On Equitable Power Domination Number of Some Graphs

A. Parthiban¹, G. Samdanielthompson², K. Sathish Kumar³

¹Department of Mathematics, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara-144 411, Punjab, India
²Department of Mathematics, Hindustan College of Arts and Science, Padur, Chennai-603 103, Tamil Nadu, India
³Department of Mathematics, Madras Christian College (Autonomous), Chenna-600 059, Tamil Nadu, India

Corresponding author: parthiban.23447@lpu.co.in

Abstract

Let G (V, E) be graph. A set S ⊆ V is said to be a power dominating set (PDS) if every vertex u ∈ V − S is observed by certain vertices in S by the following rules: (i) if a vertex v in G is in PDS, then it dominates itself and all the adjacent vertices of v and (ii) if an observed vertex v in G has k > 1 adjacent vertices and if k − 1 of these vertices are already observed, then the remaining one non-observed vertex is also observed by v in G. A power dominating set S ⊆ V in G (V, E) is said to be an equitable power dominating set (EPDS), if for every vertex v ∈ V − S there exists an adjacent vertex u ∈ S such that the difference between the degree of u and degree of v is less than or equal to 1, i.e., |d(u) − d(v)| ≤ 1. The minimum cardinality of an equitable power dominating set of G is called the equitable power domination number of G, denoted by γ_{epd}(G). “An edge is said to be subdivided if the edge xy is replaced by the path: xwy, where w is the new vertex. A graph obtained by subdividing each edge of a graph G is called subdivision of G, and is denoted by S(G)”. In this paper we establish the equitable power domination number of subdivision of graphs. We also obtain the equitable power domination number of the generalized Petersen graphs and balanced binary tree.

Keywords
Power dominating set, Power domination number, Equitable power dominating set, Equitable power domination number, Generalized Petersen graphs, Balanced binary tree, and Subdivision graph.

1. Introduction

Only simple, finite, undirected, and connected graphs are considered in this paper. A dominating set of a graph G = (V, E) is a set S of vertices such that every vertex v in V − S has at least one neighbor in S. The minimum cardinality of a dominating set of G is called the domination number of G, denoted by γ_d(G) [8]. For a few other variants of dominating set refer to [9, 10].

A power dominating set S ⊆ V in G (V, E) is said to be an equitable power dominating set, if for every vertex v ∈ V − S there exists an adjacent vertex u ∈ S such that the difference between the degree of u
and degree of \( v \) is less than or equal to 1, that is \( |d(u) - d(v)| \leq 1 \). The “minimum cardinality” of an equitable power dominating set of \( G \) is called the equitable power domination number of \( G \), denoted by \( \gamma_{epd}(G) \) [2]. For more results one can refer to [3, 4]. In this paper, we obtain the equitable power domination number of the generalized Petersen graphs and balanced binary tree.

2. Main Results

For the sake of convenience, by EPDS and EPDN we mean an equitable power dominating set and the equitable power domination number, respectively.

2.1 EPDN of the Generalized Petersen Graphs and Balanced Binary Tree

First we recall the definition of the generalized Petersen graph for the sake of completeness.

**Definition 1** [1]

“The generalized Petersen graph \( GP(n, k) \) is defined to be a graph with \( V(GP(n, k)) = \{a_i, b_i: 0 \leq i \leq n - 1\} \) and \( E(GP(n, k)) = \{a_ia_{i+1}, a_ib_i, b_ib_{i+k}: 0 \leq i \leq n - 1\} \), where the subscripts are expressed as integers modulo \( n \) (\( n \geq 5 \)) and \( k \) (\( k \geq 1 \)).”

**Note:**
1. \( GP(n, k) \) is isomorphic to \( GP(n, n - k) \).
2. Without restriction of generality, one may consider the generalized Petersen graph \( GP(n, k) \) with \( k \leq \lfloor (n-1)/2 \rfloor \).

**Theorem 2**

Let \( GP(n, k) \) be the generalized Petersen graph.

Then \( \gamma_{epd}(GP(n, k)) = \begin{cases} 2, & \text{for } k = 1, 2 \text{ and } m \geq 4 \\ 3, & \text{for } m \geq 10 \text{ and } k \geq 3. \end{cases} \)

**Proof.**

Let \( GP(n, k) \) be the given GPG with \( V = \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\} \) and edge set \( E(GP(n, k)) = \{a_ia_{i+1}, a_ib_i, b_ib_{i+k}: 0 \leq i \leq n - 1\} \). To obtain the equitable power domination number of \( GP(n, k) \), we consider the following two cases:

**Case 1:** For \( k = 1, 2 \) and \( m \geq 4 \).
Without loss of generality, we choose any one of $b_i$'s, $1 \leq i \leq n$, to be in $S$, say $b_1$. Note that $b_1$ equitably power dominates $b_3$, $a_1$, and $b_{n-1}$. Now the observed vertices $b_3$, $a_1$ and $b_{n-1}$ have more than one non-observed vertices and so fail to observe their neighboring vertices which leads to choose another vertex to be in EPDS. Then one can choose either $b_2$ or $b_n$ to be in $S$ for the sake of minimum cardinality. Now it is easy to see that all the remaining non-observed vertices are observed by their respective neighbors and therefore $|S| = 2$.

**Case 2:** For $m \geq 10$ and $k \geq 3$

Construction of EPDS is similar to Case 1.

### 2.2 Equitable Power Domination Number of the Balanced Binary Tree

We recall a few relevant definitions needed for this section for the sake of convenience.

**Definition 3** [5]

“A graph without cycles is called an acyclic graph and a connected acyclic graph is called as a tree.”

**Definition 4** [5]

A binary tree is a tree in which each vertex has at most 2 pendant vertices.

**Definition 5** [5]

A balanced binary tree is a binary tree in which the left and right sub trees of every vertex differ in height by no more than one.
Theorem 6

Let $B(1, k)$ be a balanced binary tree. Then $\gamma_{cpd}(B(1, k)) = \sum_{n=0}^{n=k} 2^n - 2^{n-1}$.

Proof.

Let $B(1, k)$ be the given balanced binary tree on $k$ levels with vertex set $V = \{a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, ..., a_n\}$ where $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, ..., a_n$ are the pendant vertices. To obtain an equitable power dominating set $S$, without loss of generality, we choose $a_0$ to be in $S$. The vertex $a_0$ equitably power dominates $a_1$ and $a_2$. Now the vertices $a_1$ and $a_2$ have two non-observed vertices $a_1', a_2'$ and $a_3', a_4'$, respectively. So one has to choose any one between $a_1$ and $a_2$, say $a_1$, then $a_2$ is observed by $a_0$. Again as $a_2$ has two non-observed vertices $a_3'$ and $a_4'$, so one has to choose any one between $a_3'$ and $a_4'$, say $a_3'$ . Also $a_1$ in $S$ observes $a_1'$ and $a_2'$. Proceeding in the same way, finally we need to choose $a_1', a_2', a_3', a_4', a_5', a_6', a_7', a_8', ..., a_n'$ as they are the pendant vertices and there are no adjacent vertices satisfying the desired equitable property. Thus we obtain the sequence of vertices, namely $a_0, a_1, a_2, a_3', a_4', a_5', a_6', a_7', ...$ and so on. That is,

$\gamma_{cpd}(B(1, 1)) = 1$

$\gamma_{cpd}(B(1, 2)) = 1 + 2$

$\gamma_{cpd}(B(1, 3)) = 1 + 2 + 2^3$

$\gamma_{cpd}(B(1, 4)) = 1 + 2 + 2^2 + 2^4$

$\gamma_{cpd}(B(1, 5)) = 1 + 2 + 2^2 + 2^3 + 2^5$

Thus $\gamma_{cpd}(B(1, k)) = \sum_{n=0}^{n=k} 2^n - 2^{n-1}$.

2.2 Equitable Power Domination Number of Subdivision of Certain Classes of Graphs

The concept of subdivision in graphs was introduced by Trudeau, Richard J in 1993 [11]. We recall the definition of subdivision of a graph.

Definition 7 [11]

“An edge is said to be subdivided if the edge $uv$ is replaced by the path $uwv$, where $w$ is the new vertex. A graph obtained by subdividing each edge of a graph $G$ is called subdivision of the graph $G$, and is denoted by $S(G)$.”

![Fig.3: A graph G and subdivision of G, S(G)](image-url)
Theorem 8
Let $G$ be a graph on $n$ vertices. Then $\gamma_{epd}(S(G)) \geq \gamma_{epd}(G)$.

Proof.
Let $G$ be the given graph with $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_n\}$. Obtain the subdivision of $G$, denoted $S(G)$, as follows: $V(S(G)) = V(G) \cup E(G)$ and $E(S(G)) = \{(v_i e_i), (e_i v_j): 1 \leq i \leq n$ and $i + 1 \leq j \leq m - 1\}$. We consider the following two cases in obtaining an EPDS of $S(G)$.

Case 1: For a vertex $v_i$ incident with $e_i$ for which $|d(v_i) - d(e_i)| \geq 1$ for at least one ‘i’. Then one has to choose $e_i$ to be in $S$. Thus $\gamma_{epd}(S(G)) \geq \gamma_{epd}(G)$.

Case 2: For a vertex $v_i$ incident with $e_i$ for which $|d(v_i) - d(e_i)| < 1$ for $1 \leq i \leq n$. Then $S$ remains the same. Thus $\gamma_{epd}(S(G)) = \gamma_{epd}(G)$.

Theorem 9 [2]
Let $C_n$, $n \geq 3$ be a cycle. Then $\gamma_{epd}(C_n) = 1$.

Theorem 10
Let $C_n$, $n \geq 3$ be a cycle. Then $\gamma_{epd}(S(C_n)) = 1$.

Proof.
Let $C_n$ be a cycle with $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. When one performs the subdivision on $C_n$, the resultant graph is again a cycle on $2n$ vertices. So by Theorem 9, $\gamma_{epd}(S(C_n)) = 1$.

Theorem 11 [2]
Let $P_n$, $n \geq 1$ be a path. Then $\gamma_{epd}(P_n) = 1$.

Theorem 12
Let $P_n$, $n \geq 3$ be a path. Then $\gamma_{epd}(S(P_n)) = 1$.

Proof.
Let $P_n$ be a path with $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. An easy check shows that when one performs the subdivision on $P_n$, the resultant graph is again a path on $2n$ -1 vertices. So by Theorem 11, we deduce that $\gamma_{epd}(S(P_n)) = 1$.

Definition 13 [5]
“Any two distinct vertices of a graph $G$ are adjacent then $G$ is said to be complete graph and it is denoted by $K_n$.”

Theorem 14 [2]
For a complete graph $K_n$, $\gamma_{epd}(K_n) = 1$.

Theorem 15
Let $S(K_n)$ be the subdivision of a complete graph $K_n$. Then $\gamma_{epd}(S(K_n)) = m + n$, for $n \geq 5$. 

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Proof.

Let $K_n$ be a complete graph with $V(K_n) = \{v_1, v_2, ..., v_n\}$ and $E(K_n) = \{e_1, e_2, ..., e_m\}$. By the definition of a complete graph, the degree of each vertex $v_i, d(v_i) = n - 1$ for $1 \leq i \leq n$. Obtain the subdivision of a complete graph $K_n$, denoted by $S(K_n)$ as follows: $V(S(K_n)) = V_1 \cup V_2$, where $V_1 = V(K_n) = \{v_1, v_2, ..., v_n\}$ and $V_2 = E(K_n)$. One can notice that the subdivided graph of a complete graph $K_n$ gives rise to the graph such that no two adjacent vertices with $|d(u) - d(v)| \leq 1$ and violate the equitable property. So to obtain an equitable power dominating set, one has to choose the entire vertex set to be in EPDS. Thus $|S| = m + n$.

References


