Statistical Inference and Hypothesis Testing Procedures for Two Reliability Functions of Moore and Bilikam Family of Lifetime Distributions Under Type 2 and 1 Censoring Scheme

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Abstract
A family of lifetime distributions of Moore and Bilikam[33] is considered, which covers many probabilistic models as specific cases. Discussing important reliability measures as, viz. R(t)=P(X > t) and P=P(X>Y). Furthermore, the uniformly minimum variance unbiased estimators (UMVUES) and maximum likelihood estimators (MLEs) are used to derive the point estimators of the parameter based on Type 2 and Type 1 censoring scheme. Constructing the exact confidence interval for MLE & UMVUE for θ, R(t) & P under type 2 censoring scheme. Also the asymptotic confidence interval for the parameter θ and β under Type 2 censoring scheme have been constructed. The variance expression for UMVUE & MLE under both Type2 censoring and Type 1 censoring scheme are derived. Hypothised test procedures for different parametric functions are developed. Lastly, the simulation study of two reliability procedures and real life data is done

Keywords: Family of lifetime distributions, estimation method, censoring techniques, hypothised test, UMVUE and MLE

1 Introduction
The process of failure free operation with uncertainty until it reaches time t is known as reliability function R(t). R(t)=P(X > t), where, X is a random variable which is considered as lifetime of an item. P=P(X>Y), is the stress-strength reliability, which is stated as the random strength X subject to random stress Y in term of reliability. As far as for both the reliability, dozens of work has been made in point estimation under complete sample and censored samples. For reviewing the literature on can refer to Pugh[37], Basu[5], Bartholomew[4-5], Tong[42-43], Johnson[23], Kelley, Kelley and Schucany[25], Sathe and Shah[40], Chao[9], Constantine, Karson and Tse[21], Awad and Gharraf[2], Tyagi and Bhattacharya[44], Chaturvedi and Rani[13-14], Chaturvedi and Surinder[17], Chaturvedi and Tomar[18-19], Chaturvedi and Singh[15-16], Chaturvedi and Pathak[10-11-12] and other.

The purpose work are present in many-fold and the probabilistic model of Moore and Bilikam[33] is considered which covers various specific distributions. We develop point estimation procedures under type II and type I censoring scheme. Hypothesised test procedures are also proposed. Considering the point estimation, UMVUE and MLE are derived. A new formulation has been introduced to obtaining power, reliability and stress-strength reliability. For a specified point the probability density function are obtained by derivate the reliability functions. For same and different families of distributions, derive the stress-strength reliability estimator are obtained. Considering the UMVUES and MLES to obtain the variance expression under type 2 & type I censoring scheme and also construct exact & asymptotic confidence interval for UMVUE & MLE under type II censoring scheme.

In Section 2, we defined the family of lifetime distributions which covers many probabilistic models. In Section 3 and 4, we provide the position estimators under type 2 and type I censoring scheme also derive the variance. Again in section 3, exact confidence interval and asymptotic confidence interval is obtained. In Section 5, we develop hypothesis testing procedures. In Section 6 simulation result are made and in section 7 real data study is done. In the last, conclusion is given in section 8.

2 Preliminaries, Notations and Definitions
Moore and Bilikam[33] introduced the following family of lifetime distribution:Let the random variable X follows the distribution having the p.d.f

\[ f(x; \beta, \theta) = \frac{\theta}{\beta} g(x) \theta^{\beta-1}(x) \exp \left( -\frac{\theta g(x)}{\beta} \right), x > 0, (\beta, \theta) > 0 \quad (1) \]
where \( g(x) \) is real-valued-differentiable-strictly increasing function of \( x \) such that \( g(x) = 0 \) \& \( g'(x) \) is derivative of \( g(x) \). \( \beta \) is known and \( \theta \) is unknown. Some of the most important and specific probabilistic models for equation(1) is defined as,

(1) For \( g(x)=x \) and \( \beta = 1 \), we get Exponential distribution [Johnson and Kotz [24]].
(2) For \( g(x)=x-a \) and \( \beta = 1 \), we get the linear Exponential distribution [Mahmoud and Al-Nagar [32]].
(3) For \( g(x)=bx + \frac{a}{2}x^2; \lambda > 0, b > 0 \) and \( \beta = 1 \), we get the two parameter Exponential distribution [Ahsanullah [1]].
(4) For \( g(x)=x \) and \( \beta = 2 \), we get Rayleigh distribution.
(5) For \( g(x)=x \), we get Weibull distribution,
(6) For \( g(x)=(x^r)^{\exp(\gamma)} \), \( \gamma > 0, \nu > 0, x > 0 \) and \( \beta = 1 \), we obtained modified Weibull distribution [Lai et al [28]].
(7) For \( g(x)=(x^r)^{\exp(\gamma)} - 1; \lambda > 0, b > 0 \) and \( \beta = 1 \), it gives the generalized power Weibull distribution [Nikulin and Haghihi [34]].
(8) For \( g(x)=log(1 + x^b), b > 0 \) and \( \beta = 1 \), it leads us to Burr distribution [Burr [7], Burr and Cislak [8]].
(9) For \( g(x)=log(1 + x^b); b > 0, \nu > 0 \) and \( \beta = 1 \), we get Burr distribution with scale parameter \( \nu \) [Tadikamalla [41]].
(10) For \( g(x)=log(\frac{x^r}{\alpha}) \) and \( \beta = 1 \), it gives Pareto distribution.
(11) For \( g(x)=(x-a) + \nu \log(\frac{x^r+\nu}{a+\nu}); \nu > 0, \lambda > 0 \) and \( \beta = 1 \), it gives generalized Pareto distribution [Ljubo [30]].
(12) For \( g(x)=log(1 + \frac{x^r}{\nu}); \lambda > 0 \) and \( \beta = 1 \), then it becomes Lomax distribution [Lomax [31]].
(13) For \( g(x)=\frac{x^r}{b}(e^{bx} - 1); \alpha > 0, b > 0 \) and \( \beta = 1 \) we get the Gompertz distribution [Khan and Zia [26]].
(14) For \( g(x)=(e^{bx} - 1); b > 0 \) and \( \beta = 1 \), this gives Chen distribution [Chen [20]].

Reliability function \( R(t) \) at a specified mission time \( t \) and Stress-Strength Reliability \( P \) are defined as
\[
R(t) = P(X > t) = \int_t^\infty f(x; \beta, \theta) \, dx
\]
\[
R(t) = e^{-\frac{\theta}{\theta+\theta_2}}. \tag{2}
\]
\[
P = P(X > Y) = \int_y^\infty \int_x^y f(x) f(y) \, dx \, dy
\]
\[
P = \frac{\theta_2}{\theta + \theta_2}. \tag{3}
\]

3 Type II Censoring Scheme for Point Estimators

Recording the \( r \) item after termination of test when \( n \) items are introduced for preforming the test. Let us denote by "\( 0 < X_1 \leq X_2 \leq \ldots \leq X_r \)", \( 0 < r < n \), be the lifetime of 1st \( r \) failures. Obviously, \( (n-r) \) items survived until \( X_r \). Before proving main theorem in this section, we first state lemma.

**Lemma 3.1** Let \( S_r = \sum_{i=1}^r g^\beta(x_i) + (n-r)g^\beta(x_r) \). The \( S_r \) is complete and sufficient. \( S_r \) is probability density function is defined as
\[
g(S_r, \theta) = \frac{s_r^{\beta-1} \exp(-\frac{s_r}{\theta})}{\theta^{s_r}}, S_r > 0. \tag{4}
\]

**Proof.** From the p.d.f given in equation (1), the joint p.d.f of "\( 0 < X_1 \leq X_2 \leq \ldots \leq X_n \)" is
\[
f^*(x_1, x_2, \ldots, x_n; \beta, \theta) = n! \left(\frac{\theta}{\beta}\right)^n \prod_{i=1}^n g(x_i) g^{\beta-1}(x_i) \exp\left(-\frac{\theta}{\beta}\right). \tag{5}
\]
Integrating out \( x_{(r+1)}, x_{(r+2)}, \ldots, x_n \) from equation (5) over the region \( x_r \leq x_{(r+1)} \leq \ldots \leq x_n \), the joint p.d.f of \( 0 < x_1 \leq x_2 \leq \ldots \leq x_r \) comes out to be
\[
h(x_1, x_2, \ldots, x_r; \beta, \theta) = n(n-1) \ldots (n-r+1) \left(\frac{\theta}{\beta}\right)^n \prod_{i=1}^n g(x_i) g^{\beta-1}(x_i) \exp\left(-\frac{s_r}{\theta}\right). \tag{6}
\]
It follows form equation (1) and \( F \) N factorization theorem [see Rohatgi [38]] that \( S_r \) is sufficient for the family of distributions. Let us make the transformation \( U = g^\beta(x) \). It can be verified that \( U \) follows exponential distribution with mean life \( \theta \). Let the transformation "\( Z_i = (n-i+1)(U_i - U_{(i+1)}), i = 1, 2, \ldots, r \)" is considered and \( Z_i \)'s are independent and identically distributed random variable with exponential distribution. Since \( \sum_{i=1}^r Z_i = S_r \), Result of equation (4) holds by using the reproductive property of gamma distribution [see Johnson and Kotz [24]]. From (6), \( S_r \) is sufficient for equation (1). \( S_r \)'s distribution is to exponential family of distributions, as well as complete [see Rohatgi [38]].
3.1 Uniformly minimum variance unbiased estimators

Theorem 3.2 For $q \in (-\infty, \infty)$, the UMVUE of $\theta^q$ is
\[ \hat{\theta}^q_{U} = \begin{cases} \frac{1}{\Gamma(r)} \int_0^\infty S_r^{r-q-1} e\left(-\frac{S_r}{\theta}\right) ds_r, & r-q > 0 \\ 0, & otherwise \end{cases} \] (7)

Proof. From (4),
\[ E\left(\theta^{-q}\right) = \frac{1}{\Gamma(r)} \int_0^\infty S_r^{r-q-1} e\left(-\frac{S_r}{\theta}\right) ds_r \]
and equation (7) follows from Lehmann-Scheffe theorem [see Rohatgi[38]].

Theorem 3.3 The UMVUE of $R(t)$ at time $t$ is
\[ \hat{R}_{II}(t) = \begin{cases} \left(1 - \frac{g^\beta(t)}{s_r}\right)^{r-1}, & g^\beta(t) < s_r \\ 0, & otherwise \end{cases} \] (8)

Proof. Suppose the random variable is defined as
\[ V = \begin{cases} 1, & X_1 > t \\ 0, & otherwise \end{cases} \] (9)
Unbiasedness can easily be checked for the defined function in equation (9) for reliability $R(t)$, equation (9) follows after applying Rao-Blackwell theorem,
\[ \hat{R}_{II}(t) = E(V/S_r) = P(X_1 > t/S_r) = \frac{1}{\theta^{(r-1)}} \int_{\frac{g^\beta(x)}{s_r}}^{g^\beta(t)} (1 - \nu_1)^{(r-2)} d\nu_1. \] (10)
It follows from equation (2.1) that $V_1 = \frac{g^\beta(x)}{s_r}$ has beta-I kind distribution with the parameter $(1, r - 1)$. Using Basu’s theorem [see Rohatgi and Saleh[39]], from equation (3.7), we have
\[ \hat{R}_{II}(t) = \frac{1}{\theta^{(r-1)}} \int_{\frac{g^\beta(x)}{s_r}}^{\infty} (1 - \nu_1)^{(r-2)} d\nu_1. \] (11)
and the theorem holds.

Corollary 3.4 The UMVUE of $f(x; \beta, \theta)$ at a particular point $x$ is
\[ \hat{f}_{II}(x; \beta, \theta) = \begin{cases} \frac{(r-1)g^\beta(x)}{s_r}(1 - \frac{g^\beta(x)}{s_r})^{r-2}, & g^\beta(x) < s_r \\ 0, & otherwise \end{cases} \] (12)

Proof. Suppose the integral $\int_t^\omega \hat{f}(x; \beta, \theta)$ w.r.t. $S_r$ i.e.
\[ \hat{R}_{II}(t) = \int_t^\omega \hat{f}(x; \beta, \theta) dx \]
or
\[ -\frac{d}{dt} \hat{R}_{II}(t) = \hat{f}_{II}(t; \beta, \theta) \]
and hence corollary holds after follows the theorem (3.3).

Let the two independent r.v.’s $X$ and $Y$ follows the classes of distributions $f_1(x; \beta_1, \theta_1)$ & $f_2(y; \beta_2, \theta_2)$ respectively and also suppose “n” items on X and m items on Y are put on a life test & the execution for X and Y are r & s, where
\[ f_1(x; \beta_1, \theta_1) = \frac{\beta_1}{\theta_1} g(x)g^{\beta_1-1}(x) e\left(-\frac{g^{\beta_1}(x)}{\theta_1}\right), x > 0 \]
and
\[ f_2(y; \beta_2, \theta_2) = \frac{\beta_2}{\theta_2} h(x)h^{\beta_2-1}(x) e\left(-\frac{h^{\beta_2}(x)}{\theta_2}\right), x > 0. \]

and
\[ S_r = \sum_{i=1}^r g^{\beta_1}(x_i) + (n - r)g^{\beta_1}(x_r) \]
and
\[ T_s = \sum_{j=1}^s h^{\beta_2}(y_j) + (m - s)h^{\beta_2}(y_s). \]

Theorem 3.5 The UMVUE of $P$ is given by
Theorem 3.7

Proof. From corollary 3.4 that the UMVUEs of $f_1(x; \beta_1, \theta_1)$ and $f_2(y; \beta_2, \theta_2)$ at particular point x and y,

\[
\hat{f}_{1H}(x; \beta_1, \theta_1) = \begin{cases} 
\frac{\beta_1(r-1)g(x)g^{\beta_1-1}(x)}{s_r} & \text{if } g^{\beta_1}(x) < S_r \\
0 & \text{otherwise}
\end{cases}
\]

\[
\hat{f}_{2H}(y; \beta_2, \theta_2) = \begin{cases} 
\frac{\beta_2(s-1)h(y)h^{\beta_2-1}(y)}{t_s} & \text{if } h^{\beta_2}(y) < T_s \\
0 & \text{otherwise}
\end{cases}
\]

Considered the similar argument that are used in proving theorem 3.3, UMVUE of P is given by

\[
\hat{P}_{HI} = \int_0^{\infty} \int_0^{\infty} \hat{f}_{1H}(x; \beta_1, \theta_1)\hat{f}_{2H}(y; \beta_2, \theta_2) \, dx \, dy
\]

applying theorem (3.3) and (15) gives that

\[
\hat{P}_{HI} = \int_{y=0}^{\infty} \hat{R}_{HI}(y; \beta_1, \theta_1) \left\{ -\frac{d}{dy} \hat{P}_{2HI}(y) \right\} \, dy
\]

\[
\hat{P}_{HI} = \int_{y=0}^{\infty} \left( 1 - \frac{g^{\beta_1}(y)}{s_r} \right)^{r-2} \beta_2(s-1)h(y)h^{\beta_2-1}(y)
\]

\[
\left( 1 - \frac{h^{\beta_2}(y)}{t_s} \right)^{r-2} \, dy,
\]

where $M = \min(g^{-1}(S_r), h^{-1}(T_s))$.

Taking into consideration the two cases, putting $\frac{h^{\beta_2}(y)}{t_s} = z$ and hence theorem holds.

Corollary 3.6 When $\beta_1 = \beta_2 = \beta$,

\[
\hat{P}_{HI} = \begin{cases} 
(s - 1) \sum_{i=0}^{s-2} s - 2i (-1)^i \left( \frac{S_r}{T_s} \right)^{(i+1)} B(i + 1, r); & S_r < T_s \\
(s - 1) \sum_{j=0}^{r-1} r - 1j (-1)^j \left( \frac{T_s}{S_r} \right)^{(j+1)} B(j + 1, s - 1); & S_r > T_s
\end{cases}
\]

Proof. Using theorem (3.5) and applying the condition $S_r < T_s$, we have,

\[
\hat{P}_{HI} = (s - 1) \int_0^{T_s} \left( 1 - z \right)^{(s-2)^{r-1}} \left[ 1 - \frac{S_r}{T_s} \right] \, dz
\]

and the first assertion follows.

Furthermore, when $S_r > T_s$,

\[
\hat{P}_{HI} = (s - 1) \sum_{j=0}^{r-1} r - 1j (-1)^j \left( \frac{T_s}{S_r} \right)^{(j+1)} \int_0^1 u^j (1 - u)^{r-1} \, du
\]

and the second assertion follows.

Theorem 3.7 The Variance of $\hat{R}_{HI}(t)$ is given is

\[
\text{Var}(\hat{R}_{HI}(t)) = \left( \frac{g^R(t)}{\theta^2} \right)^r \exp \left( -\frac{g^R(t)}{\theta} \right) \left[ a_{r-1} \left( \frac{\theta}{g^R(t)} \right) \right]
\]

\[
+ (a_{r-2}) \exp \left( \frac{g^R(t)}{\theta} \right) \left\{ -E_i \left( -\frac{g^R(t)}{\theta} \right) \right\}
\]
where $a_i = (-1)^i 2r - 2$ and $-E_i(-x) = \int_x^\infty e^{-u} du$.

**Proof.** Here,

$$\text{Var}(R_H(t)) = E(R_H(t))^2 - (E(R_H(t)))^2.$$  

From equation (4),

$$E(R_H(t))^2 = \frac{1}{\Gamma(r)} \exp\left(-\frac{g^\beta(t)}{\theta}\right) (\frac{g^\beta(t)}{\theta})^r \int_0^\infty \frac{u^{2r-2}}{(1+u)^{r-1}} du$$

$$= \frac{1}{\Gamma(r)} \exp\left(-\frac{g^\beta(t)}{\theta}\right) (\frac{g^\beta(t)}{\theta})^r I,$$  

where

$$I = \int_0^\infty \frac{u^{2r-2}}{(1+u)^{r-1}} du$$

$$= \sum_{i=0}^{2r-2} a_i \int_0^\infty \frac{u^{i}}{(1+u)^{r-1}} du + \sum_{i=0}^{2r-2} a_i \int_0^\infty (1 + u)^{i-r+1} \exp\left(-\frac{ug^\beta(t)}{\theta}\right) du,$$

and $a_i = (-1)^i 2r - 2$.

Using the following result (Erdelyi[22]) that

$$-E_i(-x) = \int_x^\infty e^{-u} du$$

and

$$\int_0^\infty \frac{\exp(-up)}{(u+a)^r} du = \sum_{i=0}^{2r-2} \frac{(-1)^i m^{i-1} (m-1) \ldots (m-i+1)}{(r-1)!} \exp(ap) (E_i) (-ap),$$

we have,

$$\int_0^\infty \frac{1}{(1+u)^{r-i}} \exp\left(-\frac{ug^\beta(t)}{\theta}\right) du = \sum_{i=0}^{2r-2} \frac{(-1)^i m^{i-1} (m-1) \ldots (m-i+1)}{(r-1)!} \exp\left(\frac{g^\beta(t)}{\theta}\right)$$

$$- \frac{1}{(r-i-2)!} \left(-\frac{g^\beta(t)}{\theta}\right)^{r-i-2} E_i\left(-\frac{g^\beta(t)}{\theta}\right),$$

**Furthermore,**

$$\int_0^\infty \frac{1}{(1+u)^{r}} \exp\left(-\frac{ug^\beta(t)}{\theta}\right) du = \exp\left(\frac{g^\beta(t)}{\theta}\right) \int_1^\infty \frac{1}{u} \exp\left(-\frac{ug^\beta(t)}{\theta}\right) du$$

$$= \frac{g^\beta(t)}{\theta} \exp\left(-\frac{g^\beta(t)}{\theta}\right) \int_1^\infty \frac{e^{-z}}{z} dz$$

$$= - \frac{g^\beta(t)}{\theta} \exp\left(-\frac{g^\beta(t)}{\theta}\right) E_i\left(-\frac{g^\beta(t)}{\theta}\right),$$

$$\int_0^\infty \exp\left(-\frac{ug^\beta(t)}{\theta}\right) du = \frac{\theta}{g^\beta(t)}.$$  

Also,

$$\int_0^\infty (1 + u)^{i-r+1} \exp\left(-\frac{ug^\beta(t)}{\theta}\right) du = \sum_{w=0}^{i-r+1} i - r + 1 \ w$$

$$\int_0^\infty u^w \exp\left(-\frac{ug^\beta(t)}{\theta}\right) du$$

and

$$(26)$$
\[ = \sum_{w=0}^{t-r+1} i - r + 1 \ w! \left( -\frac{\theta}{g^\theta(t)} \right)^{w+1}. \] (27)

Substituting equation (22), (23), (24) and (24) in (3.19) and then in (20), the theorem follows.

### 3.2 When \( \beta \) is known, Maximum likelihood estimators

Using equation (6), the MLE of \( \theta \) under Type 2 censoring is

\[ \hat{\theta} = \frac{S_r}{r} \] (28)

From equation(28) and one - one property of MLEs, following theorem give the MLE of \( R(t) \).

**Theorem 3.8**

\[ \tilde{R}(t)_{II} = e^{-\frac{rg^\theta(t)}{S_r}}. \] (29)

**Corollary 3.9** The MLE of \( f(x; \beta, \theta) \) at a particular point \( x \) is

\[ \tilde{f}_I(x; \beta, \theta) = \frac{r}{S_r} \beta g(x) g^{\beta-1}(x) \exp \left( -\frac{rg^\theta(x)}{S_r} \right). \] (30)

**Proof.** Considering the fact that given below

\[-\frac{d}{dt} \tilde{R}_I(t) = \tilde{f}_I(t; \beta, \theta) \]

the theorem follows and on using this we get similar result as given in corollary 3.4.

**Theorem 3.10** The MLE of \( P \) is given is

\[ \tilde{P}_I = \int_{y=0}^{\infty} \exp \left[ \left( -\frac{r}{s_r} \right) g^\beta \left( \frac{h^2 + T^2}{s} \right) \right] e^{-\frac{z}{s_r}} dz \] (31)

**Lemma 3.11** The MLE of \( P \) when \( X \) and \( Y \) belongs to same families of distribution i.e. \( \beta = \beta_1 = \beta_2 \) is given by

\[ \tilde{P}_I = \left( \frac{s_{S_r}}{S_{S_r} + rT_s} \right). \] (32)

**Theorem 3.12** The variance of \( \tilde{R}_I(t) \) is defined as

\[ \text{var} \left( \tilde{R}_I(t) \right) = \frac{2}{(r-1)!} \left( 2r g^2(\theta) \right)^2 K_r \left( 2 \frac{2rg^\theta(\theta)}{\theta} \right) \]

\[- \frac{2}{(r-1)!} \left( \frac{rg^\theta(\theta)}{\theta} \right)^2 K_r \left( 2 \frac{rg^\theta(\theta)}{\theta} \right)^2 \] (33)

where \( K_r(\cdot) \) is the modified Bessel function of the kind- II with order \( r \).

**Proof.** Applying equation (4), (6) and theorem3.12, we get

\[ E \left( \tilde{R}_I(t) \right) = \int_0^{\infty} \exp \left( -\frac{rg^\theta(t)}{S_r} \right) \left( \frac{s_r}{\theta} \right)^{r-1} \exp \left( -\frac{s_r}{\theta} \right) dS_r \]

\[ = \frac{1}{\Gamma(r)} \int_0^{\infty} y^{r-1} \exp \left( -\left( y + \frac{rg^\theta(t)}{\theta} \right) \right) dy \] (34)

Applying the result of Watson[45] that

\[ \int_0^{\infty} x^{-r} \exp \left( -\left( ax + \frac{b}{x} \right) \right) dx = 2a^{\frac{r-1}{2}} \ K_{r-1} \left( 2\sqrt{ab} \right), \]

it is to be noted that \( K_{-r}(\cdot) = K_r(\cdot) \) for “\( r = 0, 1, 2, \ldots \)” “\( a = 1, \ -r = n - 1, n = r - 1 \)” and \( b = \frac{r g^\theta(t)}{\theta} \). After simplification from equation(3.31), we get

\[ E \left( \tilde{R}_I(t) \right) = \frac{2}{(r-1)!} \left( \frac{rg^\theta(t)}{\theta} \right)^2 K_r \left( 2 \left( \frac{rg^\theta(t)}{\theta} \right)^2 \right). \] (35)

Similarly, we can develop the expression of second order moment given as under

\[ E \left( \tilde{R}_I(t) \right)^2 = \frac{2}{(r-1)!} \left( \frac{2rg^\theta(t)}{\theta} \right)^2 K_r \left( 2 \left( \frac{2rg^\theta(t)}{\theta} \right)^2 \right). \] (36)

The theorem follows by combing equation(3.32) and (3.33).

### 3.3 Exact Confidence Interval Under type 2 censoring scheme

Construction of two sided confidence interval for MLE and UVMUE of \( \theta \), \( R(t) \) and \( P \) under type 2 censoring scheme is considered in this part. First, we construct the two-sided confidence interval problem for MLE of \( \theta \), using the pivotal quantity \( 2\theta S_r \). By defining \( \chi^2 \) as the value of \( \chi^2 \) such that

\[ P \left( \chi^2 > \chi^2_\alpha \right) = \int_{\chi^2_\alpha}^{\infty} p(\chi^2) d\chi^2 = \alpha, \] (37)

where \( p(\chi^2) \) is the probability distribution function of \( \chi^2 \) -distribution with 2r degree of freedom.

Knowing the fact that, \( 2\theta S_r \sim \chi^2_{(2r)}(\cdot) \)
Now we will next derive confidence interval for UMVUE and MLE for P. UMVUE is given as

\[ P \left[ \chi^2(2r) \left( 1 - \frac{a}{2} \right) \leq \chi^2(P) \leq \chi^2(2r) \left( \frac{a}{2} \right) \right] = 1 - \alpha \]

Also, since \( P \) is unbiased estimator of \( \theta \), one can obtain the confidence interval for MLE of \( P \) as

\[ 100(1 - \alpha) \% \text{ confidence interval for MLE of } P \text{ is} \]

\[ \left[ \frac{\bar{r} \chi^2(P) \left( 1 - \frac{a}{2} \right)}{2(r-1) \chi^2(P)} \right] \]

(38)

In order to construct the problem of two-sided confidence interval for UMVUE of \( \theta \), the unbiased estimator of \( \theta \) for UMVUE is given as \( \hat{\theta} = \frac{\bar{r} \chi^2(P)}{\chi^2(P)} \). Proceeding in similar way as above, 100(1 - \( \alpha \)) % confidence interval for UMVUE of \( \theta \) is of the form

\[ \left[ \frac{\theta}{2(r-1) \chi^2(P)} \right] \]

(39)

According to theorem (3.3), we have

\[ \hat{R}_{UVMUE}(t) = (1 - \frac{\theta}{\theta})^{r-1} \]

Thus, 100(1 - \( \alpha \)) % confidence interval for UMVUE of \( \theta \) is as under

\[ \left[ \left( 1 - \frac{2\theta \chi^2(P) \left( \frac{a}{2} \right)}{2(r-1) \chi^2(P)} \right) \right] \]

(40)

Now we will next derive confidence interval for UMVUE and MLE for P. Since \( \bar{\theta}_1 = \frac{\bar{r}}{\bar{T}^2} \) and \( \bar{\theta}_2 = \frac{s_r}{T^2} \), and also \( P = \frac{\theta_1 + \theta_2}{\theta_1 + \theta_2} \) = MLE of \( \theta \). From two random quantities which are independent and we have, \( \frac{\bar{r} \chi^2(P)}{\chi^2(P)} \sim F_{2r,2s} \); a scaled F distribution. Following that \( \hat{P}_{MLE} = \frac{1}{\frac{\theta_1 + \theta_2}{\theta_1 + \theta_2}} \), by simple transformation techniques one can obtain the confidence interval for MLE of P as

\[ P \left[ F_{2r,2s} \left( 1 - \frac{a}{2} \right) \leq F_{2r,2s} \leq F_{2r,2s} \left( \frac{a}{2} \right) \right] = 1 - \alpha \]

\[ P \left[ F_{2r,2s} \left( 1 - \frac{a}{2} \right) \leq \theta \frac{s_r}{T^2} \leq F_{2r,2s} \left( \frac{a}{2} \right) \right] = 1 - \alpha \]

\[ P \left[ \frac{(\bar{r} \chi^2(P) \left( 1 - \frac{a}{2} \right)}{s s_r} + 1 \right] \leq \frac{\theta_1 + \theta_2}{s_r} \leq \left( \frac{\bar{r} \chi^2(P) \left( 1 - \frac{a}{2} \right)}{s s_r} + 1 \right)^{-1} \]

Therefore, 100(1 - \( \alpha \)) % confidence interval of P for MLE is as under

\[ \left[ \frac{(\bar{r} \chi^2(P) \left( 1 - \frac{a}{2} \right)}{s s_r} + 1 \right] \]

(41)

Now we have considered UMVUE for constructing the problem of confidence interval for P, we know that

\[ (s - 1) \int_0^1 \left[ 1 - \left( 1 - \frac{\theta_1 + \theta_2}{s_r} \right)^{-1} \right] (1 - z) dz; \quad S_r > T_s. \]

Also,

\[ \frac{\bar{r} \chi^2(P)}{\chi^2(P)} \sim F_{2r,2s} \]

\[ P \left[ F_{2r,2s} \left( 1 - \frac{a}{2} \right) \leq \frac{\theta_1 s_r}{\theta_2 T^2} \leq F_{2r,2s} \left( \frac{a}{2} \right) \right] = 1 - \alpha \]

or

\[ P \left[ \frac{\theta_1 s_r}{\theta_2 T^2} \leq F_{2r,2s} \left( 1 - \frac{a}{2} \right) \right] \leq \frac{\theta_1 s_r}{\theta_2 T^2} \leq \left( \frac{\theta_1 + \theta_2}{s_r} \right) \]

\[ \frac{\theta_1 s_r}{\theta_2 T^2} \leq F_{2r,2s} \left( 1 - \frac{a}{2} \right) \]

\[ (s - 1) \int_0^1 \left[ 1 - \left( 1 - \frac{\theta_1 s_r}{\theta_2 T^2} \right)^{-1} \right] (1 - z) dz; \quad S_r > T_s. \]
or

\[
P \left[ (1 - z)^{s-2} \left[ 1 - \left(1 - \frac{\theta r}{\theta + s} F_{2r,2s} \left(1 - \frac{\alpha}{2}\right) z \right) \right] \leq (1 - z)^{s-2} \left(1 - \frac{S_r}{t_s} z \right) \leq (1 - z)^{s-2} \left[ 1 - \left( \frac{\theta r}{\theta + s} F_{2r,2s} \left( \frac{\alpha}{2} \right) z \right) \right] \right] = 1 - \alpha.
\]

Thus, 100(1 - \alpha)% confidence interval of P for UMVUE is as under

\[
\left[ (s - 1) \int_0^1 (1 - z)^{s-2} [1 - \left(1 - \frac{\theta r}{\theta + s} F_{2r,2s} \left(1 - \frac{\alpha}{2}\right) z \right)] \leq (1 - z)^{s-2} \right. \\
\left. \left(1 - \frac{S_r}{t_s} z \right) \leq (1 - z)^{s-2} \left(1 - \frac{\theta r}{\theta + s} F_{2r,2s} \left( \frac{\alpha}{2} \right) z \right) \right] \\

\]

We can get the result when \( S_r < T_s \) in similar manner.

### 3.4 Asymptotic confidence intervals

Likelihood function of family of the distribution in equation (3.3) is given by

\[
lnh = ln r! + r ln \beta - r ln \theta + \sum_{i=1}^r \ln \left( g^\beta(x_i) \right) + \sum_{i=1}^r \ln \left( g^{\beta-1}(x_i) \right) - \frac{S_r}{\theta}
\]

Estimating the parameter \( \theta \) and \( \beta \) taking first and second order partial derivation of equation (3.40), we have

\[
\frac{\partial^2 lnh}{\partial \beta^2} = \frac{r}{\theta^2} - \frac{1}{\theta} \sum_{i=1}^r \left( g^{\beta-1}(x_i) (ln(g(x_i)))^2 \right) - \frac{(n-r)}{\theta} \left( g^\beta(x_r) (ln(g(x_r)))^2 \right)
\]

(43)

Thus we can obtain an estimate of the information matrix given as

\[
I(\hat{\theta}, \hat{\beta}) = \begin{bmatrix}
\frac{\partial^2 lnh}{\partial \theta^2} & \frac{\partial^2 lnh}{\partial \theta \beta} \\
\frac{\partial^2 lnh}{\partial \beta \theta} & \frac{\partial^2 lnh}{\partial \beta^2}
\end{bmatrix}
\]

where \( \hat{\theta} = \hat{\theta}_H \) and \( \hat{\beta} = \hat{\beta}_H \) are the MLEs estimates of the parameter and \( V(\hat{\theta}) \) and \( V(\hat{\beta}) \) are elements of \( I^{-1}(\hat{\theta}, \hat{\beta}) \). The approximate \((1 - \alpha)100\%\) confidence intervals for the parameters \( \theta \) and \( \beta \) is, therefore, given as, \( \hat{\theta} = \hat{\theta}_H \pm \gamma_{a}\frac{1}{2} \sqrt{V(\hat{\beta}_H)} \) and \( \hat{\beta}_H \pm \gamma_{a}\frac{1}{2} \sqrt{V(\hat{\beta}_H)} \) respectively, where \( \gamma_{a}\frac{1}{2} \) is the upper \((\frac{a}{2})\) percentile of standard distribution.

### 4 Type I Censoring Scheme for Point Estimators

Let “0 < \( X(1) \leq X(2) \leq \ldots \leq X(n) \)” be the failure time of \( n \) items under test from equation (1). The test start at time \( X(0) = 0 \) and the system runs until "\( X(1) = x(1) \)" when the 1st failure occurs. The unsuccessful item is replaced by a latest one and the system runs till the 2nd failure occurs at time "\( X(2) = x(2) \)" and so on. The experiment is completed at time \( t_0 \). Before proving the main theorem of this section, we first state lemma.

**Lemma 4.1** If \( N(t_0) \) be the number of failures during the interval \( [0, t] \), then

\[
P[N(t_0) = r | t_0] = \exp \left( -\frac{n \theta^r}{\theta} \right) \left( \frac{n \theta^r}{\theta} \right)^r r!
\]

(44)

**Proof.** Let us make the transformation

\[
W_1 = g^\beta(X(1)), W_2 = g^\beta(X(2)) - g^\beta(X(1)), \ldots, W_n = g^\beta(X(n)) - g^\beta(X(n-1)).
\]

The p.d.f. of \( W_1 \) is

\[
h(w_1) = \frac{n}{\theta} \exp \left( -\frac{n w_1}{\theta} \right).
\]

Moreover \( W_1, W_2, \ldots, W_n \) are iid random variable. Using the monotonically property of \( g^\beta(x) \).

\[
P[N(t_0) = r | t_0] = P(X(r) \leq t_0) - P(X(r+1) \leq t_0)
\]

\[
= P \left( g^\beta(X(r)) \leq g^\beta(t_0) \right)
\]

\[
- P \left( g^\beta(X(r+1)) \leq g^\beta(t_0) \right)
\]

(46)

\[
= P \left( W_1 + W_2 + \ldots + W_r \leq g^\beta(t_0) \right)
\]

\[
- P \left( W_1 + W_2 + \ldots + W_{r+1} \leq g^\beta(t_0) \right)
\]

(47)

Using the reproductive property of exponential distribution (see Johnson and Kotz[24]), \( U = \frac{n}{\theta} \sum_{i=1}^n W_i \) follows gamma distribution with p.d.f.

\[
h(u) = \frac{1}{\Gamma(r)} u^{r-1} e^{-u}; u > 0
\]

(48)

Using the result of Patel, Kapadia, and Owen[36] and equation (44) we achieve that
Proof. From lemma (4.1) and F-N factorization theorem (see Rohatgi[38]) that r is sufficient for estimating \( \theta \). r is complete and it is distributed to the family of exponential (see Rohatgi[38]). The theorem now follows from the result that the qth factorial moment of distribution of r is given by

\[
E(r(r-1)...(r-q+1)) = \left(n\theta^{-1}g^\beta(t(0))\right)^q
\]

Theorem 4.3 The UMVUE of \( R(t) \) is defined as

\[
\hat{R}_1(t) = \left[1 - \frac{g^\beta(t)}{ng^\beta(t_0)}\right]^T, \quad g^\beta(t) < ng^\beta(t_0)
\]

otherwise.

Proof. Find a function \( f(r) \) such that \( E(f(r)) = R(t) \), so \( f(r) \) is the UMVUE of \( R(t) \),

\[
\sum_{r=0}^{\infty} f(r) \left[\frac{|n\theta^{-1}g^\beta(t(0))|}{r!}\right] \exp \left(-\frac{ng^\beta(t_0)}{\theta}\right) = \exp \left(-\frac{g^\beta(t)}{\theta}\right)
\]

or

\[
\sum_{r=0}^{\infty} f(r) \left[\frac{|n\theta^{-1}g^\beta(t(0))|}{r!}\right] = \exp \left(\theta^{-1}ng^\beta(t) - \theta^{-1}g^\beta(t(0))\right)
\]

Equation (54) is satisfied if we choose Chosing the given as

\[
f(r) = \left[1 - \frac{g^\beta(t)}{ng^\beta(t_0)}\right]^T
\]

the equation (54) is satisfied and the theorem holds.

Corollary 4.4 The UMVUE of \( f(x; \beta, \theta) \) at a particular point \( x \) is

\[
f_1(x; \beta, \theta) = \begin{cases} \frac{r_1g(x)g^\beta_1(x)}{ng^\beta(t_0)} \left(1 - \frac{g^\beta_1(x)}{ng^\beta(t_0)}\right)^{-1}, & g^\beta(x) < ng^\beta(t_0) \\ 0, & \text{otherwise} \end{cases}
\]

Proof. Corollary (4.4) holds by adopting the similar argument using proving the corollary (3.4). Let X and Y be two independent r.v.’s following the classes of distribution \( f_1(x; \beta_1, \theta_1) \) and \( f_2(y; \beta_2, \theta_2) \) respectively.

\[
f_1(x; \beta_1, \theta_1) = \frac{\beta_1 g(x)g^\beta_1(x)e^{-\frac{g^\beta_1(x)}{\theta_1}}}{x > 0, (\beta_1, \theta_1) > 0}
\]

and

\[
f_2(y; \beta_2, \theta_2) = \frac{\beta_2 h(y)h^\beta_2(y)e^{-\frac{h^\beta_2(y)}{\theta_2}}}{y > 0, (\beta_2, \theta_2) > 0}
\]

Let n items on X and m items on Y are put on a life test. Let \( t_0 \) and \( t_{00} \) be the termination numbers for X and Y are r and s, respectively,

\[
f_1(x; \beta_1, \theta_1) = \begin{cases} \frac{r_1g(x)g^\beta_1(x)}{ng^\beta(t_0)} \left(1 - \frac{g^\beta_1(x)}{ng^\beta(t_0)}\right)^{-1}, & g^\beta_1(x) < ng^\beta_1(t_0) \\ 0, & \text{otherwise} \end{cases}
\]

and

\[
f_2(y; \beta_2, \theta_2) = \begin{cases} \frac{s_2h(y)h^\beta_2(y)}{mh^\beta_2(t_{00})} \left(1 - \frac{h^\beta_2(y)}{mh^\beta_2(t_{00})}\right)^{-1}, & h^\beta_2(y) < mh^\beta_2(t_{00}) \\ 0, & \text{otherwise} \end{cases}
\]
Theorem 4.5 The UMVEUE of $\hat{\theta}_l$ is given by

$$
\hat{\theta}_l = \left\{ \begin{array}{cl}
\frac{s \int_{0}^{z} \frac{m h^{2}(t_{00})}{m h^{2}(t_{00})} (1 - z)^{s-1} \left( 1 - \frac{g^{r}(n g^{r}(m h^{2}(t_{00})))}{m g^{r}(t_{00})} \right)^{r} \, dz}{1 - g^{r}(n g^{r}(m h^{2}(t_{00})))} & < h^{r}(m h^{2}(t_{00})), \\
\frac{s \int_{0}^{1} (1 - z)^{s-1} \left( 1 - \frac{g^{r}(h^{r}(m h^{2}(t_{00})))}{m g^{r}(t_{00})} \right)^{r} \, dz}{h^{r}(m h^{2}(t_{00}))} & < g^{r}(n g^{r}(t_{00})).
\end{array} \right.
$$

Proof. Using similar arguments which are use in theorem (3.5) we get,

$$
\hat{\theta}_l = \int_{y=0}^{y=\infty} \int_{x=y} \int_{\beta_{1}, \theta_{1}} f_{2}(x; \beta_{1}, \theta_{1}) f_{2}(y; \beta_{2}, \theta_{2}) \, dx \, dy
$$

$$
= \int_{y=0}^{\infty} \tilde{R}_{2}(y; \beta_{2}, \theta_{2}) \left\{ - \frac{d}{dy} \tilde{R}_{2}(y; \beta_{2}, \theta_{2}) \right\} \, dy.
$$

Using $\tilde{R}_{1}(t)$ and $\tilde{f}_{2}(y; \beta_{2}, \theta_{2})$, we get

$$
\tilde{R}_{1}(t) = \int_{y=0}^{\infty} \left[ 1 - \frac{g^{r}(t)}{n g^{r}(t)} \right] \frac{s \beta_{2} \cdot g^{r} \cdot g^{r}(y) \cdot h^{r-1}(y)}{m h^{2}(t_{00})} \left[ 1 - \frac{h^{r}(y)}{m h^{2}(t_{00})} \right] \, dy;
$$

$$
h^{r-1}(y) \cdot g^{r}(y) \cdot g^{r}(t) < m h^{2}(t_{00}), \quad g^{r}(t) < n g^{r}(t_{00}),
$$

$$
= \frac{s \beta_{2}}{m h^{2}(t_{00})} \int_{y=0}^{\infty} \left[ 1 - \frac{g^{r}(t)}{n g^{r}(t)} \right] \frac{s h^{r-1}(y) \cdot g^{r}(y) \cdot g^{r}(t)}{m h^{2}(t_{00})} \left[ 1 - \frac{h^{r}(y)}{m h^{2}(t_{00})} \right] \, dy;
$$

$$
h^{r-1}(y) \cdot g^{r}(y) \cdot g^{r}(t) < m h^{2}(t_{00}), \quad g^{r}(t) < n g^{r}(t_{00}),
$$

By putting $z = \frac{h^{r}(y)}{m h^{2}(t_{00})}$ the theorem holds.

Corollary 4.6 Substitute $t_{0} = t_{00}$ and $\beta_{1} = \beta_{2} = \beta$ i.e. $X$ and $Y$ belong to same family of distributions,

$$
\hat{\theta}_l = \left\{ \begin{array}{cl}
s \int_{t_{00}}^{t_{00}} \left[ 1 - \frac{m^{i}}{n^{i}} \right] f_{1}(t; \beta_{1}) \, dt & , m < n, \\
n \int_{t_{00}}^{t_{00}} (-1)^{i} \left[ 1 - \frac{m^{i}}{n^{i}} \right] \int_{y=0}^{1} z^{i} (1 - z)^{s-1} \, dz & , n < m
\end{array} \right.
$$

Proof. From Theorem (4.5) when $m < n$.

$$
\hat{\theta}_l = s \int_{y=0}^{1} \left( 1 - \frac{m}{z} \right)^{r} (1 - z)^{s-1} \, dz
$$

and 1st statement follows. Again from Theorem (4.5) when $n < m$

$$
\hat{\theta}_l = s \int_{0}^{n} \left[ 1 - \frac{u}{m} \right] u^{r} (1 - u)^{s-1} \, du
$$

and the 2nd statement follows.

Next proving variance expression for $\tilde{R}_{l}(t)$ as under

Theorem 4.7 The variance of $\tilde{R}_{l}(t)$ is given by

$$
Var \left( \tilde{R}_{l}(t) \right) = \exp \left( - \frac{g^{r}(t)}{\theta} \left( 2 - \frac{g^{r}(t)}{n g^{r}(t_{0})} \right) \right) - \exp \left( - \frac{2 g^{r}(t)}{\theta} \right).
$$

Proof. From equation (4.7),

$$
E \left( \tilde{R}_{l}(t) \right)^{2} = \sum_{r=0}^{\infty} \frac{1}{r!} \left( 1 - \frac{g^{r}(t)}{n g^{r}(t_{0})} \right)^{2r} \left( n \theta^{-1} g^{r}(t_{0}) \right)^{r} \exp \left( - n \theta^{-1} g^{r}(t_{0}) \right)
$$

$$
= \exp \left( 1 - \frac{g^{r}(t)}{n g^{r}(t_{0})} \right)^{2} \left( n \theta^{-1} g^{r}(t_{0}) \right)
$$
\[ \exp \left( -n \theta^{-1} g^\beta (t_0) \right) \] (66)

The theorem follows from (4.23) and the fact that \( \hat{R}_t(t) \) is an unbiased estimator of \( R(t) \).

### 4.2 When \( \beta \) is known, Maximum likelihood estimator

Under the sampling scheme of Bartholomew[5], it follows from equation (44) as below

\[ \hat{\theta}_t = \left( \frac{n}{r} g^\beta (t_0) \right). \] (67)

From (67) and one-one property of the MLEs, the MLE of \( R(t) \) in the form of theorem is as under.

**Theorem 4.8** The MLE of \( R(t) \) is as under

\[ \hat{R}_t(t) = \exp \left( - \frac{rg^\beta(t)}{ng^\beta(t_0)} \right). \] (68)

**Corollary 4.9** The MLE of \( f(x; \beta, \theta) \) at a particular point \( x \) is

\[ \hat{f}_x(x; \beta, \theta) = \frac{r g(x) \beta g^{\beta-1}(x)}{g^\beta(t_0)} \exp \left( - \frac{rg^\beta(t)}{ng^\beta(t_0)} \right) \] (69)

**Proof.** Using the fact that

\[ \hat{R}_t(t) = \int_t^\infty \hat{f}(x; \beta, \theta) \, dx \]

or

\[ - \frac{d}{dt} \hat{R}_t(t) = \hat{f}_x(t; \beta, \theta) \]

and corollary (4.9) follows.

**Theorem 4.10** The MLE of the \( P \) under type I censoring is given by

\[ \hat{P}_t = \int_{t_0}^\infty \exp \left( - \frac{rg_1}{ng_1(t_0)} \frac{mhz(t_0) - \beta z}{s} \right) \exp(-z) \, dz. \] (70)

**Proof.**

\[ \hat{P}_t = \int_{t_0}^\infty \int_{z_0}^\infty \hat{f}_1(x; \beta_1, \theta_1) f_2(x; \beta_2, \theta_2) \, dx \, dy \]

\[ = \int_{t_0}^\infty \hat{R}_1(y; \beta_1, \theta_1) \left[ - \frac{d}{dy} \hat{R}_2(y; \beta_2, \theta_2) \right] \, dy. \]

Using \( \hat{R}_1(t) \), corollary (4.9), putting \( z = \frac{sh - \beta z}{mh \beta_z(t_0)} \) and theorem holds.

**Corollary 4.11** Substituting \( t_0 = t_0 \) and \( \beta_1 = \beta_2 = \beta \) i.e. \( X \) and \( Y \) belong to same family of distributions,

\[ \hat{P}_t = \frac{sn}{sn + rm}. \] (71)

**Theorem 4.12** The variance of MLE \( R(t) \) is given by

\[ \text{var}(\hat{R}_t(t)) = \exp \left( \frac{ng^\beta(t_0)}{\theta} \left( \exp \left( - \frac{rg^\beta(t)}{ng^\beta(t_0)} - 1 \right) \right) \right) \]

\[ - \exp \left( \frac{2ng^\beta(t_0)}{\theta} \left( \exp \left( - \frac{rg^\beta(t)}{ng^\beta(t_0)} - 1 \right) \right) \right) \] (72)

**Proof.** Using equation (4.7) and (4.25) and the theorem follows.

### 4.3 Asymptotic confidence intervals

From Lemma (4.1) the likelihood function is defined as

\[ \ln N = r \ln(n) + r \ln(1) + r \ln(\theta) + r \ln(g(\beta(t_0))) - \ln(r!) - \frac{ng^\beta(t_0)}{\theta}. \] (73)

Estimating the parameter \( \theta \) and \( \beta \) taking first and second order partial derivation of (4.30), we have

\[ \frac{\partial^2 \ln N}{\partial \theta^2} = \frac{r}{\theta^2} - 2 \frac{ng^\beta(t_0)}{\theta^3} \] (74)

Thus we can obtain an estimate of the information matrix as given

\[ I(\hat{\theta}, \hat{\beta}) = \begin{bmatrix} \frac{\partial^2 \ln N}{\partial \theta^2} & -\frac{\partial^2 \ln N}{\partial \theta \beta} \\ -\frac{\partial^2 \ln N}{\partial \theta \beta} & \frac{\partial^2 \ln N}{\partial \beta^2} \end{bmatrix} \]

where \( \hat{\theta} = \hat{\theta}_t \) and \( \hat{\beta} = \hat{\beta}_t \) are the MLEs estimates of the parameter and \( V(\hat{\theta}) \) and \( V(\hat{\beta}) \) are elements of \( I^{-1}(\hat{\theta}, \hat{\beta}) \). The approximate \( (1 - \alpha)100\% \) confidence intervals for the parameters \( \theta \) & \( \beta \) is, therefore, given as, \( \hat{\theta} = \hat{\theta}_t \pm \sqrt{\frac{\alpha}{2}} V(\hat{\theta}_t) \) and \( \hat{\beta}_t \pm \sqrt{\frac{\alpha}{2}} V(\hat{\beta}_t) \) respectively, where \( \sqrt{\frac{\alpha}{2}} \) is the upper \( \left( \frac{\alpha}{2} \right) \) percentile of standard distribution.
5 Hypotheses testing for different procedures

In life testing experiments, the hypothesis is defined as $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$. It follows from equation (3.3) that the likelihood function observing $\theta$ is as under

$$L(\theta|x) = n(n-1) \ldots (n-r+1) \frac{(S_r)^{r}}{\theta^r} \prod_{i=1}^{r} g'(x_i)g^{\theta-1}(x_i)$$

$$\exp\left(-\frac{S_r}{\theta}\right).$$

(76)

Under $H_0$,

$$\sup_{\theta_0} L(\theta|x) = n(n-1) \ldots (n-r+1) \frac{(S_r)^{r}}{\theta_0^r} \prod_{i=1}^{r} g'(x_i)g^{\theta_0-1}(x_i)$$

$$\exp\left(-\frac{S_r}{\theta_0}\right); \Theta = (\theta = \theta_0)$$

(77)

and

$$\sup_{\theta} L(\theta|x) = n(n-1) \ldots (n-r+1) \frac{(r)^{r}}{\theta^r} \prod_{i=1}^{r} g'(x_i)g^{\theta-1}(x_i)$$

$$\exp(-r); \Theta = (\theta = \theta > 0).$$

(78)

The likelihood ratio is defined as

$$\lambda(x) = \frac{\sup_{\theta} L(\theta|x)}{\sup_{\theta_0} L(\theta|x)}$$

$$\lambda(x) = \left(\frac{S_r}{r\theta_0}\right)^{r} \exp\left(-\frac{S_r}{\theta_0} + r\right)$$

(79)

From equation (79) Ist term is increasing monotonically and the 2nd term is decreasing monotonically in $S_r$. Using the truth that $\frac{2S_r}{\theta_0} \sim \chi^2_{2r}$, the critical region is as \{0 < $S_r$ < $k_0$\} U \{$k'_0$ < $S_r$ < $\infty$\}, where $k_0$ and $k_0'$ are obtained such that $P\left[\chi^2_{2r} < \frac{2k_0}{\theta_0} \| 2k'_0 \theta_0 < \chi^2_{2r} \right] = \alpha$. Thus, $k_0 = \frac{\theta_0^2}{2} \chi^2_{2r}(1 - \alpha)$ and $k'_0 = \frac{\theta_0}{2} \chi^2_{2r}(\frac{\alpha}{2})$. Similarly, it can be shown that under type I censoring Bartholomew[5] sampling scheme, the UMPCR for hypothesized test is $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$ is as under $(r < k_1 \text{or} r > k'_1), r \sim \text{Poisson} \left(\frac{ng^\theta}{\theta} \right)$.

Now we assume to test hypothesis as $H_0 : \theta \leq \theta_0 \ \forall, H_1 : \theta > \theta_0$. It follows from equation (6) that, for $\theta_1 > \theta_2$

$$\lambda(x) = \frac{(\frac{S_r}{\theta_1})^{r}}{(\frac{1}{\theta_1} + \frac{1}{\theta_2}).S_r}.$$

(80)

As in equation (80), $\lambda(x)$ is monotonic likelihood in $S_r$, the UMPCR for testing $H_0$ against $H_1$ is define as [see Lehmann[29]]

$$\lambda(x_{(1)}, x_{(2)}, \ldots, x_{(r)}) = \begin{cases} 1, & S_r \leq k''_0 \\ 0, & \text{otherwise} \end{cases}$$

(81)

where $k''_0 = \frac{(\theta_0^2/2) \chi^2_{2r}(1 - \alpha'')}{\theta_0}$ is obtained such that

$$P\left[\chi^2_{2r} < \frac{2k''_0}{\theta_0} \right] = \alpha''.$$

Similarly, under type I censoring Bartholomew[5] sampling scheme, the UMPCR for hypothesized test is $H_0 : \theta \leq \theta_0 \ \forall, H_1 : \theta > \theta_0$ is as under

$$\lambda(r) = \begin{cases} 1, & r \geq k''_1 \\ 0, & \text{otherwise} \end{cases}$$

(82)

where "$k''_1$" is obtained such that $P[r > k''_1] = \beta$.

Next test the hypothesis as $P_0 : \theta = \theta_0$ against $P_1 : \theta \neq \theta_0$ under type 2 censoring and $P = \frac{\theta_1}{\theta_1 + \theta_2}$ when $\beta_1 = \beta_2 = \beta$. For $\delta = \frac{P_0}{1-P_0}$, $H_0 : \delta_1 = \delta \theta_2$ against $H_1 : \delta_1 \neq \delta \theta_2$. For generic constant $\eta$, the likelihood of sampled observation $x$ and $y$ is

$$L(\theta_1, \theta_2|x,y) = (K \frac{1}{\theta_1 + \theta_2}) \exp\left(-\left(\frac{S_r}{\theta_1} + \frac{T_s}{\theta_2}\right)\right).$$

(83)

Under $H_0$,

$$\hat{\theta}_{1H} = \delta \frac{S_r}{(r+s)} \text{ and } \hat{\theta}_{2H} = \frac{T_s}{(r+s)}.$$ Thus,

$$\sup_{\theta_0} L(\theta_1, \theta_2|x,y) = \frac{K^\exp(-r+s)}{\delta^r \left(\frac{S_r+T_s}{\theta_1 + \theta_2}\right)^{r+s}}$$

(84)

Also for whole parametric space $\Theta = \{(\theta_1, \theta_2)/\theta_1, \theta_2 > 0\}$,

$$\sup_{\theta} L(\theta_1, \theta_2|x,y) = \frac{K^\exp(-r+s)}{(S_rT_s)^{r+s}}$$

(85)

From (5.9) and (5.10), the likelihood ratio criterion is
\[ \lambda(\theta_1, \theta_2|\mathbf{xy}) = K \frac{(s_{y|\theta_2})^r}{(s_{F|r+y}^r+1)} \]

On using the fact that
\[ \frac{s_x}{\tau_x} \sim r\theta_2 \frac{s_{x|\theta_1}}{\theta_1}, \]

The critical region is as under
\[ \left( \left\{ \frac{s_x}{\tau_x} < k_2 \right\} \cup \left\{ \frac{s_x}{\tau_x} > k_2 \right\} \right). \]

The estimates are obtained such that
\[ P \left( \frac{s_{x|\theta_2}}{k_{2r} \tau_x} < F_{2r,2s} \cup \frac{s_{x|\theta_2}}{k_{2r} \tau_x} > F_{2r,2s} \right) = \alpha'', \]

where,
\[ k''_2 = \frac{k_{2r}}{s} F_{2r,2s} \left( 1 - \frac{\alpha''}{2} \right) \quad \text{and} \quad k''_2 = \frac{k_{2r}}{s} F_{2r,2s} \left( \frac{\alpha''}{2} \right). \]

### 6 Simulation results

This section, conduct the Monte Carlo simulation study reliability and stress strength reliability using UMVUE and MLES. Here, the inverse transformation method of simulation is used to compare the performance of two estimators under type 2 and type 1 censoring scheme and also their variances. It is shown how simulation can be helpful and illuminating way to approach problems in reliability and stress strength reliability using UMVUE and MLES. Here, the inverse transformation method of Monte Carlo simulation is used to compare the performance of two reliability estimators under type 2 and type 1 censoring scheme. We investigate the performance of the power under type 2 censoring scheme. Generating 10000 sample of size 50 from the inverse transformation method with \( \theta = 1, 1.5, 2.5; \beta = 2 \) and \( p = 2, 3 \). The study is carried out for different values of \( r = 10, 20 \) and 35.

#### Table 1: Performance of the Power estimates under type II

<table>
<thead>
<tr>
<th>( r \to )</th>
<th>P[( \theta^P )]</th>
<th>( \hat{\theta}_{II} )</th>
<th>( \hat{\theta}_{II} )</th>
<th>( \hat{\theta}_{II} )</th>
<th>( \hat{\theta}_{II} )</th>
<th>( \hat{\theta}_{II} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2[1]</td>
<td>1.0076</td>
<td>1.1083</td>
<td>0.9981</td>
<td>1.0480</td>
<td>1.0001</td>
</tr>
<tr>
<td></td>
<td></td>
<td>9.3871</td>
<td>0.5349</td>
<td>9.2132</td>
<td>0.225</td>
<td>9.1122</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.995,</td>
<td>(1.094,</td>
<td>(0.989,</td>
<td>(1.039,</td>
<td>(0.993,</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.020)</td>
<td>1.122)</td>
<td>1.007)</td>
<td>1.057)</td>
<td>1.007)</td>
</tr>
<tr>
<td></td>
<td>3[1]</td>
<td>0.9901</td>
<td>1.307</td>
<td>1.0076</td>
<td>1.1638</td>
<td>0.992</td>
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<td></td>
<td>10.0684</td>
<td>1.8522</td>
<td>9.4679</td>
<td>0.7116</td>
<td>9.3134</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.97,</td>
<td>(1.281,</td>
<td>(0.994,</td>
<td>(1.148,</td>
<td>(0.982,</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.010)</td>
<td>1.333)</td>
<td>1.022)</td>
<td>1.18)</td>
<td>1.002)</td>
</tr>
<tr>
<td></td>
<td>2[2.25]</td>
<td>2.2237</td>
<td>2.4461</td>
<td>2.2573</td>
<td>2.3701</td>
<td>2.249</td>
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<tr>
<td></td>
<td></td>
<td>5.2592</td>
<td>2.5844</td>
<td>4.1083</td>
<td>1.9154</td>
<td>3.6551</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(2.195,</td>
<td>(2.415,</td>
<td>(2.237,</td>
<td>(2.349,</td>
<td>(2.235,</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.252)</td>
<td>2.477)</td>
<td>2.278)</td>
<td>2.391)</td>
<td>2.265)</td>
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<td></td>
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<td>(3.300,</td>
<td>(4.356,</td>
<td>(3.34,</td>
<td>(3.858,</td>
<td>(3.338,</td>
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<td></td>
<td></td>
<td>3.439)</td>
<td>4.54)</td>
<td>3.433)</td>
<td>3.966)</td>
<td>3.407)</td>
</tr>
<tr>
<td></td>
<td>3[15.625]</td>
<td>15.5173</td>
<td>20.4829</td>
<td>15.6706</td>
<td>18.0995</td>
<td>15.6461</td>
</tr>
<tr>
<td></td>
<td></td>
<td>392.7883</td>
<td>376.8654</td>
<td>255.3622</td>
<td>165.0861</td>
<td>202.7628</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(15.201,</td>
<td>(20.066,</td>
<td>(15.457,</td>
<td>(17.852,</td>
<td>(15.485,</td>
</tr>
<tr>
<td></td>
<td></td>
<td>15.833)</td>
<td>20.9)</td>
<td>15.885)</td>
<td>18.347)</td>
<td>15.807)</td>
</tr>
</tbody>
</table>

In the above table 1, we workout that true estimate in square braces [], average estimate \( \hat{\theta}_{II} \) and \( \hat{\theta}_{II} \), mean square error and confidence interval in ()braces for umvue and mle. we have summarized the result for the table 1, the average estimates is nearly close to true estimate, both estimators are efficient, the mean square error(mse) is decreasing when we increase the
value of \( r \) and confidence interval lies with the interval. To the evaluate the performance of the \( R(t) \) in type-2 censoring scheme, it require to generate 10000 sample of size 50 from the inverse transformation method with \( \theta = 1.5, \beta = 1 \). The study is carried out for different values of \( r = 10, 20, 35 \) and \( t = 0.20, 0.40, 0.60, 0.80, 1 \).

Table 2: Performance of the \( R(t) \) estimates under type II when \( \theta = 1.5 \) and \( \beta = 1 \)

<table>
<thead>
<tr>
<th>( t )</th>
<th>( R_{II}(t) )</th>
<th>( \hat{R}_{II}(t) )</th>
<th>( \hat{R}_{II}(t) )</th>
<th>( \hat{R}_{II}(t) )</th>
<th>( \hat{R}_{II}(t) )</th>
<th>( \hat{R}_{II}(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20</td>
<td>0.8752</td>
<td>0.8754</td>
<td>0.8698</td>
<td>0.8750</td>
<td>0.8718</td>
<td>0.8718</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.8017</td>
<td>0.875</td>
<td>0.875</td>
<td>0.875</td>
<td>0.875</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.874, 0.876)</td>
<td>0.875, 0.876</td>
<td>0.875</td>
<td>0.876</td>
<td>0.872</td>
</tr>
<tr>
<td>0.40</td>
<td>0.7659</td>
<td>0.766</td>
<td>0.757</td>
<td>0.7659</td>
<td>0.7608</td>
<td>0.7608</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.0047</td>
<td>0.0022</td>
<td>0.0024</td>
<td>0.0012</td>
<td>0.0013</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.766, 0.768)</td>
<td>0.765, 0.767</td>
<td>0.756</td>
<td>0.767</td>
<td>0.761</td>
</tr>
<tr>
<td>0.60</td>
<td>0.6703</td>
<td>0.6715</td>
<td>0.6607</td>
<td>0.6701</td>
<td>0.6639</td>
<td>0.6639</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.0081</td>
<td>0.0037</td>
<td>0.0039</td>
<td>0.0022</td>
<td>0.0022</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.667, 0.671)</td>
<td>0.665, 0.673</td>
<td>0.666, 0.662</td>
<td>0.669</td>
<td>0.665</td>
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<tr>
<td>0.80</td>
<td>0.5866</td>
<td>0.5862</td>
<td>0.5747</td>
<td>0.5871</td>
<td>0.5805</td>
<td>0.5805</td>
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<tr>
<td></td>
<td></td>
<td>0.0109</td>
<td>0.0051</td>
<td>0.0053</td>
<td>0.0029</td>
<td>0.0029</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.584, 0.588)</td>
<td>0.585, 0.587</td>
<td>0.586</td>
<td>0.588</td>
<td>0.582</td>
</tr>
<tr>
<td>1</td>
<td>0.5134</td>
<td>0.5133</td>
<td>0.5021</td>
<td>0.5135</td>
<td>0.507</td>
<td>0.507</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.0127</td>
<td>0.0061</td>
<td>0.0062</td>
<td>0.0035</td>
<td>0.0035</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.51, 0.515)</td>
<td>0.512, 0.515</td>
<td>0.501, 0.504</td>
<td>0.512, 0.506</td>
<td>0.515, 0.508</td>
</tr>
</tbody>
</table>

The results follows for the above table 2, the true estimate in second column, the average estimate \( \hat{R}_{II} \) and \( \tilde{R}_{II} \) comes closer to the true estimate, so both the estimators are equally efficient, mean square error is decreasing when we simultaneously increase the values of \( r \) and time \( t \) and confidence interval of the average estimate in ( ) braces lies within the interval for unmvue and mle. The results are summarized in the table as under. Evaluate the performance of the \( P(X > Y) \) under type II censoring scheme. We generate 10000 sample of size 50 from the inverse transformation method with \( \theta_1 = 0.5, 1.5, \theta_2 = 1, 1.5, 2.5, n = m = 50, \beta_1 = \beta_2 = 1.5 \). The study is carried out for different values of \( r = 10, 20, 35 \).

Table 3: Performance of the \( P(X > Y) \) estimates under type II

<table>
<thead>
<tr>
<th>( r = s )</th>
<th>( [\theta_1, \theta_2] )</th>
<th>( P_{II} )</th>
<th>( \hat{P}_{II} )</th>
<th>( \bar{P}_{II} )</th>
<th>( \hat{P}_{II} )</th>
<th>( \bar{P}_{II} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.3325</td>
<td>0.3397</td>
<td>0.3322</td>
<td>0.3368</td>
<td>0.3339</td>
<td>0.336</td>
</tr>
<tr>
<td></td>
<td>0.0105</td>
<td>0.0099</td>
<td>0.0051</td>
<td>0.0049</td>
<td>0.0029</td>
<td>0.0028</td>
</tr>
<tr>
<td></td>
<td>(0.33, 0.334)</td>
<td>(0.338, 0.342)</td>
<td>(0.332, 0.335)</td>
<td>(0.335, 0.338)</td>
<td>(0.333, 0.335)</td>
<td>(0.337)</td>
</tr>
<tr>
<td>20</td>
<td>0.3998</td>
<td>0.4044</td>
<td>0.3987</td>
<td>0.4011</td>
<td>0.4002</td>
<td>0.4016</td>
</tr>
<tr>
<td></td>
<td>0.0122</td>
<td>0.0112</td>
<td>0.0058</td>
<td>0.0056</td>
<td>0.0033</td>
<td>0.0032</td>
</tr>
<tr>
<td></td>
<td>(0.398, 0.402)</td>
<td>(0.402, 0.407)</td>
<td>(0.397, 0.40)</td>
<td>(0.4, 0.403)</td>
<td>(0.399, 0.401)</td>
<td>(0.4)</td>
</tr>
<tr>
<td>35</td>
<td>0.3761</td>
<td>0.3816</td>
<td>0.375</td>
<td>0.3779</td>
<td>0.3751</td>
<td>0.3768</td>
</tr>
<tr>
<td></td>
<td>0.0118</td>
<td>0.0109</td>
<td>0.0056</td>
<td>0.0054</td>
<td>0.0032</td>
<td>0.0031</td>
</tr>
<tr>
<td></td>
<td>(0.374, 0.378)</td>
<td>(0.38, 0.384)</td>
<td>(0.374, 0.376)</td>
<td>(0.376, 0.379)</td>
<td>(0.374, 0.376)</td>
<td>(0.376)</td>
</tr>
</tbody>
</table>

Results for strength reliability in table 3, we workout that true estimate given in column second, average estimate \( \hat{R}_{II} \) and \( \tilde{R}_{II} \) comes closer when compute the true estimate which means two estimators are equally efficient, mean square error decreasing.
for different values of r & s when it increases and confidence interval for the estimates in ()braces lies within the interval for umvue and mle. Evaluating the R(t) to see the performance under type-1 censoring scheme, for it generate 10000 sample of size 50 from the inverse transformation method with $\theta = 1.5, \beta = 1$. Here $t_o$ is fixed which is a termination time, r is number of failure before time $t_o$. The study is carried out for different values of $t = 0.20, 0.40, 0.60, 0.80, 1$ and to = 0.60, 0.80, 1.

Table 4: Performance of the R(t) estimates under type I when $\theta = 1.5$ and $\beta = 1$

<table>
<thead>
<tr>
<th>t</th>
<th>$R(t)$</th>
<th>$\hat{R}(t)$</th>
<th>$\tilde{R}(t)$</th>
<th>$\hat{R}(t)$</th>
<th>$\tilde{R}(t)$</th>
<th>$\hat{R}(t)$</th>
<th>$\tilde{R}(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20</td>
<td>0.8752</td>
<td>0.8818</td>
<td>0.8821</td>
<td>0.8836</td>
<td>0.8839</td>
<td>0.8854</td>
<td>0.8856</td>
</tr>
<tr>
<td>0.40</td>
<td>0.7659</td>
<td>0.7782</td>
<td>0.7795</td>
<td>0.7806</td>
<td>0.7816</td>
<td>0.7845</td>
<td>0.7853</td>
</tr>
<tr>
<td>0.60</td>
<td>0.6703</td>
<td>0.6858</td>
<td>0.6884</td>
<td>0.691</td>
<td>0.6929</td>
<td>0.6941</td>
<td>0.6956</td>
</tr>
<tr>
<td>0.80</td>
<td>0.5866</td>
<td>0.604</td>
<td>0.608</td>
<td>0.6104</td>
<td>0.6134</td>
<td>0.6149</td>
<td>0.6173</td>
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<tr>
<td>1</td>
<td>0.5134</td>
<td>0.5343</td>
<td>0.5399</td>
<td>0.5391</td>
<td>0.5432</td>
<td>0.5429</td>
<td>0.5462</td>
</tr>
</tbody>
</table>

In the table 4, We workout the performance of the average estimate of $\hat{\theta}_{II}$ and $\tilde{\theta}_{II}$ with the true estimate given in the column 2 it is closely related so both the methods are equally efficient, mean square error is decreasing when we increase the time (t and $t_o$) and confidence interval for estimates in ()braces lies within interval for umvue and mle. Now we estimate the strength probability $P(X > Y)$ for different values of $\theta_1 = 1.15, 2.5$, $\theta_2 = 1.5, 1.2$, $to = t_o = 0.60, 0.80, 1$ and $\beta_1 = \beta_2 = 1.5$ and when $(n < m)$ $n = 30, m = 50$.

Table 5: Performance of the $P(X > Y)$ estimates under type I when $(n < m)$

<table>
<thead>
<tr>
<th>$[\theta_1, \theta_2]$</th>
<th>$P_t$</th>
<th>$\hat{P}_t$</th>
<th>$\tilde{P}_t$</th>
<th>$\hat{P}_t$</th>
<th>$\tilde{P}_t$</th>
<th>$\hat{P}_t$</th>
<th>$\tilde{P}_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[1,1.5]</td>
<td>0.4</td>
<td>0.412</td>
<td>0.4171</td>
<td>0.4177</td>
<td>0.4212</td>
<td>0.4255</td>
<td>0.4283</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.0075</td>
<td>0.0077</td>
<td>0.0046</td>
<td>0.0048</td>
<td>0.0035</td>
<td>0.0037</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.4103, 0.4137)</td>
<td>(0.4154, 0.4188)</td>
<td>(0.4164, 0.4189)</td>
<td>(0.4199, 0.4225)</td>
<td>(0.4245, 0.4266)</td>
<td>(0.4272, 0.4294)</td>
</tr>
<tr>
<td>[1.5,1]</td>
<td>0.6</td>
<td>0.5853</td>
<td>0.5903</td>
<td>0.5777</td>
<td>0.5813</td>
<td>0.5713</td>
<td>0.5741</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.0083</td>
<td>0.0082</td>
<td>0.0053</td>
<td>0.0052</td>
<td>0.0041</td>
<td>0.0040</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.5835, 0.5887)</td>
<td>(0.5886, 0.5921)</td>
<td>(0.5763, 0.5799)</td>
<td>(0.5799, 0.5826)</td>
<td>(0.5702, 0.5725)</td>
<td>(0.573, 0.5752)</td>
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<td>[2.5,2]</td>
<td>0.556</td>
<td>0.5519</td>
<td>0.5611</td>
<td>0.549</td>
<td>0.5551</td>
<td>0.5457</td>
<td>0.5503</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.0167</td>
<td>0.0167</td>
<td>0.0101</td>
<td>0.0101</td>
<td>0.0069</td>
<td>0.0068</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.5494, 0.5545)</td>
<td>(0.5586, 0.5637)</td>
<td>(0.547, 0.5509)</td>
<td>(0.5531, 0.5571)</td>
<td>(0.5441, 0.5473)</td>
<td>(0.5486, 0.5519)</td>
</tr>
</tbody>
</table>
In table 5, that the average estimates of strength reliability are closely related to true estimate which is given in column second which means both the estimators are equally efficient, mean square error (mse) is decreasing when \( t_o = t_{oo} \) is increasing, confidence coefficient of the average estimate is lies within the interval for umvue and mle when \((m < n)\). Again we see the performance of estimates of strengthreliability for differentvalues \( \theta_1 = 1,1.5,2.5, \theta_2 = 1.5,1,2, \) to = too = 0.60,0.80,1 and \( \beta_1 = \beta_2 = 1.5 \) and when \((m < n)\) \( n = 50, m = 40 \).

Table 6: Performance of the \( P(X > Y) \) estimates under type I when \((m < n)\)

<table>
<thead>
<tr>
<th>([\theta_1, \theta_2] ) ↓</th>
<th>( p_l )</th>
<th>0.60</th>
<th>0.80</th>
<th>1</th>
<th>0.60</th>
<th>0.80</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>[1.1,5]</td>
<td>0.4</td>
<td>0.4127</td>
<td>0.4108</td>
<td>0.4212</td>
<td>0.4199</td>
<td>0.4262</td>
<td>0.4252</td>
</tr>
<tr>
<td></td>
<td>0.0068</td>
<td>0.0067</td>
<td>0.0045</td>
<td>0.0044</td>
<td>0.0033</td>
<td>0.0033</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.4111,</td>
<td>(0.4092,</td>
<td>(0.42,</td>
<td>(0.4187,</td>
<td>(0.4252,</td>
<td>(0.4242,</td>
<td>(0.4262</td>
</tr>
<tr>
<td></td>
<td>0.4143)</td>
<td>0.4124)</td>
<td>0.4225)</td>
<td>0.4121)</td>
<td>0.4272)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>[1.5,1]</td>
<td>0.6</td>
<td>0.5867</td>
<td>0.5848</td>
<td>0.5804</td>
<td>0.5791</td>
<td>0.5732</td>
<td>0.5721</td>
</tr>
<tr>
<td></td>
<td>0.0065</td>
<td>0.0066</td>
<td>0.0041</td>
<td>0.0042</td>
<td>0.0032</td>
<td>0.0033</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.5851,</td>
<td>(0.5832,</td>
<td>(0.5793,</td>
<td>(0.5779,</td>
<td>(0.5722,</td>
<td>(0.5712,</td>
<td>(0.5731</td>
</tr>
<tr>
<td></td>
<td>0.5882)</td>
<td>0.5863)</td>
<td>0.5816)</td>
<td>0.5803)</td>
<td>0.5742)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>[2.5,2]</td>
<td>0.5556</td>
<td>0.5536</td>
<td>0.5501</td>
<td>0.5505</td>
<td>0.5482</td>
<td>0.5468</td>
<td>0.5451</td>
</tr>
<tr>
<td></td>
<td>0.0131</td>
<td>0.0132</td>
<td>0.0079</td>
<td>0.0079</td>
<td>0.0055</td>
<td>0.0055</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.5513,</td>
<td>(0.5479,</td>
<td>(0.5488,</td>
<td>(0.5465,</td>
<td>(0.5453,</td>
<td>(0.5436,</td>
<td>(0.5465</td>
</tr>
<tr>
<td></td>
<td>0.5558)</td>
<td>0.5524)</td>
<td>0.5522)</td>
<td>0.55)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In table 6, we have seen that the average estimates of strength reliability are closely related to true estimate which is given in column second which means both the estimators are equally efficient, mean square error (mse) is decreasing when \( t_o = t_{oo} \) is increasing, confidence coefficient of the average estimate is lies within the interval for umvue and mle when \((m < n)\). In the table 7 we compare the performance of two variances of reliability i.e. \( \text{Var}(\hat{\theta}_{II}) \) and \( \text{Var}(\hat{\theta}_{II}) \) under type-2 censoring scheme for different values of \( r \) & \( t \). Again, we conduct similar comparison to compare two variances of reliability for seeing the performance i.e. \( \text{Var}(\hat{\theta}_I) \) and \( \text{Var}(\hat{\theta}_I) \) under type I censoring scheme for different values of \( n,t \) and to in Table 8. In both the cases we have seen that when \((r,t) \) and \((n,t, to) \) increase, the variances under both type of censoring scheme decrease, both are equally efficient.

Table 7: Performance of the variance estimates under type II when \( \theta = 1.5 \) and \( \beta = 0.5 \)

<table>
<thead>
<tr>
<th>( r )</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t )</td>
<td>Var(( \hat{\theta}_{II} ))</td>
<td>Var(( \hat{\theta}_{II} ))</td>
<td>Var(( \hat{\theta}_{II} ))</td>
<td>Var(( \hat{\theta}_{II} ))</td>
<td>Var(( \hat{\theta}_{II} ))</td>
<td>Var(( \hat{\theta}_{II} ))</td>
</tr>
<tr>
<td>1</td>
<td>0.0027 56</td>
<td>0.0239 4</td>
<td>0.0127 4</td>
<td>0.0121 4</td>
<td>0.0082 7</td>
<td>0.0080 5</td>
</tr>
<tr>
<td>2</td>
<td>0.0297 3</td>
<td>0.0234 3</td>
<td>0.0142 0</td>
<td>0.0127 4</td>
<td>0.0093 1</td>
<td>0.0087 0</td>
</tr>
<tr>
<td>3</td>
<td>0.0282 2</td>
<td>0.0212 0</td>
<td>0.0136 8</td>
<td>0.0119 0</td>
<td>0.0090 3</td>
<td>0.0082 4</td>
</tr>
<tr>
<td>4</td>
<td>0.0258 2</td>
<td>0.0188 9</td>
<td>0.0126 2</td>
<td>0.0107 6</td>
<td>0.0083 5</td>
<td>0.0075 1</td>
</tr>
<tr>
<td>5</td>
<td>0.0232 9</td>
<td>0.0167 3</td>
<td>0.0114 3</td>
<td>0.0096 8</td>
<td>0.0075 8</td>
<td>0.0067 6</td>
</tr>
<tr>
<td>6</td>
<td>0.0208 9</td>
<td>0.0149 3</td>
<td>0.0102 7</td>
<td>0.0086 0</td>
<td>0.0068 2</td>
<td>0.0060 5</td>
</tr>
<tr>
<td>7</td>
<td>0.0187 1</td>
<td>0.0133 7</td>
<td>0.0092 0</td>
<td>0.0076 8</td>
<td>0.0061 1</td>
<td>0.0054 0</td>
</tr>
<tr>
<td>8</td>
<td>0.0167 7</td>
<td>0.0120 4</td>
<td>0.0082 8</td>
<td>0.0068 7</td>
<td>0.0054 0</td>
<td>0.0048 3</td>
</tr>
<tr>
<td>9</td>
<td>0.0150 4</td>
<td>0.0108 2</td>
<td>0.0073 8</td>
<td>0.0061 0</td>
<td>0.0049 7</td>
<td>0.0043 3</td>
</tr>
<tr>
<td>1</td>
<td>0.0135 9</td>
<td>0.0097 1</td>
<td>0.0066 4</td>
<td>0.0055 9</td>
<td>0.0043 9</td>
<td>0.0038 9</td>
</tr>
</tbody>
</table>
Table 8: Performance of the variance estimates under type I when $\theta = 1.5$ and $\beta = 0.5$

<table>
<thead>
<tr>
<th>$r \to$</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_o$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$t$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Var(\hat{\theta}_{II})$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.0303</td>
<td>0.0283</td>
<td>0.0113</td>
<td>0.0110</td>
<td>0.0063</td>
<td>0.0062</td>
</tr>
<tr>
<td>2</td>
<td>0.0369</td>
<td>0.0352</td>
<td>0.0133</td>
<td>0.0132</td>
<td>0.0073</td>
<td>0.0073</td>
</tr>
<tr>
<td>3</td>
<td>0.0376</td>
<td>0.0376</td>
<td>0.0133</td>
<td>0.0134</td>
<td>0.0073</td>
<td>0.0073</td>
</tr>
<tr>
<td>4</td>
<td>0.0379</td>
<td>0.0383</td>
<td>0.0127</td>
<td>0.0131</td>
<td>0.0069</td>
<td>0.0070</td>
</tr>
<tr>
<td>5</td>
<td>0.0366</td>
<td>0.0382</td>
<td>0.0119</td>
<td>0.0124</td>
<td>0.0064</td>
<td>0.0066</td>
</tr>
<tr>
<td>6</td>
<td>0.0351</td>
<td>0.0378</td>
<td>0.0109</td>
<td>0.0117</td>
<td>0.0058</td>
<td>0.0061</td>
</tr>
<tr>
<td>7</td>
<td>0.0335</td>
<td>0.0372</td>
<td>0.0100</td>
<td>0.0110</td>
<td>0.0053</td>
<td>0.0056</td>
</tr>
<tr>
<td>8</td>
<td>0.0319</td>
<td>0.0365</td>
<td>0.0092</td>
<td>0.0103</td>
<td>0.0048</td>
<td>0.0051</td>
</tr>
<tr>
<td>9</td>
<td>0.0304</td>
<td>0.0357</td>
<td>0.0084</td>
<td>0.0096</td>
<td>0.0043</td>
<td>0.0047</td>
</tr>
<tr>
<td>10</td>
<td>0.0290</td>
<td>0.0350</td>
<td>0.0077</td>
<td>0.0090</td>
<td>0.0039</td>
<td>0.0043</td>
</tr>
<tr>
<td>11</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For conforming the tests with authentication as extracting in section 5, the testing of null hypothesis $H_0: \theta = 1.5$ against $H_1: \theta \neq 1.5$ under Type II censoring, we generate a sample size 50 from inverse transformation technique with $\theta = 1.5$ and $\beta = 1$ is given as

Sample 1

0.0125 0.0245 0.0791 0.0890 0.0985 0.1419 0.2305 0.2361 0.2821 0.3066 0.3191 0.3204 0.3755 0.3761 0.3834 0.3892

0.4327 0.4880 0.6094 0.7209 0.7621 0.7868 0.8782 0.9234 1.0397 1.0547 1.1167 1.1978 1.2044 1.2104 1.3511 1.3628

1.3754 1.4796 1.5141 1.5554 1.6463 1.8141 1.8207 1.9472 2.2306 2.5127 2.6054 2.7410 2.8988 3.0866 3.4995 3.5249

4.6631 6.0554.

For $r=35$, the value of Sr=45.79257. Using $\chi^2$ table at 5% significance level, the values of $k_0 = 36.57$ and $k'_0 = 71.26$ we do not reject null hypothesis, since the value of Sr lies between $k_0$ and $k'_0$. Considering the null hypothesis which is to be tested is $H_0: \theta \leq 1.5$ V, $H_1: \theta > 1.5$, suppose $r=35$, the value of the test statistic, Sr comes out to be 45.79257. Applying the $\chi^2$ test at 5% significance level, the value of $k''_0=38.797$, null hypothesis is accept. Next we test the hypothesis as $H_0: P = P_0$, $H_1: P \neq P_0$ where $P_0 = 0.375$. We generate a sample of size m=40 as

Sample 2

0.0178 0.2952 0.3318 0.3375 0.4044 0.4429 0.4929 0.4959 0.5818 0.5955 0.6866 0.6984 0.7011 0.7346 0.9295 1.0169 1.0288

1.4375 1.4580 1.4705 1.4903 1.5577 1.6558 1.7013 1.8192 2.2640 2.3962 2.5672 2.8361 3.1123 3.1726 3.9632 4.7545


Here $\theta_1 = 1.5$, $\theta_2 = 1$, $\beta_1=\beta_2=1$. Put $s=30$, then the value of $T_{30} = 62.8036$ then the ratio $S_{35}/S_{30} = 0.7291$. Using the F-table, the value of $k''_0=0.3347$ and $k''_0=0.9985$. At 5% significance level, null hypothesis is accept. For testing, null hypothesis $H_0: \theta = 1.5$ against $H_1: \theta \neq 1.5$ under Type I censoring. We have considered n=50 as Sample 3

0.0543, 0.0644, 0.0673, 0.0765, 0.0814, 0.1942, 0.2005, 0.2380, 0.2530, 0.2881, 0.3062, 0.3410, 0.3654, 0.4094, 0.4384, 0.4520, 0.4544, 0.4554, 0.6416, 0.6895, 0.7375, 0.8202, 0.8610, 1.0595, 1.1306, 1.1718, 1.2793, 1.4223, 1.6081, 1.6444, 1.7963, 1.8922, 1.9634, 2.0082, 2.0596, 2.1200, 2.1829, 2.1879, 2.3579, 2.3674, 2.5065, 2.6202, 2.8833, 3.1969, 3.2821, 3.3161, 3.8519, 3.9688, 6.0168, 7.7712.

Using the fact that $r \sim Poisson \left( \frac{n \theta (r)}{\theta} \right)$, using Poisson table at 5% significance level, we obtain $k_1=20$ and $k'_1=48$. Hence
$H_0$ is accept at 5% significance level, as $r=24$ when $t=0.20$.

7 Real Data Analysis

In Present section, we analyze the two real data sets for explanatory purposes. Badar and Priest [4] considered this data which is given below, that represents the strength measured in GPA for single carbon fibers, and infused 1000-carbon fiber tows. Single fiber were tested under tension at gauge lengths of 20mm (Data Set 1) and 10mm (Data Set 2). After that many authors considered these data sets in their study as refereed by Raqab and Kundu [40], Kundu and Gupta [29], Asgharzadeh et al. [2], Shoae & Khorram [44] and more. For analyzing the data we subtract 0.85 from both the data sets. We have examined the fitness of two parameter weibull distribution for two data sets, separately. The two real data sets with size $n = 69$ and $m = 63$, respectively given below.

Real Data Set 1 (gauge length 20mm)

1.312, 1.341, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063,
2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435,
2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.632, 2.642, 2.648, 2.648, 2.684, 2.697, 2.726, 2.770,

Real Data Set 2 (gauge length 10mm)

1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.362, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575,
2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.977, 2.996, 3.030, 3.125,

Our computed KS test and p-values for both the data sets as $D = 0.044341$ & $D = 0.080178$ and p-value = 0.9992 & p-value = 0.8127. We come to the conclusion that both the Monte Carlo simulations and data analysis are performing well for UMVUE & MLE under type II & type I censoring scheme. This infers that both the Monte Carlo simulations and data analysis are performing well for UMVUE & MLE under type II & type I censoring scheme.

Table 9: Real Data analysis under type II when $r = 35$ and $s = 30$.

<table>
<thead>
<tr>
<th>$R(t)_{\text{true}}$</th>
<th>$R(t)_{\text{est}}$</th>
<th>MSE</th>
<th>$P_t$</th>
<th>$P(t)$</th>
<th>MSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>UMVUE</td>
<td>MLE</td>
<td>UMVUE</td>
<td>MLE</td>
<td>UMVUE</td>
<td>MLE</td>
</tr>
<tr>
<td>0.8752</td>
<td>0.9662</td>
<td>0.0083</td>
<td>0.9653</td>
<td>0.0081</td>
<td>0.6</td>
</tr>
</tbody>
</table>

Table 10: Real Data analysis under type I when $r = 35$ and $s = 30$

<table>
<thead>
<tr>
<th>$R(t)_{\text{true}}$</th>
<th>$R(t)_{\text{est}}$</th>
<th>MSE</th>
<th>$P(t)$</th>
<th>$P(t)$</th>
<th>MSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>UMVUE</td>
<td>MLE</td>
<td>UMVUE</td>
<td>MLE</td>
<td>UMVUE</td>
<td>MLE</td>
</tr>
<tr>
<td>0.8752</td>
<td>0.9695</td>
<td>0.9695</td>
<td>0.9696</td>
<td>0.0089</td>
<td>0.7143</td>
</tr>
</tbody>
</table>

8 Conclusion

As the value of $r$ is increasing the mean square error of power estimator, reliability function & strength reliability $P(X > Y)$ are decreasing under type-2 and type-1 censoring scheme as shown in table 1 to 6. Table 1 shows that the performance of MLE is improved than UMVUE for power estimate under type-2 censoring scheme. Also as depicted in table 2 the efficiency of both UMVUE and MLE are equal. In case of strength reliability the performance MLE is improved than UMVUE under type II censoring scheme as shown in table 3. The performance of UMVUE is improved than MLE for reliability function under type I censoring scheme as given in table 4. For $n < m$ case in strength reliability under type-1 censoring scheme MLE is efficiently preforming as compared to UMVUE (table 5) similarly for case $m < n$ two estimator are equally efficient (table 6). Results for variance are given in table 7 and 8 shows that when we increase the value of $(r, t)$ and $(t, t_0)$, both the estimators preform well.
Figure 1: The pdfs of $f(x)$

Figure 2: The cdfs of $F(x)$
Figure 3: The UMVUE of pdf of f under different value of r

Figure 4: The MLE of f under different value of r
Figure 5: The UMVUE, MLE of \( f \) and \( f \) for \( r=35 \)

Empirical and theoretical CDFs

Figure 6: Data Set 1 (gauge length 20mm)
Empirical and theoretical CDFs

![Graph of Empirical and Theoretical CDFs]

Figure 7: Data Set 2 (gage length 10mm)

References


