Review of Lagrange-Euler Mechanics applied for Dynamic Study of Robotic Arm

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ABSTRACT

A robotic manipulator's dynamic analysis based on various parameters such as velocity, acceleration and forces etc., is the examination of robotic arm motions in relation to a coordinate system known as a fixed frame. There exists always a challenge while solving the problem of inverse kinematics because many solutions are available for the same problem. This work focused on defined trajectory tracking in a given workspace.

1. INTRODUCTION

The origin of the robot word is come from Robota, which means "work" or "forced labor" in Czech. A robot is a multi-function control unit with reprogrammable abilities, to move materials, parts, tools or specialized devices, by means of varying programmed instructions.

Robotics is a multidisciplinary science and engineering branch that integrate mechanical, electrical, software and other fields of study and implementation. It manages the purpose, expansion, and operation of robots, as well as IOT control, sensory feedback, and information handling systems.

- **Robot Kinematics**

The kinematics is mechanical division which explores the action of things without taking account of the forces that cause action. Robot arm kinematics conduct an analytical study of robot arm geometry as a time function, regardless of force / moment, against the fixed reference coordinating system.

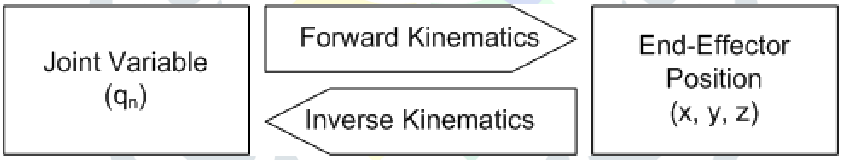

Figure 1: Schematic representing of forward kinematics and inverse kinematics

2. LITERATURE SURVEY

- **Forward Kinematics Problems**

The problems of the forward kinematics are explained by the transformation matrices between an end-effector coordinate frame and base coordinate frame. The position vector is signified by the homogeneous transformation matrix in a 3-D space along with the rotation matrix of the body. The entire homogeneous transformation matrix is generated simply by multiplying specific frame transformations fixed in contiguous chain links. **Denavit and Hartenberg [1]** were the first to make this judgment for the manipulator's spatial geometry, and its benefit is in the universal algorithm to solve the kinematics of a manipulator.

- **Inverse Kinematics Problems**

Indirect kinematics problems can be resolved by using two types of approaches first one is closed form solutions and second is numerical approach. Closed form solutions are robot dependent and faster than the numerical approach. This approach classified into two type of methods Analytical method **[2]** and Geometric method **[3]**. Analytical method it is also called as algebraic method, analytically invert the direct kinematics equations. By solving an algebraic equations scheme, the problem of inverse kinematics can be summarized. If Cosine and Sine values are evaded by fixed substitutes, robot kinematics gives an arithmetical system or a sequence of equations. Then algebraic method is used to solve these equations that are obtained. The numerical approach method can be applied to any kinematic arrangement because it is not robot dependent. It is slower and, in some case, it is not possible to compute the solution. In this approach, there are different methods like symbolic elimination method **[4]**, continuation method **[5]** and iterative method. Symbolic elimination method gives a set of nonlinear equations by eliminating variables to shorten it into a smaller set of equations. Different numbers of iterative methods are using now a day to resolve the inverse kinematics problems. Like Newton-Raphson method **[6]**, Optimization approach **[7,8]**, Cyclic coordinate descent method **[9]**, Pseudoinverse method **[10]**

3. DYNAMIC MODELING OF MANIPULATOR

The set of equations related to the dynamics of a robot arm are its motion equations. A manipulator move at consistent speed, must accelerate, and decelerate amid the work cycle. In order to adjust internal forces and external forces, the time varies from one torque to the other with the help of actuators. In this section, numerical model and properties of dynamic motion equations for the robot arm dynamics have been developed Lagrange-Euler approach.

Lagrange-Euler Mechanics

In physics, Lagrangian mechanics is widely used to solve mechanical problems. In optimisation problems of dynamic systems Lagrange's equations are applied. The Lagrange function $'L'$ which is a scaler function is given by

$$L = Kinetic\ Energy - Potential\ Energy \tag{3.1}$$

To define the manipulator variables based on a set of generalized coordinates dynamic model formulation utilising Lagrange-Euler approach is used. In the generalized coordinates the joint variables are described as displacement 'q'. For prismatic joint which is defined as linear displacement $'d'$ and while for rotary joint defined as angular displacement $'\theta'$. The velocity for prismatic joint $'\dot{q}'$ describes linear velocity $'\dot{d}'$ and velocity for rotary joint $'\dot{q}'$ describes angular velocity '$\dot{\theta}'$.

To obtain the dynamic model for robot arm based on Lagrange-Euler approach is given by the Lagrangian, as a set of equations,

$$\frac{d}{dt}\left(\frac{dL}{d\dot{q}_i}\right) - \frac{dL}{dq_i} = \tau_i \tag{3.2}$$

Here,

L = Lagrangian function.
q_i = Generalized coordinates of the manipulator.
$\dot{q}_i$ = Generalized coordinates of the joint velocity.
τ_i = Generalized force at joint i.

- **Lagrange-Euler Formulation**

The dynamic model of a n degree of freedom robotic arm can be systematically developed using L-E model. The n degree of freedom open kinematic chain serial link manipulator has n joint position,

$$q = [q_1, \dots \dots q_i]^T. \tag{3.3}$$

The derivation of equation of motion utilizing Lagrange-Euler formulation is done in the accompanying subtopics.

Joint Velocity of a Point on the Manipulator

For computing the K.E., link velocity is required.

Let's take a point p on the link i of as revealed in figure (2). The vector r_i^i represents the location of p on the link w.r.t. frame $\{i\}$.

$$r_i^i = [x_i \quad y_i \quad z_i \quad 1]^T \tag{3.4}$$

w.r.t. frame $\{0\}$, the position of point p is given by

$$r_i^0 = T_i^0 r_i^i \tag{3.5}$$

$$T_i^0 = (T_1^0 T_2^1 \ldots\ldots T_i^{i-1}) \tag{3.6}$$

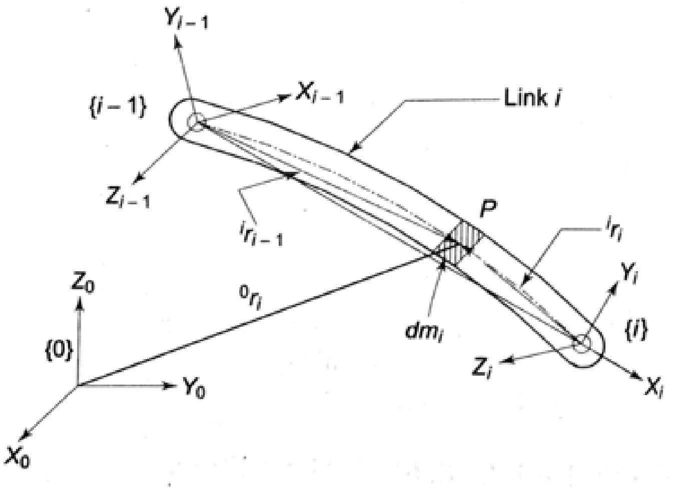

Figure 2: Manipulator's i^{th} link

If joint $'i'$ is prismatic, transformation matrix is

$$T_i^{i-1} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \cos\alpha_i & \sin\alpha_i \sin\theta_i & 0 \\ \sin\theta_i & \cos\theta_i \cos\alpha_i & \sin\alpha_i \cos\theta_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.7}$$

If joint $'i'$ is rotary, the transformation matrix is

$$T_i^{i-1} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \cos\alpha_i & \sin\alpha_i \sin\theta_i & a_i \cos\theta_i \\ \sin\theta_i & \cos\theta_i \cos\alpha_i & \sin\alpha_i \cos\theta_i & a_i \cos\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.8}$$

Considering, $a_{i,}\alpha_i$ are link parameters of robot arm and $d_{i,}\theta_i$ are joint parameters of joint i. The partial derivate of above rotary transformation matrix with respect to θ_i gives,

$$\frac{\partial T_i^{i-1}}{\partial \theta_i} = \begin{bmatrix} -\sin\theta_i & -\cos\theta_i \cos\alpha_i & \sin\alpha_i \cos\theta_i & -a_i \sin\theta_i \\ \cos\theta_i & -\sin\theta_i \cos\alpha_i & \sin\alpha_i \sin\theta_i & a_i \cos\theta_i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.9}$$

The velocity of r_i^i can be expressed as

$$v_i^0 \equiv v_i = \frac{d}{dt}(r_i^0) = \frac{d}{dt}\left(T_i^0 r_i^i \right) \tag{3.10}$$

$$\frac{d}{dt}(r_i^0) = \left(\sum\nolimits_{j=1}^{i} \frac{\partial T_i^0}{\partial q_j} \dot{q}_j \right) r_i^i \tag{3.11}$$

When equation number (3.8) and (3.9) are compared they gives a pattern, that equation (3.9) can be obtained from equation (3.8) by using some matrix operations,

Hence, the partial derivation of homogeneous transformation matrix T_i^{i-1} with respect to θ_i can be obtained using a 4×4 matrix Q_i. For revolute joint Q_i is defined as

$$Q_i = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.12}$$

For prismatic joint Q_i is defined as

$$Q_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.13}$$

and by premultiplying T_i^{i-1} with Q_i,

$$Q_i T_i^{i-1} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta_i & -\sin\theta_i \cos\alpha_i & \sin\alpha_i \sin\theta_i & a_i \cos\theta_i \\ \sin\theta_i & \cos\theta_i \cos\alpha_i & \sin\alpha_j \cos\theta_i & a_i \cos\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.14}$$

$$Q_i T_i^{i-1} = \begin{bmatrix} -\sin\theta_i & -\cos\theta_i \cos\alpha_i & \sin\alpha_i \cos\theta_i & -a_i \sin\theta_i \\ \cos\theta_i & -\sin\theta_i \cos\alpha_i & \sin\alpha_i \sin\theta_i & a_i \cos\theta_i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.15}$$

It is observed that equation (3.9) and (3.13) are same,

$$\frac{\partial T_i^{i-1}}{\partial \theta_i} = Q_i T_i^{i-1} \tag{3.16}$$

Since $T_i^0 = T_1^0 T_2^1 \ldots T_i^{i-1}$, therefore the partial derivative T_i^0 with respect to q_j,

$$\frac{\partial T_i^0}{\partial q_j} = T_1^0 T_2^1 \ldots . T_j^{j-1} \frac{\partial T_j^{j-1}}{\partial q_j} T_{j+1}^j \ldots .. T_i^{i-1} \tag{3.17}$$

After simplifying equation (3.17), the result is valid for $j \leq i$. Therefore,

$$\frac{\partial T_i^0}{\partial q_j} = \begin{cases} T_{j-1}^0 Q_j T_i^{j-1} & for\ j \leq i \\ 0 & for\ j \succ i \end{cases} \tag{3.18}$$

The link velocity v_i as given in equation (3.11), is simplified using equation (3.18),

$$v_i = \left(\sum_{j=1}^{i} T_{j-1}^0 Q_j T_i^{j-i} \dot{q}_j \right) r_i^i \tag{3.19}$$

The Inertia Tensor of the Manipulator

During the motion of links, the mass of links contributes inertia forces. The symmetric matrix is a 4×4 matrix characterizing the mass distributions of a inflexible thing. The moment of inertia tensor is

$$I_i = \begin{bmatrix} \int x_i^2 dm_i & \int x_i y_i dm_i & \int x_i z_i dm_i & \int x_i dm_i \\ \int x_i y_i dm_i & \int y_i^2 dm_i & \int y_i z_i dm_i & \int y_i dm_i \\ \int x_i z_i dm_i & \int y_i z_i dm_i & \int z_i^2 dm_i & \int z_i dm_i \\ \int x_i dm_i & \int y_i dm_i & \int z_i dm_i & \int dm_i \end{bmatrix} \quad (3.20)$$

Where dm_i is the mass of the element on link i located at $r_i^i = [x_i \quad y_i \quad z_i \quad 1]^T$.

$$I_i = \begin{bmatrix} 0.5(-I_{xx}+I_{yy}+I_{zz}) & I_{xy} & I_{xz} & m_i\overline{x_i} \\ I_{xy} & 0.5(I_{xx}-I_{yy}+I_{zz}) & I_{yz} & m_i\overline{y_i} \\ I_{xz} & I_{yz} & 0.5(I_{xx}+I_{yy}-I_{zz}) & m_i\overline{z_i} \\ m_i\overline{x_i} & m_i\overline{y_i} & m_i\overline{z_i} & m_i \end{bmatrix} \quad (3.21)$$

Taking, i^{th} link mass = m_i and center of mass = $\overline{r_i^i} = [\overline{x_i} \quad \overline{y_i} \quad \overline{z_i} \quad 1]$.

Manipulator's Kinetic Energy

K.E. of the small mass dm_i on link i, positioned at r_i^0 and moving with velocity v_i^0 w.r.t. base frame is,

$$dk_i = \frac{1}{2} dm_i (v_i)^2 \quad (3.22)$$

$$(v_i)^2 = v_i . v_i = \dot{r}_i^0 \dot{r}_i^0 = Trace\left(\dot{r}_i^0 \dot{r}_i^{0^T}\right) = Trace(v_i . v_i^T) \quad (3.23)$$

By substituting equation (3.19) in equation (3.22),

$$dk_i = \frac{1}{2} Trace\left[\left(\sum_{j=1}^{i} T_{j-1}^0 Q_j T_i^{j-i} \dot{q}_j r_i^i\right)\left(\sum_{k=1}^{i} T_{k-1}^0 Q_k T_i^{k-i} \dot{q}_k r_i^i\right)^T\right] dm_i \quad (3.24)$$

The total kinetic energy can be given by integration of equation (3.24),

$$k_i = \int dk_i \quad (3.25)$$

$$k_i = \frac{1}{2} Trace\left[\sum_{j=1}^{i}\sum_{k=1}^{i} (T_{j-1}^0 Q_j T_i^{j-i}) \int r_i^i r_i^{i^T} dm_i (T_{k-1}^0 Q_k T_i^{k-i})^T \dot{q}_j \dot{q}_k\right] \quad (3.26)$$

Form equation (3.21), the term $\int r_i^i r_i^{i^T} dm_i$ of equation (3.26) is the moment of inertia tensor I_i, therefore the above equation (3.26) is ,

$$k_i = \frac{1}{2} Trace\left[\sum_{j=1}^{i}\sum_{k=1}^{i} (T_{j-1}^0 Q_j T_i^{j-i})\, I_i (T_{k-1}^0 Q_k T_i^{k-i})^T \dot{q}_j \dot{q}_k\right] \quad (3.27)$$

The final K.E. of manipulator is,

$$K = \sum_{i=1}^{n} k_i$$
$$= \frac{1}{2}\sum_{i=1}^{n} Trace\left[\sum_{j=1}^{i}\sum_{k=1}^{i} (T_{j-1}^0 Q_j T_i^{j-i})\, I_i (T_{k-1}^0 Q_k T_i^{k-i})^T \dot{q}_j \dot{q}_k\right] \quad (3.28)$$

By simplifying equation (3.28),

$$K = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} Trace\left[(T_{j-1}^0 Q_j T_i^{j-i})\, I_i (T_{k-1}^0 Q_k T_i^{k-i})^T\right] \dot{q}_j \dot{q}_k \quad (3.29)$$

$$K = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} Trace\left[\left(U_{ij}\right) I_i (U_{ik})^T\right]\dot{q}_j\dot{q}_k \tag{3.30}$$

Manipulator's Potential Energy

P.E for i^{th} link is

$$p_i = -m_i g\left(\overline{r_i^0}\right) = -m_i g T_i^0(\overline{r_i^i}) \tag{3.31}$$

Considering, $-$ve sign represents the work is done on the system to raise link i against gravity. $\overline{r_i^i}$ represents i^{th} link COM w.r.t. frame $\{i\}$, and $\overline{r_i^0}$ represents i^{th} link COM w.r.t. base frame. The gravity vector w.r.t. bottom frame is $g = \begin{bmatrix} g_x & g_y & g_z & 0 \end{bmatrix}^T$.

Total potential energy is given as,

$$p = \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} -m_i g T_i^0(\overline{r_i^i}) \tag{3.32}$$

Manipulator's Motion Equation

Motion equation of manipulator is formulated as following. After substituting kinetic energy equation (3.30) and potential energy equation (3.32) in Lagrange-Euler equation (3.1), $L = K - P$,

$$L = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} Trace\left[\left(U_{ij}\right) I_i (U_{ik})^T\right]\dot{q}_j\dot{q}_k - \left(\sum_{i=1}^{n} -m_i g T_i^0(\overline{r_i^i})\right) \tag{3.33}$$

The generalized torque τ_i of actuator at joint i, as described in equation (3.2). by substituting above Lagrange-Euler equation (3.33) in equation (3.2), the final equation of motion is,

$$\tau_i = \sum_{j=1}^{n} M_{ij}\,(q)\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n} h_{ijk}\,\dot{q}_j\dot{q}_k + G_i \tag{3.34}$$

Where,

$$M_{ij} = \sum_{p=\max(i,j)}^{n} Trace\left[\left(U_{pj}\right) I_i \left(U_{pk}\right)^T\right] \tag{3.35}$$

$$h_{ijk} = \sum_{p=\max(i,j,k)}^{n} Trace\left[\frac{\partial(d_{pk})}{\partial q_p} I_p U_{pi}^T\right] \tag{3.36}$$

$$G_i = -\sum_{p=i}^{n} m_p g U_{pi}\left(\overline{r_p^p}\right) \tag{3.37}$$

$$U_{ij} = \begin{cases} T_{j-1}^0 Q_j T_i^{j-1} & for\ j \le i \\ 0 & for\ j \succ i \end{cases} \tag{3.38}$$

$$\frac{\partial U_{ij}}{\partial q_k} = \begin{cases} T_{j-1}^0 Q_j T_{k-1}^{j-1} Q_k T_i^{k-1} & for\ i \ge k \ge j \\ T_{k-1}^0 Q_k T_{j-1}^{k-1} Q_j T_i^{j-1} & for\ i \ge j \ge k \\ 0 & for\ i < j\ or\ i < k \end{cases} \tag{3.39}$$

The dynamic model for a manipulator is given in equation (3.34)

DEVELOPMENT OF DYNAMIC MODEL FOR 5-DOF MANIPULATOR

The following is the example to develop dynamic model using Lagrange-Euler equation for 5- link manipulator which tracks square trajectory

Let's consider a 5-link manipulator which is having all revolute joint. The physical dimensions of the links of the manipulator and the link parameter are given in table (3.1).

Table no 3.1:- Physical dimensions and parameters of 5-link manipulator

No. of links (i)	Link Masses $(m)(kg)$	Link Parameters				
		Initial Joint Angle $(\theta_i)(deg)$	Link Length $(a_i)(m)$	Joint Offset Distance $(d_i)(m)$	Twist Angle $(\alpha_i)(deg)$	Initial Joint Angle Velocity $(\dot{\theta}_i)(deg/min)$
1	40	70	0.8	0	0	1
2	20	-30	0.7	0	0	1
3	30	15	0.7	0	0	1
4	20	-40	0.52	0	0	1
5	20	2.5	0.3	0	0	1

Dynamic model for 5 link manipulators as discussed in above section can be formulated using MATLAB

The final homogeneous transformation matrices with respect to base frame {0} is,

$$T_5^0 = \begin{bmatrix} 0.9537 & -0.3007 & 0 & 1.9997 \\ 0.3007 & 0.9537 & 0 & 1.9999 \\ 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 \end{bmatrix} \quad (3.51)$$

$$M = \begin{bmatrix} 389.2631 & 245.9620 & 131.7287 & 41.4152 & 7.2257 \\ 245.9620 & 168.7943 & 93.3589 & 32.7960 & 5.7647 \\ 131.7287 & 93.3589 & 55.4902 & 20.9589 & 3.8246 \\ 41.4152 & 32.7960 & 20.9589 & 10.9277 & 2.1585 \\ 7.2257 & 5.7647 & 3.8246 & 2.1585 & 0.6000 \end{bmatrix} \quad (3.64)$$

4. NUMERICAL SIMULATIONS

In this present section numerical simulations are carry out for a manipulator having five degrees of freedom, all the joint is rotary joint. The link lengths, masses, joint angles, max joint limits are specified in following table (5.1).

Table 5.1: Parameters of 5-DOF manipulator

No. of Links (i)	Link Lengths $(l_i)(m)$	Link Masses $(m_i)(kg)$	Joint Angle $(\theta_i)(deg)$	Max, Joint Limit $(\theta_{max})(deg)$
1	0.8	40	70	180
2	0.7	20	-30	120
3	0.7	30	15	120
4	0.52	20	-40	150
5	0.3	20	2.5	150

The manipulator tracks the trajectory of four lines which forms a square, manipulator starts form a point (2,2) and reaches to same point. The square trajectory for the 1st manipulation variable $r_1^d(t)$ is the constant velocity motion along the square from point (2,2) at time $t = 0\ min$, to the point (2,2) at time $t = 1\ min$.

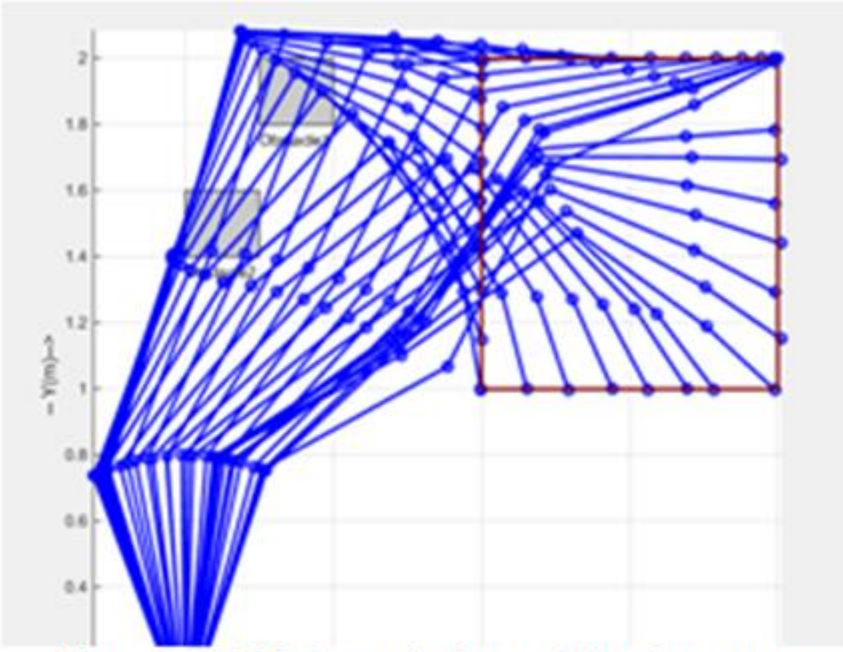

Figure 3 :5-DOF manipulator following square trajectory

From the above numerical simulation, it is observed that the manipulator due to its dynamics study can fulfil the task given.

5. CONCULSION

In this chapter, the idea of dynamic modeling is utilized, which helps to achieve defined manipulator motion in a given workspace

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