2</td>
<td>0.7</td>
<td>20</td>
<td>-30</td>
<td>120</td>
</tr>
<tr>
<td>3</td>
<td>0.7</td>
<td>30</td>
<td>15</td>
<td>120</td>
</tr>
<tr>
<td>4</td>
<td>0.52</td>
<td>20</td>
<td>-40</td>
<td>150</td>
</tr>
<tr>
<td>5</td>
<td>0.3</td>
<td>20</td>
<td>2.5</td>
<td>150</td>
</tr>
</tbody>
</table>

The manipulator tracks the trajectory of four lines which forms a square, manipulator starts form a point (2,2) and reaches to same point. The square trajectory for the 1st manipulation variable \( r_1^d(t) \) is the constant velocity motion along the square from point (2,2) at time \( t = 0 \) min, to the point (2,2) at time \( t = 1 \) min.
From the above numerical simulation, it is observed that the manipulator due to its dynamics study can fulfil the task given.

5. CONCLUSION

In this chapter, the idea of dynamic modeling is utilized, which helps to achieve defined manipulator motion in a given workspace.

REFERENCES


