Superior Eccentric Domination in some Quadrilateral Snake Graphs

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ABSTRACT

In 2017 we define superior eccentric domination in graphs. A superior dominating set S of vertices of G is called a superior eccentric dominating set if every vertex of V (G) – S has some superior eccentric vertex in S. A superior eccentric dominating set of G of minimum cardinality is a minimum superior eccentric dominating set and its cardinality is called the superior eccentric domination number and is denoted by \( \gamma_{sed}(G) \). In this paper we initiate the study of superior eccentric dominating sets in quadrilateral snake graphs, alternate quadrilateral snake graphs, double quadrilateral snake graphs and double alternate quadrilateral snake graphs.

Keywords: Superior eccentric vertex, superior dominating set, superior eccentric dominating set, Superior eccentric dominating set of some quadrilateral snake graphs.

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1 INTRODUCTION

Let G be a finite, simple, undirected (a, b) graph with vertex set V(G) and edge set E(G), |V(G)| = u, |E(G)| = v. For graph theoretic terminology refer Harary [3], Buckley and Harary [1].

In 2010, Janakiraman, Bhanumathi and Muthammai defined eccentric domination in graphs [5]. K. M Kathiresan and G.Marimuthu introduced the superior domination in graphs and superior distance in graphs [5,6].

A set D \( \subseteq V \) is said to be a dominating set in G, if every vertex in V– D is adjacent to some vertex in D. The minimum cardinality of a dominating set is called the domination number and is denoted by \( \gamma(G) \). For two vertices u and v in a graph G, the distance from u to v is denoted by d(u,v) and defined as the length of a shortest u–v path in graph G. Let G be a connected graph and v be a vertex of G. The eccentricity e(v) of v is the distance to a vertex farthest from v. Thus, e(v) = max{d(u,v) : u \in V}. A vertex v (\( \neq u \)) is called a superior neighbor of u if e(u) = d(u,v). A vertex u is said to superior dominate a vertex v if v is a superior neighbor of u. A set S of vertices of G is called a superior dominating set if every vertex of V(G) – S is superior.

2. Superior Eccentric Dominating Set.

For distinct vertices u and v of a non-trivial connected graph G, let D_{u, v} = N(u) \cup N(v). We define a D_{u, v}– walk as a u– v walk in G that contains every vertex of D_{u, v}. The superior distance d_{D}(u, v) from u to v is the length of a shortest D_{u, v} walk. For each vertex u \in V(G), define d_{D}(u) = \min\{d_{D}(u, v) : v \in V(G) – \{u\}\}. A vertex v (\( \neq u \)) is called a superior neighbor of u if d_{D}(u, v) = d_{D}(u). A vertex u is said to superior dominate a vertex v if v is a superior neighbor of u. A set S of vertices of G is called a superior dominating set if every vertex of V(G) – S is superior.
dominated by some vertex in S. A superior dominating set of G of minimum cardinality is a minimum superior dominating set and its cardinality is called the superior domination number of G and is denoted by \( \gamma_{sd}(G) \). we define the superior eccentricity of v as \( e_D(v) = \max\{d_D(u,v) : u \in V(G)\} \). A vertex v of a graph G is said to be a superior eccentric vertex of a vertex u if \( d_D(u,v) = e_D(u) \). A vertex u is superior eccentric vertex of G if it is a superior eccentric vertex of some vertex v.

2. Superior Eccentric Domination in Some Graphs:

**Definition 2.1**

A superior dominating set S of vertices of G is called a superior eccentric dominating set if every vertex of V(G) – S has some superior eccentric vertex in S.

**Quadrilateral Snake Graph:**

The Quadrilateral Snake Graph \( Q_n \) is obtained from the path \( P_n \) by replacing each edge of the path by a cycle \( C_4 \). The minimum cardinality of a superior eccentric domination in quadrilateral snake graph is \( \gamma_{sed}(Q_n) \).

![Quadrilateral Snake Graph](image)

Superior distance \( d_D(v_1, v_2) = 7 \),

Superior distance \( d_D(v_1, u_1) = 3 \),

Superior distance \( d_D(v_1, v_n) = 3 + d(v_2, v_{n-1}) + 3 = 6 + n - 3 = n + 3 \),

Superior distance \( d_D(v_i, v_n) = 4 + d(v_i, v_{i+1}) + 3 = 7 + n - i - 1 = n - i + 6 \) (for \( i \geq 2 \)),

Superior distance \( d_D(v_i, v_j) = 4 + d(v_i, v_j) + 3 = 4 + (j - 1 - i) + 3 = 7 + (j - 1 - i) = 6 + j - i \) (for \( i \geq 2, j \leq n - 1 \)),

Superior distance \( d_D(u_1, u_2) = 7 \),

Superior distance \( d_D(u_1, u_2) = 7 \),

Superior distance \( d_D(u_1, u_3) = 8 \),

Superior distance \( d_D(u_1, u_4) = 9 \),

Superior distance \( d_D(u_1, u_3) = 8 \),

Superior distance \( d_D(u_1, u_n) = 4 + d(v_2, v_{n-1}) + 3 = 7 + n - 1 - 2 = n + 4 \),

Superior distance \( d_D(u_1, v_{n-1}) = 4 + d(v_2, v_{n-2}) + 7 = 11 + n - 2 - 2 = n + 7 \),

Superior distance \( d_D(v_1, v_2) = 7 \),

Superior distance \( d_D(u_2, u_2) = 6 \),

Superior distance \( d_D(u_1, u_n) = 3 + d(v_1, v_n) + 3 = 3 + n - 1 + 3 = n + 5 \),

Superior distance \( d_D(u_1, u_n) = 3 + d(v_1, v_{n-1}) + 3 = 6 + n - i - 1 = 5 + n - i \),

Superior distance \( d_D(u_1, u_n) = 3 + d(v_1, v_{j-1}) + 7 \).

Hence \( v_2 \) and \( v_{n-1} \) are the superior eccentric vertices of other vertices, \( u_1 \) is superior adjacent to \( v_1, u_1 \) and \( u_2 \) are superior adjacent to each other, \( v_2 \) is superior adjacent to \( u_2' \). \( v_{n-1} \) is superior adjacent to \( u_{n-1} \).

\( S = \{v_1, v_2, u_1, u_2, u_2', \ldots, v_{n-1}, u_{n-1}\} \) are the superior eccentric dominating set of the quadrilateral snake graph. S is also minimum with this property.

**Double Quadrilateral Snake Graph:**

A Double Quadrilateral Snake Graph \( DQ_n \) consists of two triangular snakes that have a common path. The minimum cardinality of a superior eccentric domination in Double Quadrilateral Snake Graph is \( \gamma_{sed}(DQ_n) \).
Double Alternate Quadrilateral Snake Graph:
A Double alternate Quadrilateral Snake Graph that have a common path. That is, obtained from a path \( v_1, v_2, \ldots, v_n \) by joining \( v_i \) and \( v_{i+1} \) to new vertices \( v_i \) and \( w_i \), respectively and adding the edges \( v_i w_i \) and \( x_i y_i \). The minimum cardinality of a superior eccentric domination in double alternate quadrilateral snake graph is \( \gamma_{sed}(DAQ_n) \).

(i) \( n \) is even
(ii) $n$ is even

$$S = \{u_1, u_3, u_5, \ldots, u_{n-1}\} \cup \{w_2, w_4, \ldots, w_n\}$$

is a superior dominating set. $S_1 = S \cup \{v_2, v_{n-1}\}$ is a superior eccentric dominating set of the double alternate quadrilateral snake graph. $|S| = \frac{n}{2} + \frac{n}{2} = n$. 
Superior distance \( d_D(v_1, v_2) = 7 \),
Superior distance \( d_D(v_1, v_3) = 8 \),
Superior distance \( d_D(v_1, v_n) = 2 + d(v_2, v_{n-1}) + 6 = 8 + n - 1 - 2 = n + 5 \),
Superior distance \( d_D(u_2, u_3) = 3 \),
Superior distance \( d_D(w_1, v_1) = 4 \),
Superior distance \( d_D(v_i, v_n) = 8 + n - 1 - i + 2 = 10 - 1 + n - i = 9 + n - i \),
Superior distance \( d_D(v_2, u_2) = 5 \),
Superior distance \( d_D(u_2, v_n) = 3 + d(v_2, v_{n-1}) + 2 = 5 + n - 1 + 2 = n + 6 \),
Superior distance \( d_D(u_2, u_3) = 1 + d(v_2, v_{n-1}) + 6 = 7 + n - 1 - 2 = n + 4 \),
Superior distance \( d_D(v_i, v_j) = 8 + d(v_i, v_{j-1}) + 8 = 16 + j - 1 - i = 15 + j - i \text{if} \, i \geq 2, \, j \leq n - 1 \),
Superior distance \( d_D(u_2, u_{n-1}) = 1 + d(v_2, v_{n-1}) + 1 = 2 + n - 3 = n - 1 \),
Superior distance \( d_D(v_2, v_n) = 6 + d(v_2, v_{n-1}) + 1 = 7 + n - 1 - 2 = 7 + n - 3 = n + 4 \),
Superior distance \( d_D(v_2, v_{n-1}) = 6 + d(v_2, v_{n-1}) + 6 = 12 + n - 1 - 2 = 12 + n - 3 = n + 9 \).

Hence \( v_2 \) and \( v_{n-1} \) are superior eccentric vertices of the other vertices. \( S = \{ u_2, u_3, \ldots, u_{n-2}, u_{n-1} \} \cup \{ w_2, w_3, \ldots, w_{n-2}, w_{n-1} \} \) are the superior dominating set \( S_1 = S \cup \{ v_2, v_{n-1} \} \) are the superior eccentric dominating set the double alternate quadrilateral snake graph.

(iii) \( n \) is odd

\[ \text{Superior distance} \quad d_D(v_1, v_2) = 7, \]
\[ \text{Superior distance} \quad d_D(v_1, v_3) = 8, \]
\[ \text{Superior distance} \quad d_D(v_1, v_n) = 2 + d(v_2, v_{n-1}) + 6 = 8 + n - 1 - 2 = n + 5, \]
\[ \text{Superior distance} \quad d_D(v_i, v_n) = 8 + d(v_i, v_{n-1}) + 5 = 13 + n - 1 - i = 11 + n - i \text{for} \, i \geq 2, \]
\[ \text{Superior distance} \quad d_D(v_i, v_j) = 2 + d(v_i, v_{j-1}) + 2 = 4 + (j - i - 1) = 3 + j - i, \]
\[ \text{Superior distance} \quad d_D(u_2, u_3) = 3, \]
\[ \text{Superior distance} \quad d_D(u_2, u_n) = 3 + d(v_2, v_n) + 5 = 8 + n - 2 = n + 6, \]
\[ \text{Superior distance} \quad d_D(u_i, u_n) = 4 + d(u_i, u_{n-1}) + 3 = 7 + n - i - 1 = 6 + n - i \text{for} \, i \geq 2, \]
\[ \text{Superior distance} \quad d_D(v_2, v_n) = 6 + n - 2 + 6 = 10 + n, \]
\[ \text{Superior distance} \quad d_D(v_2, u_2) = 5, \]
\[ \text{Superior distance} \quad d_D(u_2, v_n) = 3 + d(v_2, v_{n-1}) + 3 = 6 + n - 1 - 2 = n + 3, \]
\[ \text{Superior distance} \quad d_D(v_2, v_{n-1}) = 7 + d(v_3, v_{n-2}) + 7 = 14 + n - 2 - 3 = n + 9, \]
\[ \text{Superior distance} \quad d_D(u_2, v_{n-1}) = 3 + d(v_2, v_{n-2}) + 2 = 5 + n - 2 - 2 = n + 1. \]

Hence \( v_2 \) and \( v_{n-1} \) are superior eccentric vertices of the other vertices. \( S = \{ v_2, v_4, \ldots, v_{n-3}, v_{n-1} \} \cup \{ u_3, u_5, \ldots, u_n \} \cup \{ w_3, w_5, \ldots, w_n \} \) are the superior dominating set \( S_1 = S \cup \{ v_2, v_{n-1} \} \) are the superior eccentric dominating set the double alternate quadrilateral snake graph.
Alternate quadrilateral snake graphs:
An alternate quadrilateral snake graph $A(Q_n)$ is obtained from a path $u_1, u_2, \ldots, u_n$ by joining $u_i$ to new vertices $v_i, w_i$ respectively and then joining $v_i$ and $w_i$. That is every alternate edge of a path is replaced by a cycle $C_4$. The minimum cardinality of a superior eccentric domination in quadrilateral snake graph is $\gamma_{sed}(A(Q_n))$.

(i) $n$ is odd

(ii) $n$ is even

Superior distance $d_D(v_1, v_2) = 5$,
Superior distance $d_D(v_1, v_3) = 6$,
Superior distance $d_D(v_1, v_n) = 2 + d(v_2, v_{n-1}) + 4 = 6 + n - 3 = n + 3$,
Superior distance $d_D(v_i, v_n) = 4 + d(v_i, v_{n-1}) + 3 = 7 + n - i - 1 = 6 + n - i$ (for $i \geq 2$),
Superior distance $d_D(v_i, v_{j}) = 2 + d(v_i, v_{j-1}) + 2 = 4 + (j - i - 1) + 2 = 4 + (j - i - 1) = 4 + j - i - 1$ \(= 3 + j - i \) (for $i \geq 2, j \leq n - 1$),
Superior distance $d_D(v_2, u_3) = 3$,
Superior distance $d_D(u_2, u_n) = 3 + d(v_1, v_n) + 3 = 3 + n - 1 + 3 = n + 5$,
Superior distance $d_D(u_1, u_n) = 4 + d(u_1, u_{n-1}) + 3 = 7 + n - i - 1 = 6 + n - i$ (for $i \geq 2$),
Superior distance $d_D(v_2, v_n) = 4 + n - 2 + 3 = n + 5$,
Superior distance $d_D(v_2, u_2) = 5$,
Superior distance $d_D(u_2, v_n) = 3 + d(v_2, v_{n-1}) + 3 = 6 + n - 2 = n + 3$,
Superior distance $d_D(v_2, v_{n-1}) = 5 + d(v_3, v_{n-2}) + 5 = n + 5$,
Superior distance $d_D(u_2, v_{n-1}) = 3 + d(v_2, v_{n-2}) + 2 = 5 + (n - 2 - 2) = n - 4 + 5 = n + 1$.

Hence $v_2$ and $v_{n-1}$ are the superior eccentric vertices of other vertices.

$S^* = \{ v_4, v_6, v_8, \ldots, v_{n-3} \} \cup \{ u_3, u_5, u_7, \ldots, u_{n-2}, u_n \}$ is a superior dominating set.

$S = \{ v_1, v_2, v_{n-1} \} \cup \{ v_4, v_6, v_8, \ldots, v_{n-3} \} \cup \{ u_3, u_5, u_7, \ldots, u_{n-2}, u_n \}$ is a superior eccentric dominating set of the double alternate quadrilateral snake graph.

Therefore $|S| = n - 3 - 4 + n - 3 + 3 = 2n - 7$.

(ii) $n$ is even

Superior distance $d_D(v_1, v_2) = 5$,
Superior distance $d_D(v_1, v_3) = 8$,
Superior distance $d_D(v_1, v_n) = 4 + d(v_2, v_{n-1}) + 3 = 6 + n - 2 = 6 + n - 3 = n + 3$,
Superior distance $d_D(v_i, v_n) = 2 + d(v_i, v_{n-1}) + 3 = 5 + n - 1 - i = 4 + n - i$ (for $i \geq 2$),
Superior distance $d_D(v_i, v_{j}) = 2 + d(v_i, v_{j-1}) + 2 = 4 + (j - i - 1) = 3 + j - i$ (for $i \geq 2, j \leq n - 1$),
Superior distance $d_D(v_2, v_n) = 4 + n - 2 + 4 = n + 6$,
Superior distance $d_D(v_2, u_2) = 5$. 
Superior distance \( d_D(u_1, u_2) = 3 \),
Superior distance \( d_D(u_1, u_n) = 3 + d(v_1, v_n) + 3 = 6 + n - 1 = 5 + n \),
Superior distance \( d_D(u_2, v_n) = 3 + d(v_2, v_{n-1}) + 3 = 3 + n - 1 = 2 - 3 + 3 = n + 3 \),
Superior distance \( d_D(v_2, v_{n-1}) = 4 + d(v_3, v_{n-2}) + 5 = 9 + n - 2 - 2 = n + 5 \),
Superior distance \( d_D(u_2, v_{n-1}) = 3 + d(v_2, v_{n-2}) + 5 = 8 + n - 4 = n + 4 \),
Superior distance \( d_D(u_1, v_n) = 4 + d(v_2, v_{n-1}) + 3 = 7 + n - 3 = n + 4 \),
Superior distance \( d_D(u_1, v_{n-1}) = 4 + d(v_2, v_{n-2}) + 5 = 9 + n - 2 = n + 5 \),
Superior distance \( d_D(u_i, u_{n-1}) = 3 + d(u_i, u_{n-i}) + 3 = 6 + n - i - 1 = 5 + n - i \) (for \( i \geq 2 \)),
Superior distance \( d_D(u_i, u_j) = 3 + d(u_i, u_{j-i}) + 3 = 6 + j - 1 - i = 5 + j - i \) (for \( i \leq 2, j \geq n - 1 \))

Hence \( v_2 \) and \( v_{n-1} \) are the superior eccentric vertices of other vertices.

\[ S = \{ u_1, u_3, u_5, \ldots, u_{n-1} \} \cup \{ v_4, v_6, \ldots, v_{n-2} \} \cup \{ v_2, v_{n-1} \} \]
are the superior eccentric dominating set of the alternate quadrilateral snake graph.

Therefore \( |S| = n + 2 \).

(iii) \( n \) is even


d_D(v_1, v_2) = 5,
d_D(v_1, v_3) = 6,
d_D(v_1, v_n) = 2 + n - 1 + 2 = n + 3,
N_D(v_i, v_n) = 6 + d(v_i, v_{n-1}) + 2 = 8 + n - 1 - i = 7 + n - i \) (for \( i \geq 2 \)),
d_D(v_i, v_j) = 6 + d(v_i, v_{j-i}) + 6 = 12 + j - i - 1 = 11 + j - i \) (for \( i \geq 2, j \leq n - 1 \)),
d_D(u_2, u_3) = 3,
d_D(u_2, u_{n-1}) = 1 + d(v_2, v_{n-1}) + 1 = 2 + n - 3 = n - 1 ,
d_D(v_2, u_n) = 4 + n - 3 = n + 1 ,
d_D(v_2, u_2) = 5,
d_D(v_2, v_n) = 3 + d(v_2, v_{n-1}) + 3 = 6 + n - 1 - 2 = 6 + n - 3 = n + 3 ,
d_D(v_2, v_{n-1}) = 5 + d(v_3, v_{n-2}) + 5 = 10 + n - 2 - 3 = n + 5 ,
d_D(u_2, v_{n-1}) = 3 + d(v_2, v_{n-2}) + 5 = 8 + n - 2 - 2 = n + 4 ,

Hence \( v_2 \) and \( v_{n-1} \) are the superior eccentric vertices of the other vertices.

\[ S = \{ u_1, u_2, u_5, \ldots, u_{n-1} \} \cup \{ v_4, v_6, \ldots, v_{n-2} \} \cup \{ v_2, v_{n-1} \} \]
are the superior eccentric dominating set of the alternate quadrilateral snake graph.

Therefore \( |S| = n + 2 \).

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