Construction of Non-Dyadic Wavelets Family and their Integral for Multiscale Approximation of Unknown Function

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Abstract

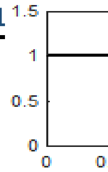
The main focus of the current work is to construct a compactly supported non-dyadic orthonormal wavelet family with scale factor 3. Orthonormal wavelet families are very much helpful in solving the various problems arises in the field of science and technology. For the construction of non-dyadic wavelet family, multi-resolution analysis (MRA) technique is used on trivial Haar scale 3 type function given by C.K Chui. Integrals of members of non-dyadic wavelet family have been calculated for their usein multiscale approximation of unknown function running in various types differential or integral equation. Matrices of Haar Scale 3 wavelets and their integrals have constructed for their use in solving the various types of differential and integral equations. Two numerical experiments have been conducted to test the efficiency of the given wavelet family in approximating the unknown function.

1 Haar Scale 3 Wavelets Family

Let us Consider any two arbitrary integers $\mathcal{A}, \mathcal{B}$ such that $\mathcal{B} > A$. let J be the maximum level of resolution to be considered for phenomena under study. Define new quantities $M, j, k. p$ such that $M = 3^J, p = 3^j$, $j = 0,1,2, \dots J$, $k = 0,1,2, \dots, p-1$, where j denotes the level of resolution and $k$the translation in wavelets. Now divide the interval $[\mathcal{A}, \mathcal{B})$ into $3M$ uniform subinterval of equal length $\Delta t = \frac{\mathcal{B}-\mathcal{A}}{3M}$. When J=0, $\mathcal{A} = 0$ and $\mathcal{B} = 1$ then we have the following members in Haar Scale 3 wavelet family[1] (Haar Scale 3 wavelet family).

$$h_1(t) = \psi^0(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & elsewhere \end{cases} \quad (1.1)$$

$$h_2(t) = \psi^1(t) = \frac{1}{\sqrt{2}} \begin{cases} -1 & 0 \le t < \frac{1}{3} \\ 2 & \frac{1}{3} \le t < \frac{2}{3} \\ -1 & \frac{2}{3} \le t < 1 \\ 0 & elsewhere \end{cases} \quad (1.2)$$



$$h_3(t) = \psi^2(t) = \sqrt{\frac{3}{2}} \begin{cases} 1 & 0 \le t < \frac{1}{3} \\ 0 & \frac{1}{3} \le t < \frac{2}{3} \\ -1 & \frac{2}{3} \le t < 1 \\ 0 & elsewhere \end{cases} \qquad (1.3)$$

Figure 1.1:Haar scale 3 Function

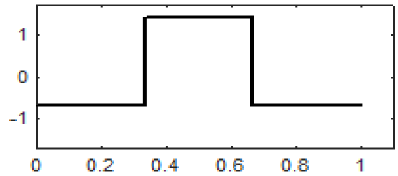


Figure 1.2:Haar wavelet $\boldsymbol{\psi^1(t)}$with dilation 3

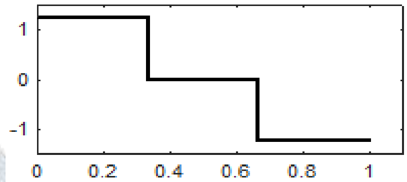


Figure 1.3:Haar wavelet $\boldsymbol{\psi^2(t)}$with dilation 3

Where the function $h_1(t)$ is called father wavelet, $h_2(t)$, $h_3(t)$ are called mother wavelets.

1.1 Construction of Orthonormal Function Spaces[2], [3]

Consider $t \in [0,1)$,then we get

$$\int_{-\infty}^{\infty} \psi^0(t)\psi^0(t)dt = \int_0^1 1.1\, dt = 1 \qquad (1.4)$$

$$\int_{-\infty}^{\infty} \psi^0(t)\psi^0(t-1)dt = \int_0^1 1.0\, dt + \int_1^2 0.1\, dt = 0 + 0 = 0 \qquad (1.5)$$

It follows from Eqs (1.4)-(1.5) and the geometric structure of Haar function that translations of Haar functions are orthonormalbut the scaled translations are only orthogonal, not orthonormal i.e

$$\int_{-\infty}^{\infty} \psi^0(t-k_1)\psi^0(t-k_2)dt = \delta_{m-n} = \begin{cases} 1 & k_1 = k_2 \\ 0 & k_1 \neq k_2 \end{cases}. \qquad (1.6)$$

$$\int_{-\infty}^{\infty} \psi^0(3^j t - k_1)\psi^0(3^j t - k_2)dt = \begin{cases} \frac{1}{3^j} k_1 = k_2 \\ 0 \quad k_1 \neq k_2 \end{cases} \qquad (1.7)$$

where$\psi^0\left(3^j t - k\right) = \begin{cases} 1 & \frac{k}{3^j} \le t < \frac{k+1}{3^j} \\ 0 & elsewhere \end{cases}$ for any integer j and k

But the orthogonal scaled translations can be normalized using the following way

$$\int_{-\infty}^{\infty}\left(3^{\frac{j}{2}}\psi^{0}(3^{j}t-k_1)\right)\left(3^{\frac{j}{2}}\psi^{0}(3^{j}t-k_2)\right)dt=\begin{cases}1 & k_1=k_2\\ 0 & k_1\neq k_2\end{cases} \quad (1.8)$$

Let V_0 be space spanned by the set$\{\psi^0(t),\psi^0(t-1),\psi^0(t-2)\ldots,\}$ which will act as a bases forV_0.Therefore

$$V_0=span\{\psi^0(t-k):k\epsilon\mathbb{Z}\} \quad (1.9)$$

Let V_1 be space spanned by the set$\{\sqrt{3}\,\psi^0(3t),\sqrt{3}\,\psi^0(3t-1),\sqrt{3}\,\psi^0(3t-2),\ldots,\}$ which will act as a bases for$V_1$and we denote it as

$$V_1=span\left\{3^{\frac{1}{2}}\,\psi^0(3t-k):k\epsilon\mathbb{Z}\right\} \quad (1.10)$$

Similarly

$$V_2=span\left\{3^{\frac{2}{2}}\,\psi^0(3^2t-k):k\epsilon\mathbb{Z}\right\} \quad (1.11)$$

$$V_3=span\left\{3^{\frac{3}{2}}\,\psi^0(3^3t-k):k\epsilon\mathbb{Z}\right\} \quad (1.12)$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

$$V_j=span\left\{3^{\frac{j}{2}}\psi^0(3^jt-k):k\epsilon\mathbb{Z}\right\}=\psi^0_{j,k}(\mathrm{t}) \quad (1.13)$$

It is always possible to write the bases of V_0 itself as a linear combination of bases of V_1

$$\psi^0(t)=\frac{1}{\sqrt{3}}\left(\sqrt{3}\,\psi^0(3t)+\sqrt{3}\,\psi^0(3t-1)+\sqrt{3}\,\psi^0(3t-2)\right) \quad (1.14)$$

$$\psi^0(t-1)=\frac{1}{\sqrt{3}}\left(\sqrt{3}\,\psi^0(3t-3)+\sqrt{3}\,\psi^0(3t-4)+\sqrt{3}\,\psi^0(3t-5)\right) \quad (1.15)$$

$$\psi^0(t-2)=\frac{1}{\sqrt{3}}\left(\sqrt{3}\,\psi^0(3t-6)+\sqrt{3}\,\psi^0(3t-7)+\sqrt{3}\,\psi^0(3t-8)\right) \quad (1.16)$$

Also, the bases of V_1 itself as a linear combination of bases of V_2

$$\sqrt{3}\,\psi^0(3t)=\frac{1}{\sqrt{3}}\left(3\psi^0(9t)+3\psi^0(9t-1)+3\psi^0(9t-2)\right) \quad (1.17)$$

$$\sqrt{3}\,\psi^0(3t-1)=\frac{1}{\sqrt{3}}\left(3\psi^0(9t-3)+3\psi^0(9t-4)+3\psi^0(9t-5)\right) \quad (1.18)$$

$$\sqrt{3}\,\psi^0(3t-2)=\frac{1}{\sqrt{3}}\left(3\psi^0(9t-6)+3\psi^0(9t-7)+3\psi^0(9t-8)\right) \quad (1.19)$$

and so, on

Therefore, we can write

$$\ldots V_{-1}\subset V_0\subset V_1\subset V_2\ldots\subset V_\infty \quad (1.20)$$

Where

$$\psi^0(3^j t - k) = \begin{cases} 1 & \frac{k}{3^j} \le t < \frac{k+1}{3^j} \\ 0 & elsewhere \end{cases} \tag{1.21}$$

Now we will test the orthonormality of mother wavelets $\psi^1(t)$,$\psi^2(t)$ where

$$\psi^1(t) = \frac{1}{\sqrt{2}}\begin{cases} -1 & t \in \left[0, \frac{1}{3}\right) \\ 2 & t \in \left[\frac{1}{3}, \frac{2}{3}\right) \\ -1 & t \in \left[\frac{2}{3}, 1\right) \end{cases}, \quad \psi^2(t) = \sqrt{\frac{3}{2}}\begin{cases} 1 & t \in \left[0, \frac{1}{3}\right) \\ 0 & t \in \left[\frac{1}{3}, \frac{2}{3}\right) \\ -1 & t \in \left[\frac{2}{3}, 1\right) \end{cases} \tag{1.22}$$

Consider $t \in [0,1)$,then

$$\int_{-\infty}^{\infty} \psi^1(t)\psi^1(t)dt =$$

$$\int_0^{\frac{1}{3}} \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right) dt + \int_{\frac{1}{3}}^{\frac{2}{3}} \left(\frac{2}{\sqrt{2}}\right)\left(\frac{2}{\sqrt{2}}\right) dt + \int_{\frac{2}{3}}^{1} \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right) dt = \frac{1}{3}\left(\frac{1}{2} + \frac{4}{2} + \frac{1}{2}\right) \tag{1.23}$$

$$= 1$$

$$\int_{-\infty}^{\infty} \psi^2(t)\psi^2(t)dt =$$

$$\int_0^{\frac{1}{3}} \left(\sqrt{\frac{3}{2}}\right)\left(\sqrt{\frac{3}{2}}\right) dt + \int_{\frac{1}{3}}^{\frac{2}{3}} (0)(0)\, dt + \int_{\frac{2}{3}}^{1} \left(-\sqrt{\frac{3}{2}}\right)\left(-\sqrt{\frac{3}{2}}\right) dt = \frac{1}{3}\left(\frac{3}{2} + 0 + \frac{3}{2}\right) \tag{1.24}$$

$$= 1$$

$$\int_{-\infty}^{\infty} \psi^1(t)\psi^2(t)dt =$$

$$\int_0^{\frac{1}{3}} \left(\frac{-1}{\sqrt{2}}\right)\left(\sqrt{\frac{3}{2}}\right) dt + \int_{\frac{1}{3}}^{\frac{2}{3}} \left(\frac{2}{\sqrt{2}}\right)(0)\, dt + \int_{\frac{2}{3}}^{1} \left(\frac{-1}{\sqrt{2}}\right)\left(-\sqrt{\frac{3}{2}}\right) dt \tag{1.25}$$

$$= \frac{1}{3}\left(-\frac{\sqrt{3}}{2} + 0 + \frac{\sqrt{3}}{2}\right) = 0$$

It follows from Eqs (1.23)-(1.25) and the geometric structure of the wavelets that Set $\{\psi^1(t-k_1), \psi^2(t-k_2): k_1, k_2 \epsilon \mathbb{Z}\}$ is orthonormal. Let W_0 be space spanned by the set of bases $\{\psi^1(t), \psi^2(t), \psi^1(t-1), \psi^2(t-2), \dots\}$.

$$\int_{-\infty}^{\infty} \psi^1(3t)\psi^1(3t)dt =$$

$$\int_0^{\frac{1}{9}} \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right) dt + \int_{\frac{1}{9}}^{\frac{2}{9}} \left(\frac{2}{\sqrt{2}}\right)\left(\frac{2}{\sqrt{2}}\right) dt + \int_{\frac{2}{9}}^{\frac{3}{9}} \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right) dt = \frac{1}{9}\left(\frac{1}{2} + \frac{4}{2} + \frac{1}{2}\right) = \frac{1}{3}$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} \sqrt{3}\,\psi^1(3t)\sqrt{3}\,\psi^1(3t)dt = 1 \tag{1.26}$$

$$\int_{-\infty}^{\infty} \psi^2(3t)\psi^2(3t)dt =$$

$$\int_0^{\frac{1}{9}} \left(\sqrt{\frac{3}{2}}\right)\left(\sqrt{\frac{3}{2}}\right) dt + \int_{\frac{1}{9}}^{\frac{2}{9}} (0)(0)\, dt + \int_{\frac{2}{9}}^{\frac{3}{9}} \left(-\sqrt{\frac{3}{2}}\right)\left(-\sqrt{\frac{3}{2}}\right) dt = \frac{1}{9}\left(\frac{3}{2} + 0 + \frac{3}{2}\right) = \frac{1}{3}$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} \sqrt{3}\,\psi^2(3t)\sqrt{3}\,\psi^2(3t)dt = 1 \qquad (1.27)$$

$$\int_{-\infty}^{\infty} \psi^1(3t)\psi^2(3t)dt =$$

$$\int_0^{\frac{1}{9}} \left(\frac{-1}{\sqrt{2}}\right)\left(\sqrt{\frac{3}{2}}\right) dt + \int_{\frac{1}{9}}^{\frac{2}{9}} \left(\frac{2}{\sqrt{2}}\right)(0)\, dt + \int_{\frac{2}{9}}^{\frac{3}{9}} \left(\frac{-1}{\sqrt{2}}\right)\left(-\sqrt{\frac{3}{2}}\right) dt = \frac{1}{9}\left(-\frac{\sqrt{3}}{2} + 0 + \frac{\sqrt{3}}{2}\right) = 0$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} \sqrt{3}\,\psi^1(3t)\sqrt{3}\,\psi^2(3t)dt = 0 \qquad (1.28)$$

Set $\{\sqrt{3}\psi^1(3t-k_1), \sqrt{3}\psi^2(3t-k_2): k_1, k_2 \epsilon \mathbb{Z}\}$ is orthonormal. Let W_1 be space spanned by the set of bases$\{\sqrt{3}\,\psi^1(3t), \sqrt{3}\psi^2(3t), \sqrt{3}\psi^1(3t-1), \sqrt{3}\psi^2(3t-1), \ldots\}$.

From the geometric properties, it has been verified that the linear combinations of the elements of W_1 are not able to create W_0. i.e W_0 is not a subset of W_1 .But

$$\int_{-\infty}^{\infty} \psi^1(t)\psi^1(3t)dt =$$

$$\int_0^{\frac{1}{9}} \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right) dt + \int_{\frac{1}{9}}^{\frac{2}{9}} \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{2}{\sqrt{2}}\right) dt + \int_{\frac{2}{9}}^{\frac{3}{9}} \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right) dt = \frac{1}{9}\left(\frac{1}{2} - \frac{2}{2} + \frac{1}{2}\right) \qquad (1.29)$$

$$= 0$$

$$\int_{-\infty}^{\infty} \psi^2(t)\psi^2(3t)dt =$$

$$\int_0^{\frac{1}{9}} \left(\sqrt{\frac{3}{2}}\right)\left(\sqrt{\frac{3}{2}}\right) dt + \int_{\frac{1}{9}}^{\frac{2}{9}} \left(\sqrt{\frac{3}{2}}\right)(0)\, dt + \int_{\frac{2}{9}}^{\frac{3}{9}} \left(\sqrt{\frac{3}{2}}\right)\left(-\sqrt{\frac{3}{2}}\right) dt \qquad (1.30)$$

$$= \frac{1}{9}\left(\frac{3}{2} + 0 - \frac{3}{2}\right) = 0$$

$$\int_{-\infty}^{\infty} \psi^1(t)\psi^2(3t)dt =$$

$$\int_0^{\frac{1}{9}} \left(\frac{-1}{\sqrt{2}}\right)\left(\sqrt{\frac{3}{2}}\right) dt + \int_{\frac{1}{9}}^{\frac{2}{9}} \left(\frac{-1}{\sqrt{2}}\right)(0)\, dt + \int_{\frac{2}{9}}^{\frac{3}{9}} \left(\frac{-1}{\sqrt{2}}\right)\left(-\sqrt{\frac{3}{2}}\right) dt \tag{1.31}$$

$$= \frac{1}{9}\left(-\frac{\sqrt{3}}{2} - 0 + \frac{\sqrt{3}}{2}\right) = 0$$

$$\int_{-\infty}^{\infty} \psi^2(t)\psi^1(3t)dt =$$

$$\int_0^{\frac{1}{9}} \left(\sqrt{\frac{3}{2}}\right)\left(\frac{-1}{\sqrt{2}}\right) dt + \int_{\frac{1}{9}}^{\frac{2}{9}} \left(\sqrt{\frac{3}{2}}\right)\left(\frac{2}{\sqrt{2}}\right) dt + \int_{\frac{2}{9}}^{\frac{3}{9}} \left(\sqrt{\frac{3}{2}}\right)\left(\frac{-1}{\sqrt{2}}\right) dt \tag{1.32}$$

$$= \frac{1}{9}\left(-\frac{\sqrt{3}}{2} + \frac{2\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) = 0$$

Therefore $W_0 \perp W_1$.

Similarly, if

$$W_2 = span\{3\,\psi^1(3^2t - k_1), 3\,\psi^2(3^2t - k_2): k_1, k_2 \epsilon \mathbb{Z}\} \tag{1.33}$$

$$W_3 = span\left\{3^{\frac{3}{2}}\psi^1(3^3t - k_1), 3^{\frac{3}{2}}\psi^2(3^3t - k_2): k_1, k_2 \epsilon \mathbb{Z}\right\} \tag{1.34}$$

$$\vdots$$

$$W_j = span\left\{3^{\frac{j}{2}}\psi^1(3^jt - k_1), 3^{\frac{j}{2}}\psi^2(3^jt - k_2): k_1, k_2 \epsilon \mathbb{Z}\right\} \tag{1.35}$$

Then, it can easily be proved

$$\cdots W_{-2} \perp W_{-1} \perp W_0 \perp W_1 \perp W_2 \cdots \tag{1.36}$$

Also

$$W_j = span\left\{3^{\frac{j}{2}}\psi^1(3^jt - k_1), k_1 \epsilon \mathbb{Z}\right\} \oplus \left\{3^{\frac{j}{2}}\psi^2(3^jt - k_2):, k_2 \epsilon \mathbb{Z}\right\} = W_j^1 \oplus W_j^2$$

$$W_j = W_j^1 \oplus W_j^2 \tag{1.37}$$

Where

$$W_j^1 = span\left\{3^{\frac{j}{2}}\psi^1(3^jt - k_1), k_1 \epsilon \mathbb{Z}\right\}, W_j^2 = span\left\{3^{\frac{j}{2}}\psi^2(3^jt - k_2):, k_2 \epsilon \mathbb{Z}\right\}$$

$$\psi^1(3^jt - k_1) = \frac{1}{\sqrt{2}}\begin{cases} -1 & \frac{k_1}{3^j} \le t < \frac{3k_1+1}{3^{j+1}} \\ 2 & \frac{3k_1+1}{3^{j+1}} \le t < \frac{3k_1+2}{3^{j+1}} \\ -1 & \frac{3k_1+2}{3^{j+1}} \le t < \frac{3k_1+3}{3^{j+1}} \\ 0 & elsewhere \end{cases} \tag{1.38}$$

$$\psi^2(3^jt-k_2)=\sqrt{\frac{3}{2}}\begin{cases}1 & \frac{k_2}{3^j}\leq t<\frac{3k_2+1}{3^{j+1}}\\ 0 & \frac{3k_2+1}{3^{j+1}}\leq t<\frac{3k_2+2}{3^{j+1}}\\ -1 & \frac{3k_2+2}{3^{j+1}}\leq t<\frac{3k_2+3}{3^{j+1}}\\ 0 & elsewhere\end{cases} \quad (1.39)$$

As $V_0\subset V_1$ then what is missing in V_0 in comparison with V_1 .Now we will investigate it with the help of the following function graph. Consider the arbitrary function $f(t)$

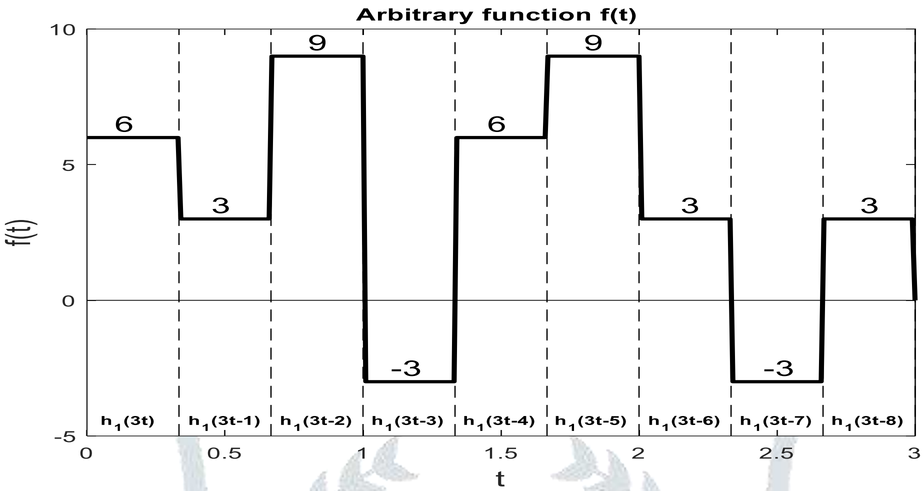


Figure 1.4:Arbitrary Function in 2D

Now we will express $f(t)$ in terms of the bases of V_1 as follows

$$\begin{aligned}f(t)&=6\psi^0(3t)+3\psi^0(3t-1)+9\psi^0(3t-2)-3\psi^0(3t-3)+6\psi^0(3t-4)\\&\quad+9\psi^0(3t-3)+3\psi^0(3t-3)-3\psi^0(3t-3)+3\psi^0(3t-3)\end{aligned} \quad (1.40)$$

It can be seen from the function graph that terms of the above function in the different unit intervals can be expressible in terms of the bases of V_0, W_0^1, W_0^2 as follows

$$6\psi^0(3t)+3\psi^0(3t-1)+9\psi^0(3t-2)=6\psi^0(t)-\frac{3\sqrt{2}}{2}\psi^1(t)-\sqrt{\frac{3}{2}}\psi^2(t) \quad (1.41)$$

$$\begin{aligned}&9\psi^0(3t-5)+6\psi^0(3t-4)-3\psi^0(3t-3)\\&\qquad=4\psi^0(t-1)+\sqrt{2}\psi^1(t-1)-2\sqrt{6}\psi^2(t-1)\end{aligned} \quad (1.42)$$

$$\begin{aligned}&3\psi^0(3t-6)-3\psi^0(3t-7)+3\psi^0(3t-8)\\&\qquad=\psi^0(t-2)-2\sqrt{2}\psi^1(t-2)-0\,\psi^2(t-2)\end{aligned} \quad (1.43)$$

$$\begin{aligned}f(t)&=6\psi^0(t)-\tfrac{3\sqrt{2}}{2}\psi^1(t)-\sqrt{\tfrac{3}{2}}\psi^2(t)+4\psi^0(t-1)+\sqrt{2}\psi^1(t-1)-\\&2\sqrt{6}\psi^2(t-1)+\psi^0(t-2)-2\sqrt{2}\psi^1(t-2)-0\,\psi^2(t-2)\end{aligned} \quad (1.44)$$

Now f(t) has been completely expressed in terms of the bases of V_0, W_0^1, W_0^2.Similar way it can easily be proved that V_0 is also orthogonal to W_0^1, W_0^2.

Therefore, any function in V_1 can be expressed in terms of the bases of V_0 and W_0 where $W_0 = W_0^1 \oplus W_0^2$.

Mathematically one can say

$$V_1 = V_0 \oplus W_0 = V_0 \oplus W_0^1 \oplus W_0^2 \qquad (1.45)$$

Similarly, using similar arguments we can prove

$$V_2 = V_1 \oplus W_1 = V_0 \oplus W_1^1 \oplus W_1^2 \qquad (1.46)$$

$$V_3 = V_2 \oplus W_2 = V_2 \oplus W_2^1 \oplus W_2^2 \qquad (1.47)$$

$$\vdots$$

$$\vdots$$

$$V_{j-1} = V_{j-2} \oplus W_{j-2} = V_{j-2} \oplus W_{j-2}^1 \oplus W_{j-2}^2 \qquad (1.48)$$

$$V_j = V_{j-1} \oplus W_{j-1} = V_{j-1} \oplus W_{j-1}^1 \oplus W_{j-1}^2 \qquad (1.49)$$

$$V_j = V_0 \oplus W_0^1 \oplus W_0^2 \oplus \cdots\cdots\cdots \oplus W_{j-2}^1 \oplus W_{j-2}^2 \oplus W_{j-1}^1 \oplus W_{j-1}^2 \qquad (1.50)$$

which mean each square-integrable function $f(t)$ can be expressible in terms of the bases of V_0 ,W_j^1, W_j^2 $j =$ 01,2,3 … … …

1.2 Multi-resolution analysis (MRA)

Now it is clear from the above discussion that the sequence of closed subspaces of $W_j, V_j \subset L_2(R), j \in \mathbb{Z}$ of $L_2(R)$ space satisfies the following properties

a) $\psi^0(t) \in V_0 \Rightarrow \psi^0(3^j t) \in V_j$

b) $\psi^0(t) \in V_0 \Rightarrow \psi^0(3^j t - k) \in V_j$

c) $\psi^i(t) \in W_0^i,\ i = 1,2 \Rightarrow \psi^i(3^j t) \in W_j^i$

d) $\psi^i(t) \in W_0^i,\ i = 1,2 \Rightarrow \psi^i(3^j t - k) \in W_j^i$

e) $W_J = W_j^1 \oplus W_j^2 = \oplus W_j^i$, $i = 1,2$

f) $\cdots \subset V_0 \subset V_1 \subset V_2 \subset V_3 \subset V_4 \subset \cdots$

g) $\cdots \perp W_0 \perp W_1 \perp W_2 \perp W_3 \perp W_4 \perp \cdots$

h) $V_j = V_0 \oplus \sum_{i=0}^{j-1} W_j^1 \oplus \sum_{i=0}^{j-1} W_j^2$

i) $\psi^0(t) \in V_0 \Rightarrow \psi^0(\mathrm{t}-\mathrm{k}) \in V_0; k \in \mathbb{Z}$ is a Riesz Basis in V_0

Process of designing the orthonormal wavelet family using the sequence of closed subspace $W_j, V_j, j \in \mathbb{Z}$ of $L_2(R)$, which satisfies the above set of properties is also known as Multi-resolution analysis (MRA)[4].The approximation of arbitrary function using the members of these orthonormal wavelet families is known as a multiscale approximation (MSA). The multiscale approximation is one of the modern numerical frameworks to find the solution of various types of differential and integral equation arises in the field of science and technology.

Now the above-described process, Haar Scale 3 wavelet family is obtained as follows:

$For\ i = 1$

$$h_i(t) = \varphi(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & elsewhere \end{cases} \tag{1.51}$$

$For \quad i = 2,4, \dots 3p - 1$

$$h_i(t) = \psi^1(3^j t - k) = \frac{1}{\sqrt{2}} \begin{cases} -1 & \alpha_1(i) \le t < \alpha_2(i) \\ 2 & \alpha_2(i) \le t < \alpha_3(i) \\ -1 & \alpha_3(i) \le t < \alpha_4(i) \\ 0 & elsewhere \end{cases} \tag{1.52}$$

$For \quad i = 3,5,7 \dots 3p$

$$h_i(t) = \psi^2(3^j t - k) = \sqrt{\frac{3}{2}} \begin{cases} 1 & \alpha_1(i) \le t < \alpha_2(i) \\ 0 & \alpha_2(i) \le t < \alpha_3(i) \\ -1 & \alpha_3(i) \le t < \alpha_4(i) \\ 0 & elsewhere \end{cases} \tag{1.53}$$

where $\alpha_1(i) = \frac{k}{p}$, $\alpha_2(i) = \frac{3k+1}{3p}$, $\alpha_3(i) = \frac{(3k+2)}{3p}$,$\alpha_4(i) = \frac{k+1}{p}$, $p = 3^j$, $j = 0,1,2, \dots$, $k = 0,1,2, \dots, p - 1$.

Table 1.1:Relationship between the wavelet number i ,dilation parameter j and translation parameter k for even members of wavelet family

i	2	4	6	8	10	12	14	16	18	20	22	24	26
j	0	1	1	1	2	2	2	2	2	2	2	2	2
k	0	0	1	2	0	1	2	3	4	5	6	7	8
m	1	3	3	3	9	9	9	9	9	9	9	9	9

Table 1.2:Relationship between the wavelet number i ,dilation parameter j and translation parameter k for odd members of wavelet family

i	1	3	5	7	9	11	13	15	17	19	21	23	25	27
j	0	0	1	1	1	2	2	2	2	2	2	2	2	2
k	0	0	0	1	2	0	1	2	3	4	5	6	7	8
m	1	1	3	3	3	9	9	9	9	9	9	9	9	9

The wavelet number $i > 1$ is calculated from the relation $i = p + 2k + 2$ (for even index) and $i = p + 2k + 1$(for odd index). j represents the level of dilation of the and k represents the translation parameters of the wavelet. The function $h_1(t)$ is called father wavelet, $h_2(t)$, $h_3(t)$ mother wavelet and all other functions $h_4(t)$, $h_5(t), h_6(t) \dots \dots$ are generated from the translation and dilation of the mother wavelet. For even index $i, \psi^1(t)$ will be considered and for odd index $i, \psi^2(t)$ will be considered.

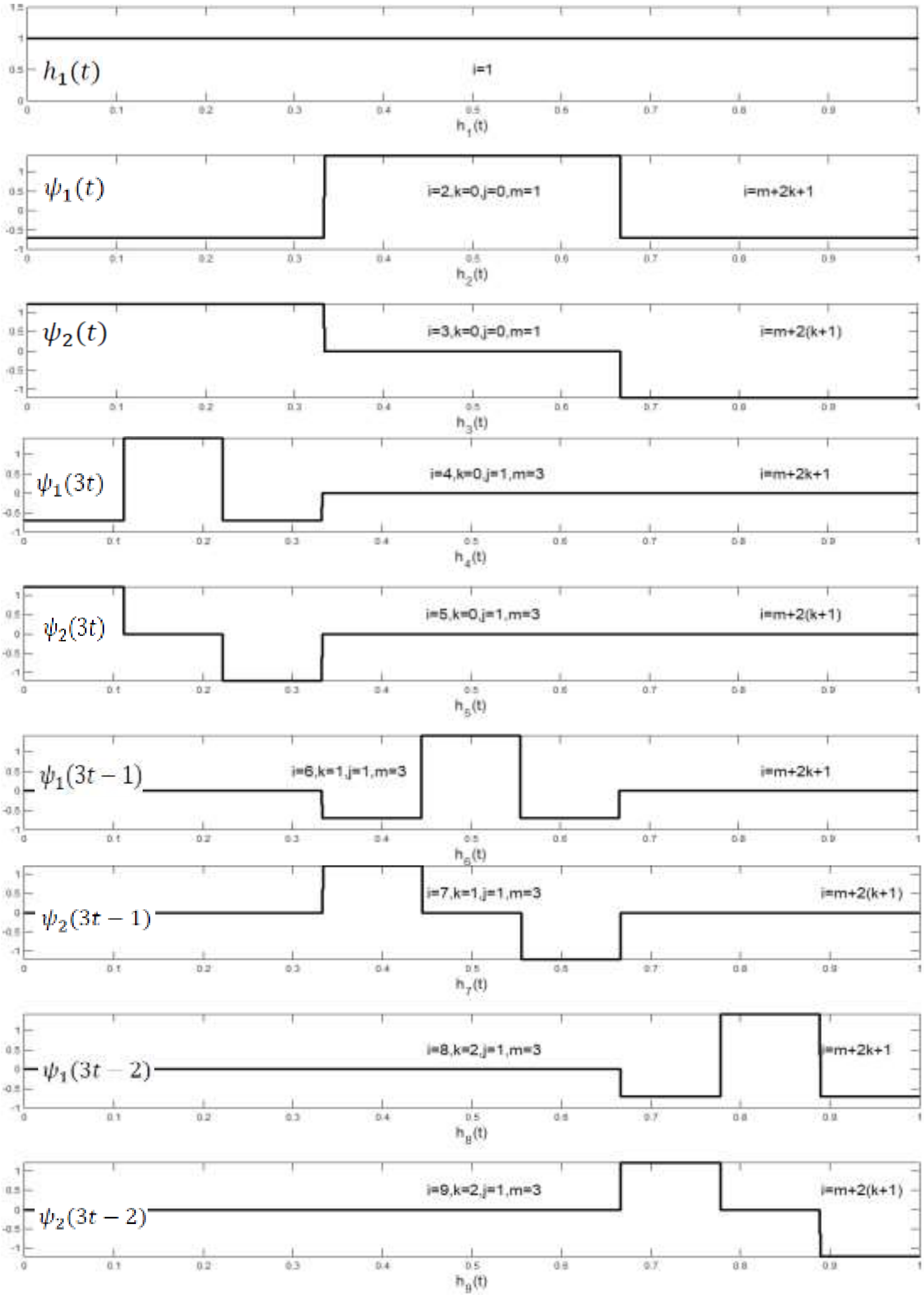


Figure 1.5: Members of the Haar Scale 3 wavelet family ($h_1(t) - h_9(t)$) at $J = 0$ and $J = 1$

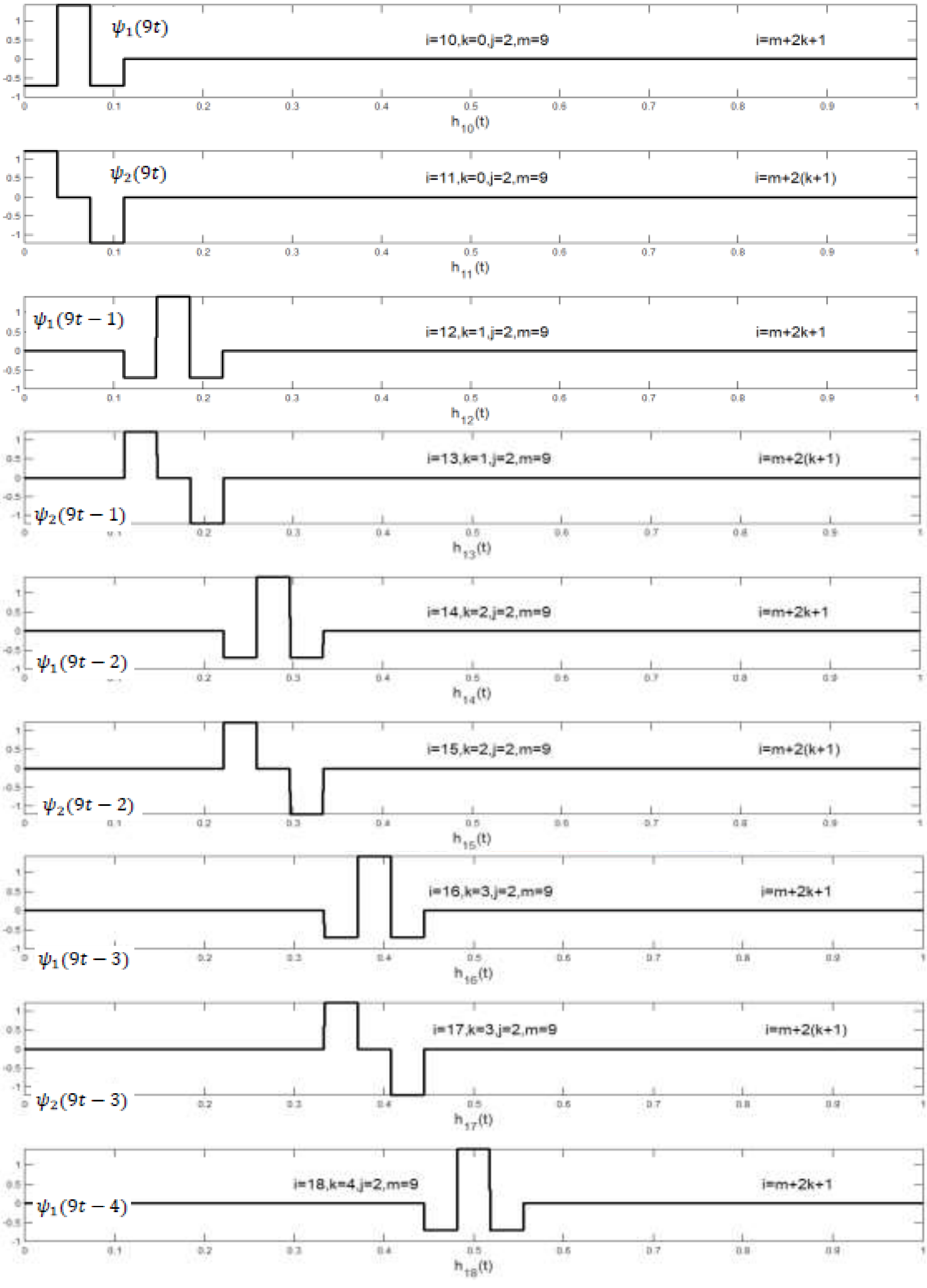


Figure 1.6: Members of the Haar Scale 3 wavelet family ($h_{10}(t) - h_{18}(t)$) at third level of resolution $J = 2$

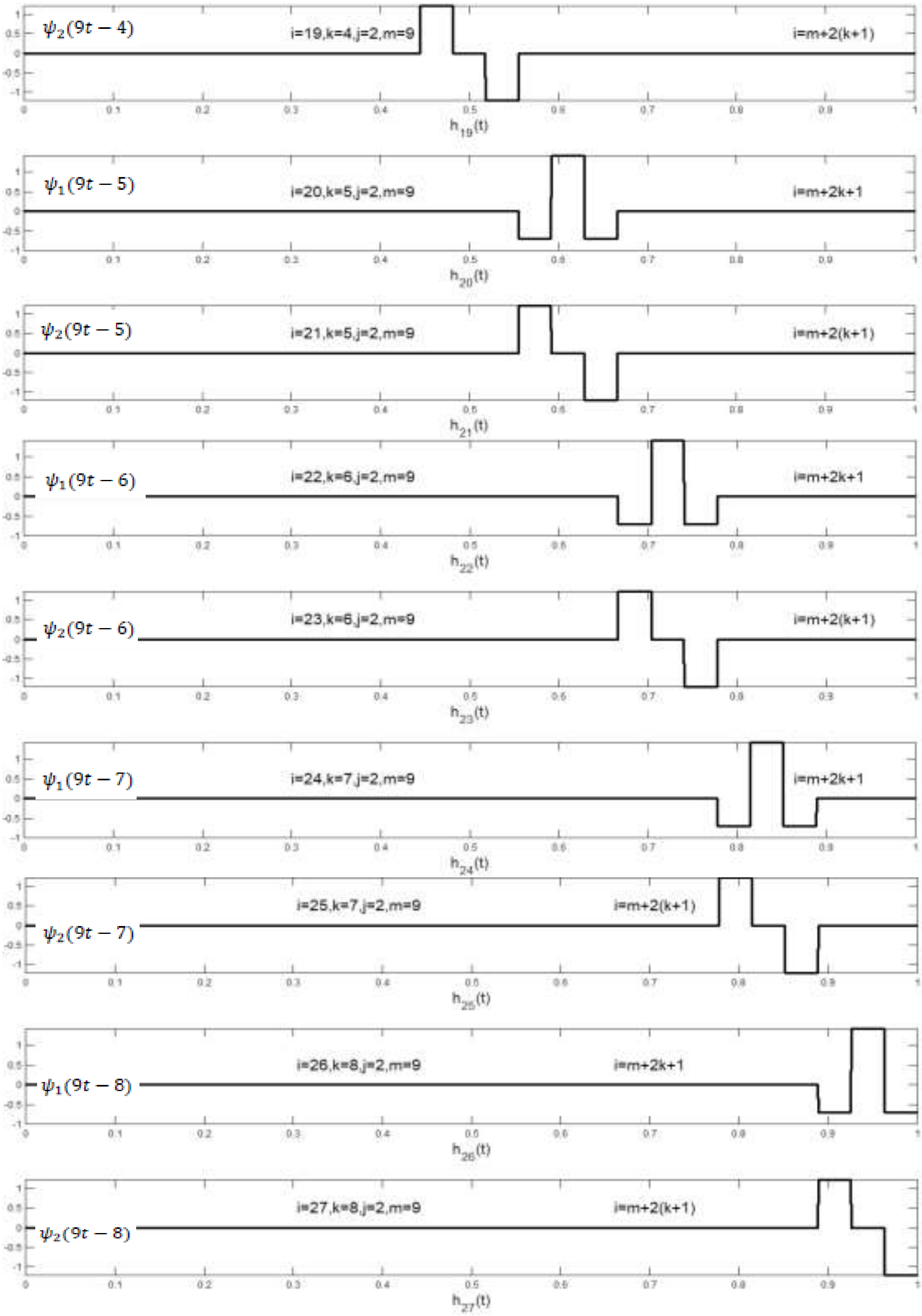


Figure 1.7: Members of Haar Scale 3 wavelet family ($h_{19}(t) - h_{27}(t)$) at the third level of resolution $J = 2$

2 Integrals of Haar Scale 3 Wavelet Family

we can integrate over the interval [A, B) as many times as required by using the formula given below

$$q_{\beta,i}(t) = \int_A^t \int_A^t \int_A^t \cdots\cdots\cdots \beta \ times \cdots\cdots\cdots \int_A^t h_i(x)(dx)^\beta$$

$$q_{\beta,i}(t) = \frac{1}{(\beta-1)!}\int_A^t (t-x)^{\beta-1} h_i(x)dx$$
$$\forall\ \beta = 1,2,3 \dots\dots \ , \quad i = 1,2,3,\dots\dots 3p \qquad (2.1)$$

For $i = 1$ the value of $h_i(t) = \begin{cases} 1 & A \le t < B \\ 0 & elsewhere \end{cases}$

Therefore

$$q_{1,1}(t) = \frac{1}{(1-1)!}\int_A^t (t-x)^{1-1} h_1(x)dx = \int_A^t 1\, dx = (t-A)$$
$$q_{2,1}(t) = \frac{1}{(2-1)!}\int_A^t (t-x)^{2-1} h_1(x)dx = \int_A^t (t-x)\, dx = \frac{1}{2!}(t-A)^2$$
$$q_{3,1}(t) = \frac{1}{(3-1)!}\int_A^t (t-x)^{3-1} h_1(x)dx = \frac{1}{2}\int_A^t (t-x)^2 dx = \frac{1}{3!}(t-A)^3$$

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$$q_{\beta,1}(t) = \frac{1}{(\beta-1)!}\int_A^t (t-x)^{\beta-1} h_i(x)dx \qquad (2.2)$$
$$= \frac{1}{(\beta-1)!}\int_A^t (t-x)^{\beta-1}dx = \frac{1}{\beta!}(t-A)^{\beta} \forall\ \beta$$

For an even integer $i = 2,4,6,8,\cdots,3p-1$

$$h_i(t) = \psi^1(3^j t - k) = \frac{1}{\sqrt{2}}\begin{cases} -1 & \alpha_1(i) \le t < \alpha_2(i) \\ 2 & \alpha_2(i) \le t < \alpha_3(i) \\ -1 & \alpha_3(i) \le t < \alpha_4(i) \\ 0 & elsewhere \end{cases}$$

$$q_{1,i}(t) = \frac{1}{(1-1)!}\int_A^t (t-x)^{1-1} h_i(x)dx = \int_A^t h_i(x)\, dx \qquad (2.3)$$
$$\forall\ i = 2,4,6,8,\cdots,3p-1$$

When $t \in [A, \alpha_1(i))$ then

$$q_{1,i}(t) = \int_A^t h_i(x)\, dx = \int_A^t 0\, dx = 0$$

When $t \in [\alpha_1(i),\ \alpha_2(i))$ then

$$q_{1,i}(t) = \int_A^{\alpha_1(i),} h_i(x)\, dx + \int_{\alpha_1(i),}^t h_i(x)\, dx$$

$$= \int_{A}^{\alpha_1(i),} 0\, dx + \int_{\alpha_1(i)}^{t} \frac{-1}{\sqrt{2}}\, dx = \frac{-1}{\sqrt{2}} \frac{\left(t - \alpha_1(i)\right)}{1!}$$

When $t \in [\alpha_2(i),\ \alpha_3(i))$ then

$$q_{1,i}(t) = \int_{A}^{\alpha_1(i),} h_i(x)\, dx + \int_{\alpha_1(i),}^{\alpha_2(i)} h_i(x)\, dx + \int_{\alpha_2(i),}^{t} h_i(x)\, dx$$

$$= \int_{A}^{\alpha_1(i),} 0\, dx + \int_{\alpha_1(i),}^{\alpha_2(i)} \frac{-1}{\sqrt{2}}\, dx + \int_{\alpha_2(i),}^{t} \frac{2}{\sqrt{2}}\, dx$$

$$= -\frac{1}{\sqrt{2}}\left(\alpha_2(i) - \alpha_1(i)\right) + \frac{2}{\sqrt{2}}\left(t - \alpha_2(i)\right) = \frac{1}{\sqrt{2}}(2t - 3\alpha_2(i) + \alpha_1(i)$$

$$= \frac{1}{\sqrt{2}}\left(-\left(t - \alpha_1(i)\right)^1 + 3\left(t - \alpha_2(i)\right)^1\right)$$

When $t \in [\alpha_3(i), \alpha_4(i))$ then

$$q_{1,i}(t) = \int_{A}^{\alpha_1(i)} h_i(x)\, dx + \int_{\alpha_1(i)}^{\alpha_2(i)} h_i(x)\, dx + \int_{\alpha_2(i)}^{\alpha_3(i)} h_i(x)\, dx + \int_{\alpha_3(i)}^{t} h_i(x)\, dx$$

$$= \frac{1}{\sqrt{2}}\left(\int_{A}^{\alpha_1(i)} 0\, dx + \int_{\alpha_1(i)}^{\alpha_2(i)} -1\, dx + \int_{\alpha_2(i)}^{\alpha_3(i)} 2\, dx + \int_{\alpha_3(i)}^{t} -1\, dx\right)$$

$$= \frac{1}{\sqrt{2}}\left(0 - \left(\alpha_2(i) - \alpha_1(i)\right) + 2\left(\alpha_3(i) - \alpha_2(i)\right) - (t - \alpha_3(i))\right)$$

$$= \frac{1}{\sqrt{2}}\left(-t + \alpha_1(i) - 3\alpha_2(i) + 3\alpha_3(i)\right) = \frac{1}{\sqrt{2}}\left(-t + \alpha_1(i) - 3\alpha_2(i) + 3\alpha_3(i)\right)$$

$$= \frac{1}{\sqrt{2}}\left(-\left(t - \alpha_1(i)\right)^1 + 3\left(t - \alpha_2(i)\right)^1 - 3\left(t - \alpha_3(i)\right)^1\right)$$

When $t \in [\alpha_4(i), B)$ then

$$q_{1,i}(t) = \int_{A}^{\alpha_1(i)} h_i(x)\, dx + \int_{\alpha_1(i)}^{\alpha_2(i)} h_i(x)\, dx + \int_{\alpha_2(i)}^{\alpha_3(i)} h_i(x)\, dx + \int_{\alpha_3(i)}^{\alpha_4(i)} h_i(x)\, dx + \int_{\alpha_4(i)}^{t} h_i(x)\, dx$$

$$= \frac{1}{\sqrt{2}}\left(\int_{A}^{\alpha_1(i)} 0\, dx + \int_{\alpha_1(i)}^{\alpha_2(i)} -1\, dx + \int_{\alpha_2(i)}^{\alpha_3(i)} 2\, dx + \int_{\alpha_3(i)}^{\alpha_4(i)} -1\, dx + \int_{\alpha_4(i)}^{t} 0\, dx\right)$$

$$=\frac{1}{\sqrt{2}}\left(0-\left(\alpha_2(i)-\alpha_1(i)\right)+2\left(\alpha_3(i)-\alpha_2(i)\right)-\left(\alpha_4(i)-\alpha_3(i)\right)+0\right)$$

$$=\frac{1}{\sqrt{2}}\left(\alpha_1(i)-3\alpha_2(i)+3\alpha_3(i)-\alpha_4(i)\right)$$

$$=\frac{1}{\sqrt{2}}\left(-\left(t-\alpha_1(i)\right)^1+3\left(t-\alpha_2(i)\right)^1-3\left(t-\alpha_3(i)\right)^1+(t-\alpha_4(i))^1\right)$$

Therefore, the first integral of the even members Haar Scale 3 wavelets family is given by

$$q_{1,i}(t)=\frac{1}{\sqrt{2}}\begin{cases} 0 & for \quad 0\le t\le\alpha_1(i) \\ \frac{-1}{1!}\left(t-\alpha_1(i)\right)^1 & for \quad \alpha_1(i)\le t\le\alpha_2(i) \\ \frac{1}{1!}\left[-\left(t-\alpha_1(i)\right)^1+3\left(t-\alpha_2(i)\right)^1\right] & for \quad \alpha_2(i)\le t\le\alpha_3(i) \\ \frac{1}{1!}\left[-\left(t-\alpha_1(i)\right)^1+3\left(t-\alpha_2(i)\right)^1-3\left(t-\alpha_3(i)\right)^1\right] & for\ \alpha_3(i)\le t\le\alpha_4(i) \\ \frac{1}{1!}\left[-\left(t-\alpha_1(i)\right)^1+3\left(t-\alpha_2(i)\right)^1-3\left(t-\alpha_3(i)\right)^1+\left(t-\alpha_4(i)\right)^1\right] & for\ \alpha_4(i)\le t\le 1 \end{cases} \tag{2.4}$$

Now second integrals take $\beta=2$

$$q_{2,i}(t)=\frac{1}{(2-1)!}\int_A^t (t-x)^{2-1}h_i(x)dx=\int_A^t (t-x)h_i(x)dx \tag{2.5}$$

$\forall\ i=2,4,6,8,\cdots,3p-1$

When $t\in[A,\alpha_1(i))$ then

$$q_{2,i}(t)=\int_A^t (t-x)h_i(x)dx=\int_A^t (t-x)\ 0\ dx=0$$

When $t\in\left[\alpha_1(i),\ \alpha_2(i)\right)$ then

$$q_{2,i}(t)=\int_A^{\alpha_1(i)}(t-x)h_i(x)\ dx+\int_{\alpha_1(i)}^t (t-x)h_i(x)\ dx$$

$$=\int_A^{\alpha_1(i)}(t-x)\ 0\ dx+\int_{\alpha_1(i)}^t \frac{-1}{\sqrt{2}}(t-x)\ dx=\frac{-1}{\sqrt{2}}\frac{\left(t-\alpha_1(i)\right)^2}{2!}$$

When $t \in [\alpha_2(i),\ \alpha_3(i))$ then

$$q_{2,i}(t) = \int_A^{\alpha_1(i)} (t-x)h_i(x)\,dx + \int_{\alpha_1(i)}^{\alpha_2(i)} (t-x)h_i(x)\,dx + \int_{\alpha_2(i)}^{t} (t-x)h_i(x)\,dx$$

$$= \int_A^{\alpha_1(i),} 0\,(t-x)\,dx + \int_{\alpha_1(i),}^{\alpha_2(i)} \frac{-1}{\sqrt{2}}(t-x)\,dx + \int_{\alpha_2(i),}^{t} \frac{2}{\sqrt{2}}(t-x)\,dx$$

$$= -\frac{1}{\sqrt{2}}\frac{\left(\left(t-\alpha_2(i)\right)^2 - \left(t-\alpha_1(i)\right)^2\right)}{2!} + \frac{2}{\sqrt{2}}\frac{\left(t-\alpha_2(i)\right)^2}{2!}$$

$$= \frac{1}{\sqrt{2}} \times \frac{1}{2!}\left(-\left(t-\alpha_1(i)\right)^2 + 3\left(t-\alpha_2(i)\right)^2\right)$$

When $t \in [\alpha_3(i), \alpha_4(i))$ then

$$q_{2,i}(t) = \int_A^{\alpha_1(i)} (t-x)h_i(x)\,dx + \int_{\alpha_1(i)}^{\alpha_2(i)} (t-x)h_i(x)\,dx + \int_{\alpha_2(i)}^{\alpha_3(i)} (t-x)h_i(x)\,dx + \int_{\alpha_3(i)}^{t} (t-x)h_i(x)\,dx$$

$$= \frac{1}{\sqrt{2}}\left(\int_A^{\alpha_1(i)} 0\,dx + \int_{\alpha_1(i)}^{\alpha_2(i)} -(t-x)\,dx + \int_{\alpha_2(i)}^{\alpha_3(i)} 2(t-x)\,dx + \int_{\alpha_3(i)}^{t} -(t-x)\,dx\right)$$

$$= \frac{1}{\sqrt{2}}\left(0 + \frac{\left(\left(t-\alpha_2(i)\right)^2 - \left(t-\alpha_1(i)\right)^2\right)}{2!} - 2\frac{\left(\left(t-\alpha_3(i)\right)^2 - \left(t-\alpha_2(i)\right)^2\right)}{2!} - \frac{(t-\alpha_3(i))^2}{2!}\right)$$

$$= \frac{1}{\sqrt{2}} \times \frac{1}{2!}\left(-\left(t-\alpha_1(i)\right)^2 + 3\left(t-\alpha_2(i)\right)^2 - 3\left(t-\alpha_3(i)\right)^2\right)$$

When $t \in [\alpha_4(i), B)$ then

$$q_{2,i}(t) = \int_A^{\alpha_1(i)} (t-x)h_i(x)\,dx + \int_{\alpha_1(i)}^{\alpha_2(i)} (t-x)h_i(x)\,dx + \int_{\alpha_2(i)}^{\alpha_3(i)} (t-x)h_i(x)\,dx + \int_{\alpha_3(i)}^{\alpha_4(i)} (t-x)h_i(x)\,dx + \int_{\alpha_4(i)}^{t} (t-x)h_i(x)\,dx$$

$$= \frac{1}{\sqrt{2}}\left(\int_A^{\alpha_1(i)} 0\,\,dx + \int_{\alpha_1(i)}^{\alpha_2(i)} -(t-x)\,dx + \int_{\alpha_2(i)}^{\alpha_3(i)} 2(t-x)\,dx + \int_{\alpha_3(i)}^{\alpha_4(i)} -(t-x)\,dx + \int_{\alpha_4(i)}^{t} 0\,dx\right)$$

$$= \frac{1}{\sqrt{2}}\left(0 + \frac{\left(\left(t-\alpha_2(i)\right)^2 - \left(t-\alpha_1(i)\right)^2\right)}{2!} - 2\frac{\left(\left(t-\alpha_3(i)\right)^2 - \left(t-\alpha_2(i)\right)^2\right)}{2!} + \frac{\left(t-\alpha_4(i)\right)^2 - \left(t-\alpha_3(i)\right)^2}{2!} + 0\right)$$

$$= \frac{1}{\sqrt{2}} \times \frac{1}{2!}\left(-\left(t-\alpha_1(i)\right)^2 + 3\left(t-\alpha_2(i)\right)^2 - 3\left(t-\alpha_3(i)\right)^2 + (t-\alpha_4(i))^2\right)$$

Therefore, the second integral of the even members Haar Scale 3 wavelets family is given by $q_{2,i}(t) =$

$$\frac{1}{\sqrt{2}}\begin{cases} 0 & for \quad 0 \le t \le \alpha_1(i) \\ \frac{-1}{2!}\left(t-\alpha_1(i)\right)^2 & for \quad \alpha_1(i) \le t \le \alpha_2(i) \\ \frac{1}{2!}[-\left(t-\alpha_1(i)\right)^2 + 3\left(t-\alpha_2(i)\right)^2] & for \quad \alpha_2(i) \le t \le \alpha_3(i) \\ \frac{1}{2!}[-\left(t-\alpha_1(i)\right)^2 + 3\left(t-\alpha_2(i)\right)^2 - 3\left(t-\alpha_3(i)\right)^2] & for\ \alpha_3(i) \le t \le \alpha_4(i) \\ \frac{1}{2!}[-\left(t-\alpha_1(i)\right)^2 + 3\left(t-\alpha_2(i)\right)^2 - 3\left(t-\alpha_3(i)\right)^2 + \left(t-\alpha_4(i)\right)^2] & for\ \alpha_4(i) \le t \le 1 \end{cases} \tag{2.6}$$

Proceeding in this way, integral of any order β for the even members of non-dyadic wavelet family are given by

$q_{\beta,i}(t)'s \quad for \quad i = 2,4,6,8,\cdots,3p-1$ are given below

$q_{\beta,i}(t) =$

$$\frac{1}{\sqrt{2}}\begin{cases} 0 & for \quad 0 \le t \le \alpha_1(i) \\ \frac{-1}{\beta!}\left(t-\alpha_1(i)\right)^\beta & for \quad \alpha_1(i) \le t \le \alpha_2(i) \\ \frac{1}{\beta!}[-\left(t-\alpha_1(i)\right)^\beta + 3\left(t-\alpha_2(i)\right)^\beta] & for \quad \alpha_2(i) \le t \le \alpha_3(i) \\ \frac{1}{\beta!}[-\left(t-\alpha_1(i)\right)^\beta + 3\left(t-\alpha_2(i)\right)^\beta - 3\left(t-\alpha_3(i)\right)^\beta] & for\ \alpha_3(i) \le t \le \alpha_4(i) \\ \frac{1}{\beta!}[-\left(t-\alpha_1(i)\right)^\beta + 3\left(t-\alpha_2(i)\right)^\beta - 3\left(t-\alpha_3(i)\right)^\beta + \left(t-\alpha_4(i)\right)^\beta] & for\ \alpha_4(i) \le t \le 1 \end{cases} \tag{2.7}$$

For odd integers $i = 3,5,7 \dots 3p$

$$h_i(t) = \psi^2\left(3^j t - k\right) = \sqrt{\frac{3}{2}}\begin{cases} 1 & \alpha_1(i) \le t < \alpha_2(i) \\ 0 & \alpha_2(i) \le t < \alpha_3(i) \\ -1 & \alpha_3(i) \le t < \alpha_4(i) \\ 0 & elsewhere \end{cases}$$

$$q_{1,i}(t) = \frac{1}{(1-1)!}\int_A^t (t-x)^{1-1} h_i(x)dx = \int_A^t h_i(x)\, dx\ \forall\ i = 3,5,7 \dots 3p \tag{2.8}$$

When $t \in [A, \alpha_1(i))$ then

$$q_{1,i}(t) = \int_A^t h_i(x)\,dx = \int_A^t 0\,dx = 0$$

When $t \in [\alpha_1(i),\ \alpha_2(i))$ then

$$q_{1,i}(t) = \int_A^{\alpha_1(i),} h_i(x)\,dx + \int_{\alpha_1(i),}^t h_i(x)\,dx$$

$$= \int_A^{\alpha_1(i),} 0\,dx + \int_{\alpha_1(i)}^t \sqrt{\frac{3}{2}}\,dx = \sqrt{\frac{3}{2}}\frac{(t-\alpha_1(i))}{1!}$$

When $t \in [\alpha_2(i),\ \alpha_3(i))$ then

$$q_{1,i}(t) = \int_A^{\alpha_1(i),} h_i(x)\,dx + \int_{\alpha_1(i),}^{\alpha_2(i)} h_i(x)\,dx + \int_{\alpha_2(i),}^t h_i(x)\,dx$$

$$= \int_A^{\alpha_1(i)} 0\,dx + \int_{\alpha_1(i)}^{\alpha_2(i)} \sqrt{\frac{3}{2}}\,dx + \int_{\alpha_2(i)}^t 0\,dx$$

$$= 0 + \sqrt{\frac{3}{2}}(\alpha_2(i)-\alpha_1(i)) + 0 = \sqrt{\frac{3}{2}}\left(\frac{(t-\alpha_1(i))^1 - (t-\alpha_2(i))^1}{1!}\right)$$

When $t \in [\alpha_3(i), \alpha_4(i))$ then

$$q_{1,i}(t) = \int_A^{\alpha_1(i)} h_i(x)\,dx + \int_{\alpha_1(i)}^{\alpha_2(i)} h_i(x)\,dx + \int_{\alpha_2(i)}^{\alpha_3(i)} h_i(x)\,dx + \int_{\alpha_3(i)}^t h_i(x)\,dx$$

$$= \sqrt{\frac{3}{2}}\left(\int_A^{\alpha_1(i)} 0\,dx + \int_{\alpha_1(i)}^{\alpha_2(i)} 1\,dx + \int_{\alpha_2(i)}^{\alpha_3(i)} 0\,dx + \int_{\alpha_3(i)}^t -1\,dx\right)$$

$$= \sqrt{\frac{3}{2}}\left(0 + (\alpha_2(i)-\alpha_1(i)) + 0 - (t-\alpha_3(i))\right) = \sqrt{\frac{3}{2}}(-t-\alpha_1(i)+\alpha_2(i)+\alpha_3(i))$$

$$= \sqrt{\frac{3}{2}}\left(\frac{(t-\alpha_1(i))^1 - (t-\alpha_2(i))^1 - (t-\alpha_3(i))^1}{1!}\right)$$

When $t \in [\alpha_4(i), B)$ then

$$q_{1,i}(t) = \int_{A}^{\alpha_1(i)} h_i(x)\,dx + \int_{\alpha_1(i)}^{\alpha_2(i)} h_i(x)\,dx + \int_{\alpha_2(i)}^{\alpha_3(i)} h_i(x)\,dx + \int_{\alpha_3(i)}^{\alpha_4(i)} h_i(x)\,dx + \int_{\alpha_4(i)}^{t} h_i(x)\,dx$$

$$= \sqrt{\frac{3}{2}}\left(\int_{A}^{\alpha_1(i)} 0\,dx + \int_{\alpha_1(i)}^{\alpha_2(i)} dx + \int_{\alpha_2(i)}^{\alpha_3(i)} 0\,dx + \int_{\alpha_3(i)}^{\alpha_4(i)} -1\,dx + \int_{\alpha_4(i)}^{t} 0\,dx\right)$$

$$= \sqrt{\frac{3}{2}}\left(0 + \left(\alpha_2(i) - \alpha_1(i)\right) + 0 - \left(\alpha_4(i) - \alpha_3(i)\right) + 0\right)$$

$$= \sqrt{\frac{3}{2}}\left(-\alpha_1(i) + \alpha_2(i) + \alpha_3(i) - \alpha_4(i)\right)$$

$$= \sqrt{\frac{3}{2}}\left(\left(t - \alpha_1(i)\right)^1 - \left(t - \alpha_2(i)\right)^1 - \left(t - \alpha_3(i)\right)^1 + (t - \alpha_4(i))^1\right)$$

Therefore, the first integral of the odd members of the Haar Scale 3 wavelets family is given by $q_{1,i}(t) =$

$$\sqrt{\frac{3}{2}}\begin{cases} 0 & for \quad 0 \le t \le \alpha_1(i) \\ \frac{1}{1!}\left(t - \alpha_1(i)\right)^1 & for \quad \alpha_1(i) \le t \le \alpha_2(i) \\ \frac{1}{1!}[\left(t - \alpha_1(i)\right)^1 - \left(t - \alpha_2(i)\right)^1] & for \quad \alpha_2(i) \le t \le \alpha_3(i) \\ \frac{1}{1!}[\left(t - \alpha_1(i)\right)^1 - \left(t - \alpha_2(i)\right)^1 - \left(t - \alpha_3(i)\right)^1] & for \quad \alpha_3(i) \le t \le \alpha_4(i) \\ \frac{1}{1!}[\left(t - \alpha_1(i)\right)^1 - \left(t - \alpha_2(i)\right)^1 - \left(t - \alpha_3(i)\right)^1 + \left(t - \alpha_4(i)\right)^1] & for\, \alpha_4(i) \le t \le 1 \end{cases} \quad (2.9)$$

Now for second integrals take $\beta = 2$

$$q_{2,i}(t) = \frac{1}{(2-1)!}\int_{A}^{t} (t-x)^{2-1} h_i(x)dx = \int_{A}^{t} (t-x)h_i(x)dx \quad (2.10)$$

$$\forall\ i = 2,4,6,8,\cdots,3p-1$$

When $t \in [A, \alpha_1(i))$ then

$$q_{2,i}(t) = \int_{A}^{t} (t-x)h_i(x)dx = \int_{A}^{t} (t-x)\,0\,dx = 0$$

When $t \in \left[\alpha_1(i),\ \alpha_2(i)\right)$ then

$$q_{1,i}(t) = \int_{A}^{\alpha_1(i)} (t-x)h_i(x)\,dx + \int_{\alpha_1(i)}^{t} (t-x)h_i(x)\,dx$$

$$= \int_A^{\alpha_1(i)} (t-x)\, 0\, dx + \int_{\alpha_1(i)}^{t} \sqrt{\frac{3}{2}}(t-x)\, dx = \sqrt{\frac{3}{2}} \frac{\left(t-\alpha_1(i)\right)^2}{2!}$$

When $t \in [\alpha_2(i),\ \alpha_3(i))$ then

$$q_{1,i}(t) = \int_A^{\alpha_1(i)} (t-x)h_i(x)\, dx + \int_{\alpha_1(i)}^{\alpha_2(i)} (t-x)h_i(x)\, dx + \int_{\alpha_2(i)}^{t} (t-x)h_i(x)\, dx$$

$$= \int_A^{\alpha_1(i),} 0\,(t-x)\, dx + \int_{\alpha_1(i),}^{\alpha_2(i)} \sqrt{\frac{3}{2}}(t-x)\, dx + \int_{\alpha_2(i),}^{t} 0\,(t-x)\, dx$$

$$= 0 - \sqrt{\frac{3}{2}} \frac{\left(\left(t-\alpha_2(i)\right)^2 - \left(t-\alpha_1(i)\right)^2\right)}{2!} + 0 = \sqrt{\frac{3}{2}} \times \frac{1}{2!}\left(\left(t-\alpha_1(i)\right)^2 - \left(t-\alpha_2(i)\right)^2\right)$$

When $t \in [\alpha_3(i), \alpha_4(i))$ then

$$q_{1,i}(t) = \int_A^{\alpha_1(i)} (t-x)h_i(x)\, dx + \int_{\alpha_1(i)}^{\alpha_2(i)} (t-x)h_i(x)\, dx + \int_{\alpha_2(i)}^{\alpha_3(i)} (t-x)h_i(x)\, dx + \int_{\alpha_3(i)}^{t} (t-x)h_i(x)\, dx$$

$$= \sqrt{\frac{3}{2}}\left(\int_A^{\alpha_1(i)} 0\, dx + \int_{\alpha_1(i)}^{\alpha_2(i)} (t-x)\, dx + \int_{\alpha_2(i)}^{\alpha_3(i)} 0\, dx + \int_{\alpha_3(i)}^{t} -(t-x)\, dx\right)$$

$$= \sqrt{\frac{3}{2}}\left(0 - \frac{\left(\left(t-\alpha_2(i)\right)^2 - \left(t-\alpha_1(i)\right)^2\right)}{2!} - 0 - \frac{(t-\alpha_3(i))^2}{2!}\right)$$

$$= \sqrt{\frac{3}{2}} \times \frac{1}{2!}\left(\left(t-\alpha_1(i)\right)^2 - \left(t-\alpha_2(i)\right)^2 - \left(t-\alpha_3(i)\right)^2\right)$$

When $t \in [\alpha_4(i), B)$ then

$$q_{1,i}(t) = \int_A^{\alpha_1(i)} (t-x)h_i(x)\, dx + \int_{\alpha_1(i)}^{\alpha_2(i)} (t-x)h_i(x)\, dx + \int_{\alpha_2(i)}^{\alpha_3(i)} (t-x)h_i(x)\, dx + \int_{\alpha_3(i)}^{\alpha_4(i)} (t-x)h_i(x)\, dx + \int_{\alpha_4(i)}^{t} (t-x)h_i(x)\, dx$$

$$= \sqrt{\frac{3}{2}}\left(\int_{A}^{\alpha_1(i)} 0\ dx + \int_{\alpha_1(i)}^{\alpha_2(i)} (t-x)\,dx + \int_{\alpha_2(i)}^{\alpha_3(i)} 0\,dx + \int_{\alpha_3(i)}^{\alpha_4(i)} -(t-x)\,dx + \int_{\alpha_4(i)}^{t} 0\,dx\right)$$

$$= \sqrt{\frac{3}{2}}\left(0 - \frac{\left(\left(t-\alpha_2(i)\right)^2 - \left(t-\alpha_1(i)\right)^2\right)}{2!} - 0 + \frac{\left(t-\alpha_4(i)\right)^2 - \left(t-\alpha_3(i)\right)^2}{2!} + 0\right)$$

$$= \sqrt{\frac{3}{2}} \times \frac{1}{2!}\left(\left(t-\alpha_1(i)\right)^2 - \left(t-\alpha_2(i)\right)^2 - \left(t-\alpha_3(i)\right)^2 + (t-\alpha_4(i))^2\right)$$

Proceeding in this way, integral of any order β for the odd members of non-dyadic wavelet family are given by

$q_{\beta,i}(t)'s \;\; for \;\; i = 3,5,7,9,\cdots,3p$are given below

$$q_{\beta,i}(t) = \sqrt{\frac{3}{2}}\begin{cases} 0 & for \quad 0 \le t \le \alpha_1(i) \\ \frac{1}{\beta!}\left(t-\alpha_1(i)\right)^{\beta} & for \quad \alpha_1(i) \le t \le \alpha_2(i) \\ \frac{1}{\beta!}\left[\left(t-\alpha_1(i)\right)^{\beta} - \left(t-\alpha_2(i)\right)^{\beta}\right] & for \quad \alpha_2(i) \le t \le \alpha_3(i) \\ \frac{1}{\beta!}\left[\left(t-\alpha_1(i)\right)^{\beta} - \left(t-\alpha_2(i)\right)^{\beta} - \left(t-\alpha_3(i)\right)^{\beta}\right] & for \quad \alpha_3(i) \le t \le \alpha_4(i) \\ \frac{1}{\beta!}\left[\left(t-\alpha_1(i)\right)^{\beta} - \left(t-\alpha_2(i)\right)^{\beta} - \left(t-\alpha_3(i)\right)^{\beta} + \left(t-\alpha_4(i)\right)^{\beta}\right] & for\ \alpha_4(i) \le t \le 1 \end{cases} \tag{2.11}$$

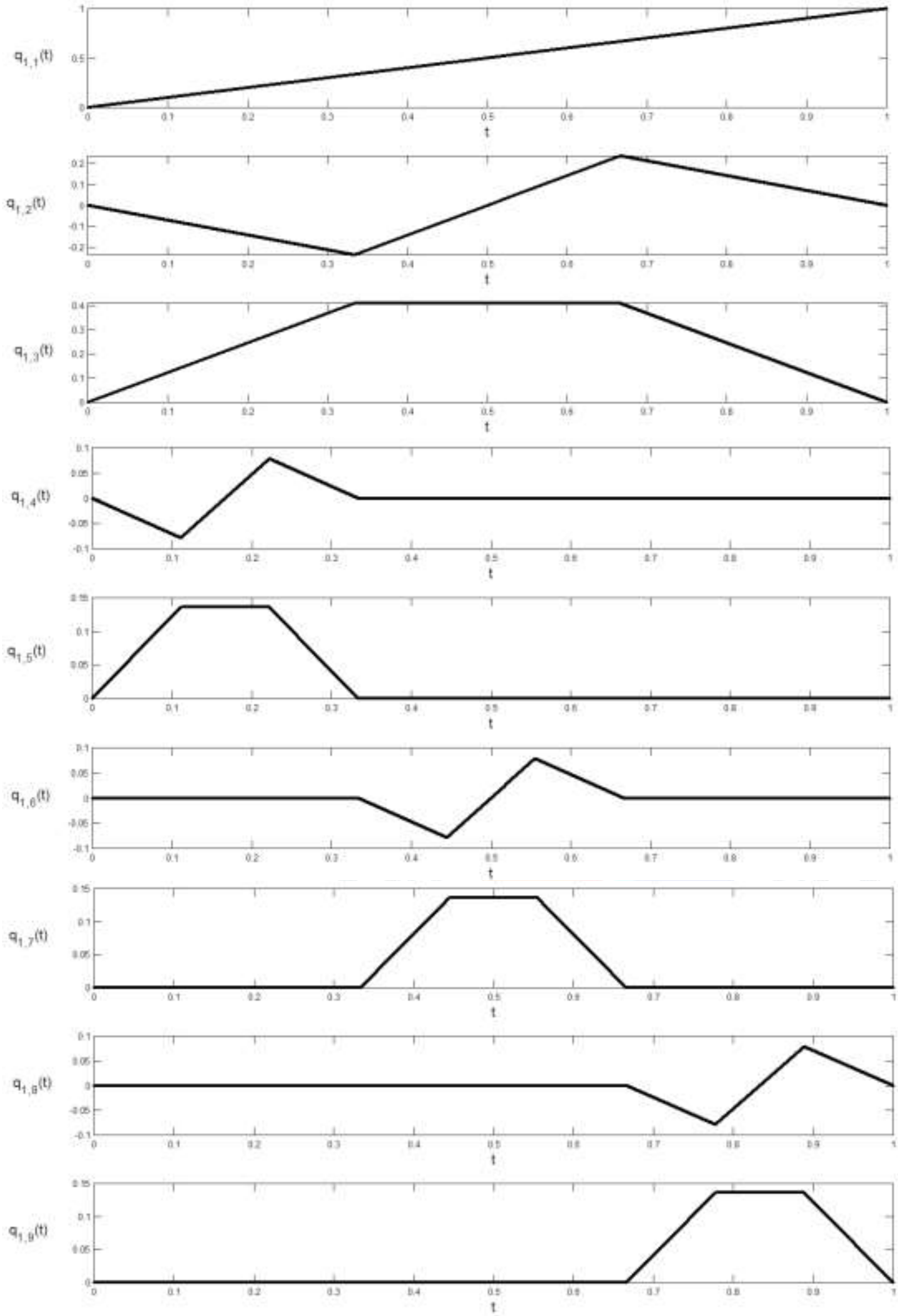

Figure 2.1:First integral of the first nine members of the Haar Scale 3 wavelet family

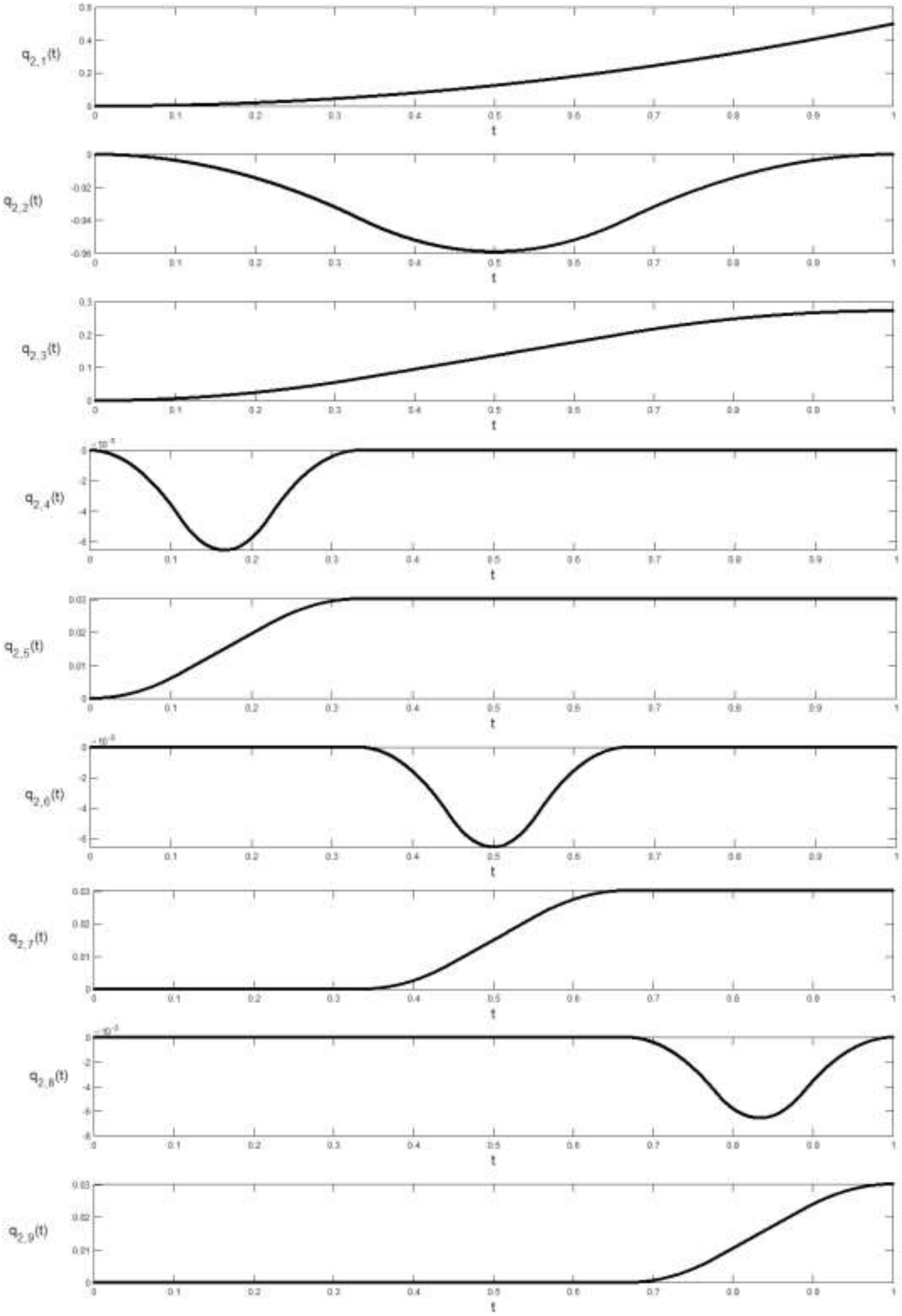

Figure 2.2: Second integral of the first nine members of the Haar Scale 3 wavelet family

3 Matrices of Haar Scale 3 Wavelets and their Integrals

In order to find the numerical solution of differential equations using Haar scale 3 wavelets, a discrete form of the Haar scale 3 wavelets seriesis required. There are many techniques to do this but we are restricting ourselves to the collocation method. Haar scale 3 wavelets are discontinuous in nature. Therefore, in order to avoid the collocation point at the point of discontinuity following approach is applied for the selection of collocation points t_l.

$$t_l = \frac{(T_{l-1} + T_l)}{2} \quad , \quad l = 1,2,3 \cdots 3p, \qquad p = 3^j \qquad (3.1)$$

Where

$$T_l = A + \frac{(B - A)}{3p} * l, \quad l = 1,2,3 \cdots 3p \, , \qquad p = 3^j \qquad (3.2)$$

Now replacing t with t_l in the above equations, a discrete form of wavelet can be obtained which can easily expressible in the matrix form. Now for $A = 0, B = 1$ and $J = 0$, we have the following Haar scale 3 matrix H and Matrices Q_1, Q_2 of their first and second integrals.

$$A = 0, B = 1, j = 0 \;\Rightarrow T_0 = \tfrac{1}{3} \quad T_2 = \tfrac{2}{6} \quad T_3 = \tfrac{5}{6} \quad T_4 = \tfrac{5}{6}$$

Haar Scale 3 Wavelet matrix for the initial level of resolution ($j = 0$)

Collocation points $\Rightarrow t_1 = \frac{1}{6} \quad t_2 = \frac{3}{6} \quad t_3 = \frac{5}{6}$

$$H = \begin{bmatrix} 1 & 1 & 1 \\ -0.70711 & 1.414214 & 0.70711 \\ 1.224745 & 0 & 1.224745 \end{bmatrix} \begin{matrix} \leftarrow h_1 \\ \leftarrow h_2 \\ \leftarrow h_3 \end{matrix}$$

First integral matrix Q_1 of Haar Scale 3 Wavelet matrix for the initial level of resolution ($j = 0$)

Collocation points $\Rightarrow t_1 = \frac{1}{6} \; t_2 = \frac{3}{6} \; t_3 = \frac{5}{6}$

$$Q_1 = \begin{bmatrix} 0.167666666666667 & 0.500000000000000 & 0.833333333333333 \\ -0.117851130197758 & 0 & 0.117851130197758 \\ 0.204124145231931 & 0.408248290463863 & 0.204124145231932 \end{bmatrix} \begin{matrix} \leftarrow q_{1,1} \\ \leftarrow q_{1,2} \\ \leftarrow q_{1,3} \end{matrix}$$

Second integral matrix Q_2 of Haar Scale 3 Wavelet matrix for the initial level of resolution($j = 0$)

Collocation points $\Rightarrow t_1 = \frac{1}{6} \quad t_2 = \frac{3}{6} \quad t_3 = \frac{5}{6}$

$$Q_2 = \begin{bmatrix} 0.167666666666667 & 0.500000000000000 & 0.833333333333333 \\ -0.117851130197758 & 0 & 0.117851130197758 \\ 0.204124145231931 & 0.408248290463863 & 0.204124145231932 \end{bmatrix} \begin{matrix} \leftarrow q_{2,1} \\ \leftarrow q_{2,2} \\ \leftarrow q_{2,3} \end{matrix}$$

Haar Scale 3 Wavelet matrix for the next level of resolution ($j = 1$)

Coll. pts $\Rightarrow \; t_1 = \frac{1}{18} \; t_2 = \frac{3}{18} \; t_3 = \frac{5}{18} t_4 = \frac{7}{18} \; t_5 = \frac{9}{18} \; t_6 = \frac{11}{18} t_7 = \frac{13}{18} \; t_8 = \frac{15}{18} \; t_9 = \frac{17}{18}$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -0.707 & -0.707 & -0.707 & 1.414 & 1.414 & 1.414 & -0.707 & -0.707 & -0.707 \\ 1.225 & 1.225 & 1.225 & 0 & 0 & 0 & -1.225 & -1.225 & -1.225 \\ -0.707 & 1.414 & -0.707 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.225 & 0 & -1.225 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.707 & 1.414 & -0.707 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.225 & 0 & -1.225 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.707 & 1.414 & -0.707 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.225 & 0 & -1.225 \end{bmatrix} \begin{matrix} \leftarrow h_1 \\ \leftarrow h_2 \\ \leftarrow h_3 \\ \leftarrow h_4 \\ \leftarrow h_5 \\ \leftarrow h_6 \\ \leftarrow h_7 \\ \leftarrow h_8 \\ \leftarrow h_9 \end{matrix}$$

First integral matrix Q_1 of Haar Scale 3 Wavelet matrix for the initial level of resolution ($j = 1$)

Coll. Pts⇒ $t_1 = \frac{1}{18}\ t_2 = \frac{3}{18}\ t_3 = \frac{5}{18} t_4 = \frac{7}{18}\ t_5 = \frac{9}{18}\ t_6 = \frac{11}{18} t_7 = \frac{13}{18}\ t_8 = \frac{15}{18}\ t_9 = \frac{17}{18}$

$$Q_1 = \begin{bmatrix} 0.056 & 0.167 & 0.278 & 0.389 & 0.500 & 0.611 & 0.722 & 0.833 & 0.944 \\ -0.039 & -0.118 & -0.196 & -0.157 & 0.000 & 0.157 & 0.196 & 0.118 & 0.039 \\ 0.068 & 0.204 & 0.340 & 0.408 & 0.408 & 0.408 & 0.340 & 0.204 & 0.068 \\ -0.039 & 0 & 0.039 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.068 & 0.136 & 0.068 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.039 & 0 & 0.039 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.068 & 0.136 & 0.068 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.039 & 0 & 0.039 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.068 & 0.136 & 0.068 \end{bmatrix} \begin{matrix} \leftarrow q_{1,1} \\ \leftarrow q_{1,2} \\ \leftarrow q_{1,3} \\ \leftarrow q_{1,4} \\ \leftarrow q_{1,5} \\ \leftarrow q_{1,6} \\ \leftarrow q_{1,7} \\ \leftarrow q_{1,8} \\ \leftarrow q_{1,9} \end{matrix}$$

First integral matrix Q_1 of Haar Scale 3 Wavelet matrix for the initial level of resolution $(j = 1)$

Coll. Pts⇒ $t_1 = \frac{1}{18}\ t_2 = \frac{3}{18}\ t_3 = \frac{5}{18} t_4 = \frac{7}{18}\ t_5 = \frac{9}{18}\ t_6 = \frac{11}{18} t_7 = \frac{13}{18}\ t_8 = \frac{15}{18}\ t_9 = \frac{17}{18}$

$$Q_2 = \begin{bmatrix} 0.002 & 0.014 & 0.039 & 0.076 & 0.125 & 0.187 & 0.261 & 0.347 & 0.446 \\ -0.001 & -0.010 & -0.027 & -0.050 & -0.059 & -0.050 & -0.027 & -0.010 & -0.001 \\ 0.002 & 0.017 & 0.047 & 0.091 & 0.136 & 0.181 & 0.225 & 0.255 & 0.270 \\ -0.001 & -0.007 & -0.001 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.002 & 0.015 & 0.028 & 0.030 & 0.030 & 0.030 & 0.030 & 0.030 & 0.030 \\ 0 & 0 & 0 & -0.001 & -0.007 & -0.001 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.002 & 0.015 & 0.028 & 0.030 & 0.030 & 0.030 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.001 & -0.007 & -0.001 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.002 & 0.015 & 0.028 \end{bmatrix} \begin{matrix} \leftarrow q_{2,1} \\ \leftarrow q_{2,2} \\ \leftarrow q_{2,3} \\ \leftarrow q_{2,4} \\ \leftarrow q_{2,5} \\ \leftarrow q_{2,6} \\ \leftarrow q_{2,7} \\ \leftarrow q_{2,8} \\ \leftarrow q_{2,9} \end{matrix}$$

4 Approximation of Function Using Haar Scale 3 Wavelet Series

Consider any square-integrable function $f(t)$ over the interval $[A, B)$. Now to approximate $f(t)$ using Haar scale 3 wavelet family, $f(t)$ can be written as

$$f(t) = \sum_{i=0}^{\infty} a_i h_i(t) = a_1 h_1(t) + \sum_{even\ i} a_i \psi^1(3^j t - k) + \sum_{odd\ i>1} a_i \psi^2(3^j t - k) \tag{4.1}$$

Here $a_i's$ are the wavelet coefficients whose values are to be determined by the proposed method. But for computational purposes, one can consider a finite number of terms. By considering the first $3p$ terms to approximate the function $f(t)$ we get

$$f(t) \approx f_{3p}(t) = \sum_{i=0}^{3p} a_i h_i(t) \quad , \quad p = 3^j \ , \qquad j = 0,1,2, \ldots \tag{4.2}$$

using the collocation points $t_l\ , l = 1,2,3 \cdots$ above equation takes the discrete form.

$$f(t_l) \approx f_{3p}(t) = \sum_{i=0}^{3p} a_i h_i(t_l) \quad , l = 1,2,3 \cdots \tag{4.3}$$

The above equation can easily be expressible into the matrix form as

$$F = AH$$

Where H is the Haar matrix of order $3p \times 3p$ and F, A are the row matrices of order $1 \times 3p$.H and F are known matrices and the value of A can be evaluated by solving the above matrix system as

$$A = FH^{-1}$$

Then by substituting the values of unknown coefficients a_i's from the matrix $A = [a_i]_{1\times 3p}$ in the above equation, wavelet approximation of $f(t)$ at the desired level of resolution can be obtained. In order to test order of accuracy of approximation a well-defined L_2 ,L_∞ and absolute errors can be calculated which are defined as

$$\text{Absolute error} = |u_{exact}(t_l) - u_{num}(t_l)|$$

$$L_\infty = \max_l |u_{exact}(t_l) - u_{num}(t_l)|$$

$$L_2 = \frac{\sqrt{\sum_{l=1}^{3p} |u_{exact}(t_l) - u_{num}(t_l)|^2}}{\sqrt{\sum_{l=1}^{3p} |u_{exact}(t_l)|^2}}$$

where t_l represents the collocation points of the domain.

5 Numerical Experiments

Numerical experiment 1:Consider a function $f(t) = t^2$ over the interval $(0,1)$.Now approximate the function $f(t)$using the Haar scale 3 wavelet series at $J = 1$ as

$$f(t) = \sum_{i=0}^{3p} a_i h_i(t) \quad , \quad p = 3^1$$

using the collocation points $t_1 = \frac{1}{18}$ $t_2 = \frac{3}{18}$ $t_3 = \frac{5}{18} t_4 = \frac{7}{18}$ $t_5 = \frac{9}{18}$ $t_6 = \frac{11}{18} t_7 = \frac{13}{18}$ $t_8 = \frac{15}{18}$ $t_9 = \frac{17}{18}$ above equation transforms to a system of simultaneous linear equations which can be further be put in the matrix system as

$$AH = F$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{bmatrix}^T \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -0.707 & -0.707 & -0.707 & 1.414 & 1.414 & 1.414 & -0.707 & -0.707 & -0.707 \\ 1.225 & 1.225 & 1.225 & 0 & 0 & 0 & -1.225 & -1.225 & -1.225 \\ -0.707 & 1.414 & -0.707 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.225 & 0 & -1.225 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.707 & 1.414 & -0.707 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.225 & 0 & -1.225 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.707 & 1.414 & -0.707 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.225 & 0 & -1.225 \end{bmatrix} = \begin{bmatrix} 0.0031 \\ 0.0278 \\ 0.0772 \\ 0.1512 \\ 0.2500 \\ 0.3735 \\ 0.5216 \\ 0.6944 \\ 0.8920 \end{bmatrix}$$

Solving the above matrix system using MATLAB software, following of Haar scale 3 coefficients are obtained

$a_1 = \mathbf{0.3323}$, $a_2 = \mathbf{-0.0524}$, $a_3 = \mathbf{-0.2722}$,$a_4 = \mathbf{-0.0058}$,$a_5 = \mathbf{-0.0302}$,$a_6 = \mathbf{-0.0058} a_7 = \mathbf{-0.0907}$, $a_8 = \mathbf{-0.0058}$,$a_9 = \mathbf{-0.1512}$

By using these values of wavelet coefficients f(t) is approximated as follows

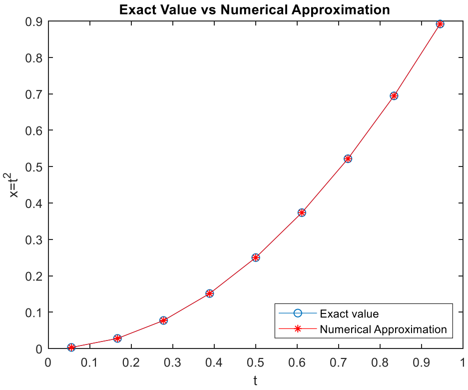

Fig.1.10a

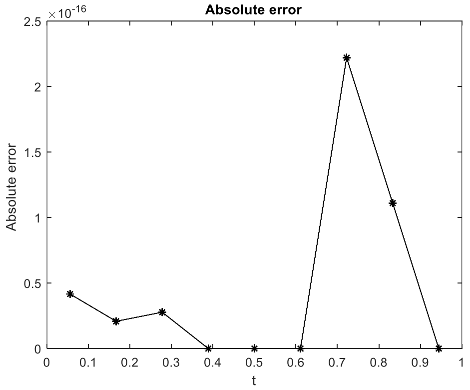

Fig.1.10b

Figure 5.1: Graph of approximation of function $f(t) = t^2$ in comparison with exact values at the different collocation points (Fig.1.10a) and absolute error(Fig.1.10b) in the approximation of a function by Haar scale 3 wavelets at $J = 1$.

Table 5.1: Comparision of the exact and approximated solution by Haar scale 3 Wavelets for Numerical experiment no. 1

t	**Exact Value ($Sin\ t$)**	**Approximated value ($Sin\ t$)**	**Absolute Error**
0.0555555555555556	0.0030864197530864	0.0030864197530864	4.16E-17
0.1666666666666670	0.0277777777777778	0.0277777777777778	2.08E-17
0.2777777777777780	0.0771604938271605	0.0771604938271605	2.78E-17
0.3888888888888890	0.1512345679012350	0.1512345679012350	0.00E+00
0.5000000000000000	0.2500000000000000	0.2500000000000000	0.00E+00
0.6111111111111110	0.3734567901234570	0.3734567901234570	0.00E+00
0.7222222222222220	0.5216049382716050	0.5216049382716050	2.22E-16
0.8333333333333330	0.6944444444444440	0.6944444444444440	1.11E-16
0.9444444444444440	0.8919753086419750	0.8919753086419750	0.00E+00

Numerical experiment 2:Consider a function $f(t) = \sin t$ over the interval $(0,1)$.Now approximate the function $f(t)$using the Haar scale 3 wavelet series at $J = 1$ as

$$f(t) = \sum_{i=0}^{3p} a_i h_i(t) \quad , \quad p = 3^1$$

using the collocation points $t_1 = \frac{1}{18}$ $t_2 = \frac{3}{18}$ $t_3 = \frac{5}{18} t_4 = \frac{7}{18}$ $t_5 = \frac{9}{18}$ $t_6 = \frac{11}{18} t_7 = \frac{13}{18}$ $t_8 = \frac{15}{18}$ $t_9 = \frac{17}{18}$ above equation transforms to a system of simultaneous linear equations which can be further be put in the matrix system as

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{bmatrix}^T \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -0.707 & -0.707 & -0.707 & 1.414 & 1.414 & 1.414 & -0.707 & -0.707 & -0.707 \\ 1.225 & 1.225 & 1.225 & 0 & 0 & 0 & -1.225 & -1.225 & -1.225 \\ -0.707 & 1.414 & -0.707 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.225 & 0 & -1.225 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.707 & 1.414 & -0.707 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.225 & 0 & -1.225 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.707 & 1.414 & -0.707 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.225 & 0 & -1.225 \end{bmatrix} = \begin{bmatrix} 0.0555 \\ 0.1659 \\ 0.2742 \\ 0.3792 \\ 0.4794 \\ 0.5738 \\ 0.6611 \\ 0.7402 \\ 0.8102 \end{bmatrix}$$

Solving the above matrix system using MATLAB software, following of Haar scale 3 coefficients are obtained

a_1 **=0.4599,** $a_2 = \mathbf{0.0124}$**,** $a_3 = \mathbf{-0.2335}$**,**$a_4 = \mathbf{0.0005}$**,**$a_5 = \mathbf{-0.0893}$**,**$a_6 = \mathbf{0.0014}$

a_7 **=-0.0795,** $a_8 = \mathbf{0.0022}$**,**$a_9 = \mathbf{-0.0609}$

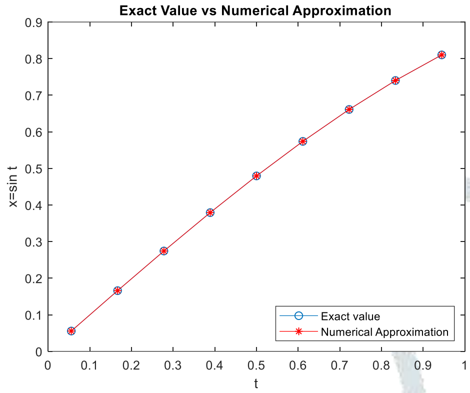

Fig.1.11a

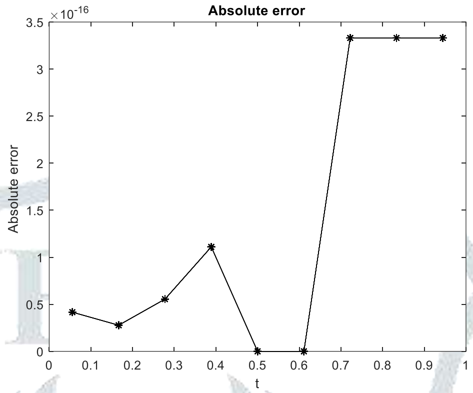

Fig.1.11b

Figure 5.2: Graph of approximation of function $f(t) = Sin\, t$ in comparison with exact values at the different collocation points (Fig.1.11a) and absolute error(Fig.1.11b) in the approximation of a function by Haar scale 3 wavelets at $J = 1$.

Table 5.2: Comparision of the exact and approximated solution by Haar scale 3 Wavelets for Numerical experiment no. 2

t	**Exact Value (*Sin t*)**	**Approximated value (*Sin t*)**	**Absolute Error**
0.0555555555555556	0.0555269820047339	0.0555269820047339	4.16E-17
0.1666666666666670	0.1658961326934150	0.1658961326934150	2.78E-17
0.2777777777777780	0.2742192892107270	0.2742192892107270	5.55E-17
0.3888888888888890	0.3791605039172600	0.3791605039172600	1.11E-16
0.5000000000000000	0.4794255386042030	0.4794255386042030	0.00E+00
0.6111111111111110	0.5737778263110660	0.5737778263110660	0.00E+00
0.7222222222222220	0.6610537218848880	0.6610537218848880	3.33E-16
0.8333333333333330	0.7401768531960370	0.7401768531960370	3.33E-16
0.9444444444444440	0.8101713960172990	0.8101713960172990	3.33E-16

6 Conclusion

Construction scheme introduced by Charles K. Chui, Jian-ao Lian has been followed to construct the compactly supported orthonormal nondyadic wavelet family. Haar Scale 3 function and their corresponding compactly supported symmetric and anti-symmetric wavelets are used and their general Integrals of nth order have been calculated by integrating then n-time. Matrices of Haar Scale 3 Wavelets and their Integrals have been calculated for the approximation of arbitrary function using the members of Haar scale 3 orthonormal wavelet families which will also be used to solve the various types of integral and differential equations in the subsequent chapters. Two functions of different types have been approximated using the Haar scale 3 wavelet families at the first level of resolution $J = 1$ and errors have been analysed. It is found that error is of the order 10^{-16} which is the default precision level of MATLAB 7 software. In the next chapters, we will extend the application of the method to solve thevarious types of Differential equations (ODEs,PDEs,FDEs). The proposed Technique is well-suited and very much helpful with the computer environment.Common programs can be used to solve various types of problems

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