STUDY ON APPROXIMATE SOLUTIONS OF FRACTIONAL ORDER BIOLOGICAL POPULATION MODEL

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Abstract : This manuscript is broadly concerned to the study of a mathematical model of biology. It encourages us to understand the dynamical procedure of population changes in biological population model and provides significant predictions. We study numerical solution of a degenerate parabolic equation of fractional order arising in the spatial diffusion of biological populations. Fractional complex transform with iterative method is used to obtain approximate solutions. The specified idea is very effective, reliable and pragmatic for fractional PDEs and could be prolonged to further physical happenings. The obtained results are illustrated in 3D graphics with the help of Mathematica.

Keywords: Biological population equation; Caputo Fractional operator; Fractional Complex Transform; Iterative method.

I. Introduction

The scope of mathematics includes mathematical modelling and esoteric mathematics. The flow of work, process, predictions and outcomes can easily be measured with the help of mathematical concepts and theory. Therefore, biologists are now extremely dependent on mathematics. Mathematical modelling of biological sciences is presented by many brilliant scientist [1,2,3]. The mathematical modelling involves biological system, integer order differential equations that show their dynamics and complex system which describes their structure. In last few decades the various biological models were studied in detail by using classical derivative, few of them see [4,5].

In recent years, the fractional-order models were given much attentions, because the biological models that involved fractional-order derivative are more realistic and accurate as compared to the classical order models. Fractional calculus have gained importance and popularity due to its various applications in fluid mechanics, visco-elasticity, biology, electrical network, fluid-dynamic traffic model with fractional derivatives, optics and signal processing and so on [6,7,8]. Except in a limited number of these problems, we have difficulty to find their exact analytic solutions. An effective and easy method for solving such equations is needed. Recently, one of the most reliable and effective technique to solve linear and nonlinear functional equations is proposed by Daftardar-Gejji and Jafari [9,10,11] known as new iterative method (NIM).

This manuscript presents the nonlinear fractional order biological population mode in the form,

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + \emptyset(u) \qquad (1.1)$$

with the initial condition

$$u(x, y, 0) = \emptyset_0(x, y)\,, \qquad (1.2)$$

where u denotes the population density, and $\emptyset$(u) represents the population supply due to birth and deaths. The derivatives in (1) are understood in the Caputo sense. Biologists believe that dispersal or emigration play a key role in the regulation of population of some species. The diffusion of a biological species in a region B is described by the following three functions of position x = (x, y) in B and time t [12], u(x, t), the population density, v(x, t), the diffusion velocity, σ(x, t), the population supply.

The field u(x, t) gives the number of individuals, per unit volume, at position x and time t, its integral over any subregion R gives the total population of R at time t. The field σ(x, t) gives the rate at which individuals are supplied (per unit volume) directly at x by births and deaths. The flow of population from point to point is described by the diffusion velocity v(x, t), which represents the average velocity of those individuals who at time t occupy x. The fields u, v and σ must be consistent with the following law of population balance: for every regular subregion R of B and for all time t.

In this manuscript, we apply the Fractional Complex Transform [13] with iterative method to solve the fractional biological population models. The advantage of this method is its capability of combing two powerful methods for obtaining exact and approximate analytical solutions for nonlinear equations.

The rest of this paper is organized as follows. In Sections II, basic definitions are presented. In Sections III we give an analysis of the fractional complex transform and new iterative method. The numerical results and graphs for the time fractional biological population equations are presented in Section IV. Finally, we give our conclusions in Section V.

II. Basic Definitions

In literature there are many definitions on fractional derivatives, but the most frequently used are as below

- **Riemann-Liouville definition:**

Suppose that $\alpha > 0, t > a, \alpha,\ a,\ t \in R$. Then fractional operator

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha+1-n}}\, dx, \qquad n-1 < \alpha < n,$$

is called the Riemann-Liouville fractional derivative or Riemann-Liouville fractional differential operator of order α.

- **Caputo's definition:**

Suppose that $\alpha > 0, t > a, \alpha, \ a, \ t \in R$. Then fractional operator

$$D_*^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha+1-n}} \, dx\,, \qquad n-1<\alpha<n,$$

is called the Caputo fractional derivative or Caputo fractional differential operator.

- **Jumaries definition**

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x) - f(0)}{(t-x)^{\alpha+1-n}} \, dx; \qquad n-1<\alpha<n$$

III. Analysis of Method

a) Fractional Complex Transform

Fractional complex transform was proposed by Li and He [14] is a very simple solution procedure to convert the fractional differential equations into ordinary differential equations, hence all the analytical methods devoted to the advanced calculus can be easily applied to the fractional calculus.

Consider the following general fractional differential equation

$$f(u, u_t^{\alpha}, u_x^{\beta}, u_y^{\gamma}, u_z^{\delta}, u_t^{2\alpha}, u_x^{2\beta}, u_y^{2\gamma}, u_z^{2\delta}, \dots) = 0$$

(3.1)

where $u_t^{\alpha} = \frac{\partial^{\alpha} u(x,y,z,t)}{\partial t^{\alpha}}$ denotes modified Riemann-Liouville derivatives.

$$0<\alpha\le 1, \ 0<\beta\le 1, \ 0<\gamma\le 1, \ 0<\delta\le 1$$

Introducing the following fractional complex transforms

$$T = \frac{p\, t^{\alpha}}{\Gamma(1+\alpha)}, \quad X = \frac{q\, x^{\beta}}{\Gamma(1+\beta)}, \quad Y = \frac{r\, y^{\gamma}}{\Gamma(1+\gamma)}, \qquad Z = \frac{s\, z^{\delta}}{\Gamma(1+\delta)}$$

where p, q, r and s are unknown constants. Using the basic properties of the fractional derivative and the above transforms, we can convert fractional derivatives into partial derivatives as:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = p\frac{\partial u}{\partial T}, \quad \frac{\partial^{\beta} u}{\partial x^{\beta}} = \frac{\partial u}{\partial X}, \quad \frac{\partial^{\gamma} u}{\partial y^{\gamma}} = \frac{\partial u}{\partial Y}, \quad \frac{\partial^{\delta} u}{\partial z^{\delta}} = \frac{\partial u}{\partial Z}.$$

Therefore, the fractional differential equations are easily converted into partial differential equations, so that anyone acquainted with simple calculus can deal with fractional calculus without any difficulty which can be solved further by new iterative method.

b) New Iterative Method

Daftardar-Gejji and Jafari have introduced a new iterative method (NIM) which is simple to understand and easy to implement in solving nonlinear equations.
In this method, consider a functional equation of the form

$$u = f + L(u) + N(u) , \quad (3.2)$$

where f is a given function, L and N are given linear and non-linear operator. Let u be a solution of Eq.(3.2) having the series form,

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t) \quad (3.3)$$

Eq.(3.2) is equivalent to $\sum_{i=0}^{\infty} u_i = f + \sum_{i=0}^{\infty} L(u_i) + \sum_{i=0}^{\infty} G_i$ (3.4)

where $G_i = N(\sum_{j=0}^{i} u_j) - N(\sum_{j=0}^{i-1} u_j), \quad i \geq 1.$
Further consider the recurrence relation as:

$$u_0 = f, \quad u_1 = L(u_0) + G_0, \quad \cdots \quad u_{m+1} = L(u_m) + G_m , \quad m = 1,2,\cdots$$

The k-term approximate solution is given by $u = u_0 + u_1 + u_2 + \cdots + u_{k-1}$.

IV. Numerical Illustration

In this section, to illustrate the efficiency and accuracy of the combination of fractional complex transform and new iterative method, we have obtained approximate solution of time fractional biological population model based on Jumarie's modified Riemann-Liouville.
To illustrate the basic idea of this method, let us consider the generalized biological population model:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + ku^a(1 - ru^b) \quad (4.1)$$

$t > 0, \quad x, y \in R, \quad 0 < \alpha \leq 1$ with the initial condition,

$$\mathrm{u(x, y, 0)} = \emptyset_0(\mathrm{x, y}) . \quad (4.2)$$

Consider the equation (4.1) with k = 1; a = 1; r = 0, we have the following fractional biological population equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + u$$

(4.3)

with the initial condition

$$u(x, y, 0) = \sqrt{\sin x \sinh y}. \tag{4.4}$$

We consider the transformation

$$T = \frac{t^\alpha}{\Gamma(1+\alpha)}$$

Hence $\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial u}{\partial T}$

Using this transformation Eq.(4.3) we get

$$\frac{\partial u}{\partial T} = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + u \tag{4.5}$$

Applying Integral operator I_T on both side of Eq.(4.5) and using initial condition we obtain the relation

$$u(x, T) = u_0(x, T) + L(u) + N(u).$$

Taking series solution as $u(x,T) = \sum_{i=0}^{\infty} u_i(x,T)$ and using iteration formula (3.3) and (3.4) with initial condition (4.4) we get

$$u_0 = \sqrt{\sin x \sinh y}$$

$$u_1 = t\sqrt{\sin x \sinh y}$$

$$u_2 = \frac{t^2}{2}\sqrt{\sin x \sinh y}$$

$$\vdots$$

In the same manner the remaining components of the Eq. (4.3) can be obtained from Mathematica software.

The series form of solution is

$$u(x, y, t) = \sqrt{\sin x \sinh y}\left[\frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots\right]$$

For $\alpha = 1$ we get exact solution as

$$u(x, y, t) = \sqrt{\sin x \sinh y}\, e^t.$$

(x, y)	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$
(0.2, 0.2)	128.8655211	66.2254250	34.0581016
(0.4, 0.4)	257.6966760	132.4331888	68.1071205
(0.6, 0.6)	386.3215763	198.5349561	102.1016280
(0.8, 0.8)	514.2927381	264.3007598	135.9233578
(1.0, 1.0)	640.7465610	329.2867863	169.3440674

Table 1: Approximate solution of Eq. (4.3) for different values of α at t = 10

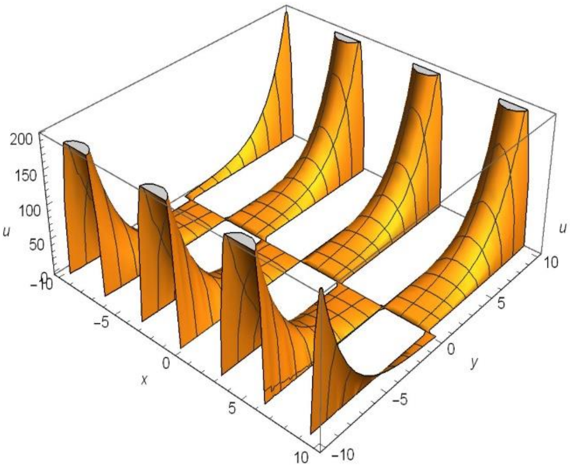

Fig 1: Apprx. Sol. at $\alpha = 1, t = 10$

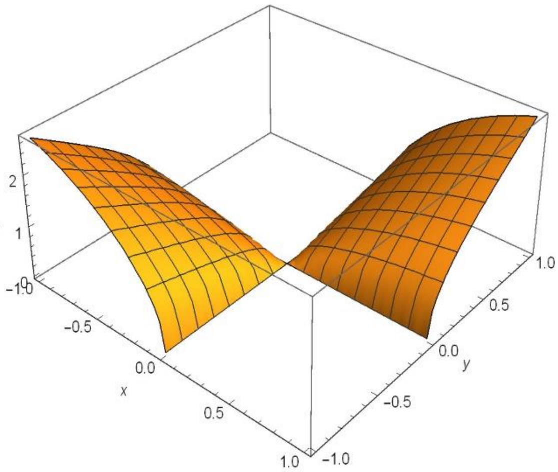

Fig 2: Apprx. Sol. at $\alpha = 0.8, t = 10$

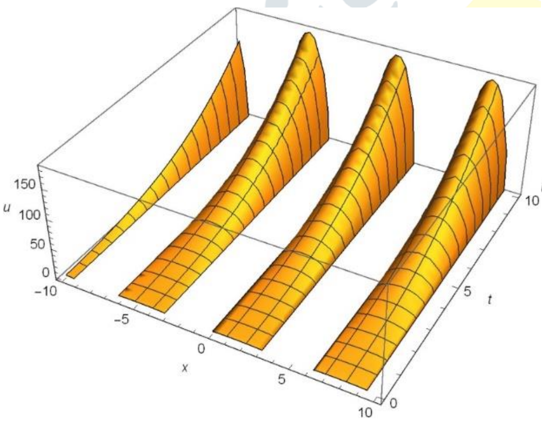

Fig 3: Apprx. Sol. at $\alpha = 0.8,\ y = 1$

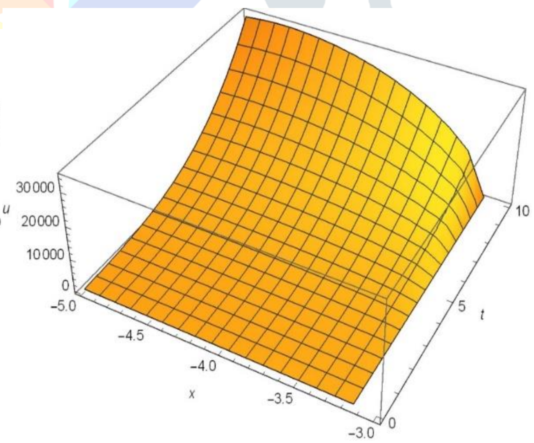

Fig 4: Apprx. Sol. at $\alpha = 0.9,\ y = 10$

V. Conclusions:

In this work, we extensively studied a mathematical model of biology. In this model, we establish a variety of exact solutions. To study the exact solutions, we used a fractional complex transform to convert the particular partial differential equation of fractional order into corresponding partial differential equation and iterative method is implemented to investigate the nonlinear equation. Graphical demonstrations along with the numerical data reinforce the efficacy of the used procedure. In simulation section of our work, it has been proved that the exact solutions of nonlinear fractional order biological population model obtained by extended new approach are in admirable agreement with the exact solutions of nonlinear fractional order biological population model. The application of this new approach into solving every kind of nonlinear time-fractional and space-fractional partial or ordinary differential equations will be our further consideration.

Finally, we conclude that the Fractional complex transform with iterative method is very powerful and efficient in finding analytical and numerical solutions for wider classes of linear and nonlinear fractional differential equations.

References:

[1] Busenberg, S., Driessche, P., Analysis of a disease transmission model in a population with varying size, J. Math. Bio. 28(1990), 65–82 (1990).

[2] El-Sayed, A.M.A., Rida, S.Z., Arafa, A.A.M.: On the solutions of time-fractional bacterial chemotaxis in a diffusion gradient chamber. Int. J. Nonlinear Sci. **7**, 485–495 (2009)

[3] Makinde, O.D.: Adomian decomposition approach to a SIR epidemic model with constant vaccination strategy. Appl. Math. Comput. 184, 842–848 (2007).

[4] Kribs-Zaleta C.M., Structured models for heterosexual disease transmission, Math. Biosci. 160 (1) (1999) 83–108.

[5] Buonomo, B. Lacitignola, D., On the dynamics of an SEIR epidemic model with a convex incidence rate, Ricerche Mat. 57(2) (2008) 261–281.

[6] He J., Nonlinear oscillation with fractional derivative and its applications, International Conference on Vibrating Engineering, Dalian, China, (1998):288-291.

[7] He J., Some applications of nonlinear fractional differential equations and their approximations, Bull. Sci. Technol, 15(1999):86-90.

[8] He J., Approximate analytical solution for seepage flow with fractional derivatives in porous media, Comput. Methods. Appl. Mech. Eng, 167(1998):57-68.

[9] Daftardar-Gejji V, Jafari H. An iterative method for solving nonlinear functional equations. J. Math. Anal. Appl. 2006; 316:753-763.

[10] Sontakke B.R., Shaikh A. Numerical Solutions of time fractional Fornberg-Whitam and modified Fornberg-Whitam equations using new iterative method. AJOMCOR. 2016;13(2):66-76.

[11] Sontakke B.R., Shaikh A. The new iterative method for approximate solutions of time fractional Kdv, K(2,2), Burgers, and cubic Boussinesq equations. Asian Research Journal of Mathematics. 2016;1(4):1-10.

[12] Gurtin M.E., Maccamy R.C., On the diffusion of biological population, Mathematical Biosciences, 33 (1977) 35–49.

[13] Li Z.B., He J.H. Fractional complex transform for fractional differential equations. Math. Comput. Appl. 15(2010), 970-973.

[14] He J.H., Li Z.B., Converting fractional differential equations into partial differential equations. Therm. Sci. 2012(16), 331-334.