Wavelets: A novel approach for transformation and finding solutions of differential equations

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Abstract

Wavelets are usually taken as a complex concept. In the light of its advantages and easy to program benefit over the other possible alternatives for solving the differential equations, wavelets are not so hard to deal with. This manuscript is highlighting the concepts of wavelets with emphasizing on the applications of wavelets.

Introduction

There are various possible ways to perform complicated calculations in mathematics. But a way to resolve the complicacy into the easy possible form is transformation. Transformation formulas are helpful in reframe the complicated problem in the simple form that can be easily solved by the known approaches. There are many famous transformations in mathematics that are used for many known applications in mathematical and physical sciences. These include the famous Fourier transform, Laplace transform and Z-transform.

Wavelets are an approach that comes as a solution to the lack of Fourier transform. Fourier transform is the dot product between real signal and different frequency of sine wave. It is defined as

$$X_k = \sum_{n=0}^{N-1} x_n \bullet e^{\frac{-i2\pi}{N}Kn}$$

$$= \sum_{n=0}^{N-1} x_n \bullet \left[\cos\left(\frac{2\pi n}{N}\right) - i \bullet \sin\left(\frac{2\pi n}{N}\right) \right]$$

With this transform, we get the frequency spectrum of the real signal. We only know about initial frequency, but don't know when that "frequency" happens and lost the resolution of the real signal.

To get both frequency and time resolution, the original signal divided into several parts and then applies a transformation to each part. This technique is named by "Short time Fourier transformation". But this technique has many problems.

Wavelet

With this transform, we get the frequency spectrum of the real signal. We only know about initial frequency, but don't know when that "frequency" Wavelets get attention in the research field as a special transform to convert time signals after its elaboration it is compared with different other transforms due to its unique properties. Its exploration leads to its applications in other fields such as it is used in compressing the image, in generating harmonic musical tones and in cleaning the noised data in data analysis. In mathematics, an expression that is used to divide the continuous-time signal into components of a different frequency is known as a wavelet. As in the concept of physics, wave notifies the disturbance in mathematics it signifies a function whose magnitude increases and decreased back to a minimum value normally taken as zero.

Wavelet transforms are generally considered in two forms as a discrete or continuous wavelet transform. The continuous wavelet transform disintegrates a given signal of either time or frequency into small wavelets, that results in small oscillations that are highly limited in time. While a discrete wavelet transform is used to transform a discrete-time signal into discrete wavelet representation. It decomposes the signal into mutually (Graps, 1995) orthogonal set of wavelets, which differs it from the continuous wavelet transform.

Mathematical expression of wavelet

In the theory of wavelets, the principle function $\psi(t)$ that is used for the transformation is known as the mother wavelet. A wavelet usually refers to a small wave that is why a wavelet which is used in the transformation process as a function a distinct area of support.

The moher wavelet $\psi(t)$ is hence a complex valued function (Daniel T.L Lee, dec 1994) defined as

$$\int_{-\infty}^{\infty} |\psi(t)|^2 \, dt < \infty \quad (1)$$

$$c_\psi = 2\pi \int_{-\infty}^{\infty} \frac{|\varphi(\omega)|^2}{|\omega|} d\omega < \infty \quad (2)$$

where φ is the fourier transform of ψ.

Applications of wavelet: Wavelet has applications in many diverse fields. Earlier it was considered as a part of physics then it was explored by mathematicians due to its distinct properties. But now it has explored its wings in the fast-developing field of computer science and data science also. The concepts of wavelets are used in the statistical study (Sibai, 2017) with a wide range of applications in this era of advanced technology. Wavelets are used commonly in the processing of signals, in the compression of data and in image processing. Nowadays it has been explored by the forensic science studies in the verification of fingerprints, in DNA and protein analysis. It has application in many medical treatments such as in measuring blood-pressure, and heart-rate and analyzing the ECG of heart. Wavelets are of utmost use in computer graphics and multiracial analysis.

Litrature review

In a study done by Wang (Wang, April 2001) the application of wavelets in the medical field is discussed. Medical science is emerging with the advancement of technology and image processing using advanced techniques. Almost all of the medical images are occurring in digital formats. Various approaches have been developed to understand, present and analyse the imaging information automatically. Among the available techniques, wavelet transforms have given significantly good results not only in biomedical imaging but also proven advantageous in the study of signal and image processing. The concept of wavelet transforms is based on the decomposition of a signal into bands of different frequency which is shown by a dyadic scheme. In the last few years, many researchers are investigating and using wavelets in different aspects of imaging processing that include compression, enhancements, analysis, classification, and retrieval.

wavelet method is treated for solving wave equation, the Burgers' equations and the Korteweg-de Vries equation as model problems.

Types of wavelets: Based on the unique characteristics, wavelets are divided into several families that have proven their applicability in major areas of science. The properties that distinguish them from each other includes the main property of *orthogonality* and *compact support*. Other properties include symmetricity, smoothness, and vanishing moments that play a very important role in choosing a wavelet for performing a task. Some of the wavelets are as follows:

Haar wavelet: Haar wavelet is a type of sequence which is "square-shaped" function and together to form a wavelet family. This wavelet is similar to Fourier analysis in that it allows a target function over an interval which is represented in term of an orthogonal basis. Basically, this wavelet is known as a mother wavelet of all wavelets. The haar(Daniel T.L Lee, Dec 1994) wavelet is the simplest and oldest kind of wavelet function. For some defined interval, this wavelet work as a step function which takes positive and negative unity and vanishes interval outside.

It is defined as

$$\psi(x)=\begin{cases}1 & ,0\le x<\frac{1}{2}\\ -1 & ,\frac{1}{2}\le x<1\\ 0 & otherwise\end{cases}$$

The scaling function of Haar wavelet is defined as

$$\phi(x)=\begin{cases}1 & ,0\le x<1\\ 0 & otherwise\end{cases}$$

Where the function $\phi(x)$ is known as Haar scaling function, whereas $\psi(x)$ is known as Haar wavelet. This function is orthogonal to its own translation and the family is

$$\psi_{m,n}(x) = 2^{\frac{-m}{2}}\psi\left(2^{-m}x - n\right) \qquad m, n \in Z$$

Where Z is real integers constitutes an orthogonal basis of $L^2(R)$.

Morlet wavelet: Morlet wavelet is also known as "Gabor wavelet" which is composed of complex exponential multiplied by Gaussian envelope. This wavelet is closely related to human perception such as hearing and vision both. This wavelet is also used in music to produce an accurate result, which was not possible by using Fourier transform techniques. This transform helps in capturing the short burst of repeating and alternative music notes with a clear initial and final time of each note. R. Kronland and J. Morlet generally used this particular function. Its Fourier transform is a shifted Gaussian, adjusted slightly so that

$$\Psi(\omega) = e^{-(\omega-\omega_0)^2/2} - e^{\frac{\omega^2}{2}} - e^{\frac{\omega_0}{2}} \qquad (1)$$

$$\psi(t) = \left(e^{-i\omega_0 t} - e^{\frac{\omega^2}{2}}\right) - e^{\frac{t^2}{2}} \qquad (2)$$

Where ω_0 the ratio of highest maximum and second highest maximum is approximately $1/2$

To identify practcal value of ω_0 is 5.for this ω_0, second term of equation 2 is small, so that it can be neglected.

Meyer wavelet: This wavelet is considered by "Yves Meyer". It is continuous wavelet which is applied in numerous cases such as adaptive filters, fractal random fields and multi fault classification. Meyer wavelet is smooth orthonormal (Daniel T.L Lee, dec 1994) wavelet and it is defied by Fourier transform $\Phi(\omega)$ of scaling function $\phi(t)$.

$$\Phi(\omega)=\begin{cases}1 & if\ |\omega|\le\frac{2\pi}{3}\\ \cos\left[\frac{\pi}{2}v\left(\frac{3}{4\pi}|\omega|-1\right)\right] & if\ \frac{2\pi}{3}\le\omega<\frac{4}{3}\\ 0 & otherwise\end{cases}$$

Where v is smooth function which satisfy the given condition,

$$v(t)=\begin{cases}0 & if\ t\le 0\\ 1 & if\ t\ge 1\end{cases}$$

With additional property

$$v(t)-v(1-t)=1$$

Daubechies wavelets: Daubechies wavelets are the natural functions, which is used to represent the solution of integral equations. They are functions of (Prakash, 2018)compact support that can locally pointwise represent lower degree polynomials. These are very similar to splines except the fact that Daubechies wavelets are orthonormal. These are a family of orthogonal wavelets defining a discrete wavelet transform and are characterized by the maximum number of moments going to zero for some defined support. The function $\psi(k)$ is defined as

$$\psi_{jk}(x)=\lambda\psi\left(2^{j}x-k\right)$$

Concluding remarks: The presneted work is to investigate the appliaction of wavelets in the diverse fiels of science and engineering. From last dedecade till now wavelts has been developed and emerged as a tool to perfrom various simplest to complicated task. The applications of wavelets are discussed with the types and how it has been implemented by researchers in their field of study.

References

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