



DOMAIN WALLS WITH DECELERATION PARAMETER IN $f(R, L_m)$ THEORY

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Abstract

In this paper, we have studied Friedmann–Robertson–Walker (FRW) space time in the presence of domain walls in the framework of $f(R, L_m)$ theory of gravitation, where R is the Ricci scalar and L_m is the matter Lagrangian. Exact solution of the field equations of the $f(R, L_m)$ theory are obtained with the help of deceleration parameter and $f(R, L_m) = \frac{R}{2} + L_m^\alpha$. It is observed that for $\eta = 0$ and $\eta = 2\sqrt{\mu\beta}$ due to domain walls, universe decay in to gravitational waves, for $\eta \neq 2\sqrt{\mu\beta}$, the domain walls tension is negative and behave like dark energy. In this model, we discussed energy conditions.

KeywordsThe Domain Wall, FRW, $f(R, L_m)$ Gravity

1 Introduction

Recent observational data indicates that the the universe is accelerating and expanding[1–3]. In the framework of General theory of relativity (GR), it is not possible to fully explain the observe accelerated expansion. For the purpose of better understand the accelerated expansion of the universe, Cosmography helps to study the accelerated expansion of the universe without assuming a specific cosmological model and is

useful for studying modified gravity theories [4]. Bamba et al. [5] studied dark energy cosmology and presented that different theoretical models will produce equivalent descriptions of the accelerated expansion of the universe through cosmographic tests. Λ CDM model is proposed to understand the phenomenon of accelerated expansion of the universe. This model basically formed by two components: cold dark matter, that is pressure less and dark energy which possesses negative pressure and related to the positive cosmological constant. It explains more properties of the universe [6, 7]. Dark energy is viewed to be the main source of the accelerated expansion of the universe at late times. Several researchers have suggested alternative theories of gravity as modifications of general relativity to explain this problem. One way is to modify the Einstein - Hilbert action by modifying it with general function $f(R)$, where R indicates the Ricci scalar describing the curvature of the space time [8, 9]. The $f(R)$ theory of gravity explained the extension of the universe in the far future [10]. Salvatore Capozziello et al. [11] examined the cosmological viability of $f(R)$ gravity and showed that the model can reproduce important cosmological phases. Amendola et al. [12] studied the cosmological viability of $f(R)$ dark energy models and explained the conditions under which these models can reproduce the observed accelerated expansion of the Universe. Nojiri and Odintsov [13] investigated a modified gravity model containing both negative and positive powers of the curvature scalar, indicating that such a framework can give a unified description of the inflationary epoch and the present accelerated expansion of the universe. Faraoni [14] studied that Solar System experiments do not yet exclude all modified gravity models and emphasized that such theories need careful observational testing. The observational features of $f(R)$ dark energy models satisfying cosmological and local gravity constraints were studied in [15]. Liu et al. [16] investigated $f(R)$ gravity using observations from the Solar System, cosmology, and binary pulsar systems. Capozziello and Tsujikawa [17] constraints on $f(R)$ dark energy models from solar system experiments combined with experiments on the violation of equivalence principle. Starobinsky [18] proposed an $f(R)$ gravity model to explain the disappearance of the cosmological constant. Nojiri and Odintsov [19] suggested two realistic $f(R)$ and one $f(G)$ modified gravities which are consistent with local tests and cosmological bounds. Nojiri and Odintsov [20] considered modified $f(R)$ gravity which may unify R^m early-time inflation with late time Λ CDM cosmological phase. Cognola et al. [21] studied a feasible class of modified $f(R)$ gravity models capable of explaining both the inflationary epoch and the late-time accelerated expansion of the universe. Nunes et al. [22] derived new observational constraints on $f(R)$ gravity using cosmic chronometer data. Energy conditions are effective tools to describe the physical work ability of modified gravity theories. The energy conditions in $f(R)$ gravity studied in [23]. The cosmographic model has been used to constrain and analyze $f(R)$ gravity models [24].

Afterward modified form of $f(R)$ gravity was suggested in which the matter Lagrangian density L_m is instantly related to the function of $f(R)$ [25]. This relation between matter and geometry generates a more force which acts on particles, producing their motion distinct from the usual geodesic path. The model was also expanded to enable more general forms of this matter - geometry connection [26]. The study of galactic rotation curves in modified gravity with non minimal matter-geometry coupling has been discussed in [27]. Harko [28] investigated the role of the matter Lagrangian and the energy-momentum tensor in modified gravity with non minimal coupling between matter and geometry. Nesseris [29] studied matter density perturbations in modified gravity models with arbitrary coupling between matter and geometry. Recently,

Harko and Lobo [30] proposed new general theory called $f(R, L_m)$ gravity. In this theory $f(R, L_m)$ is function that based on both Ricci scalar R and matter Lagrangian density L_m . This gravitational theory is one of the extension theories in Riemannian space. In $f(R, L_m)$, test particles remain stable along the usual geodesic paths because of an extra force perpendicular to their motion. Previous studies such as those by Wang and Liao [31] have studied energy conditions in $f(R, L_m)$ theory. Jaybhaye et al. [32] studied the late time cosmic expansion of the universe in the framework of $f(R, L_m)$ gravity theory. Garg [33] studied universe's behavior for a flat FLRW metric in $f(R, L_m)$ gravity. Sahlu et al. [34] investigated cosmological evolution in the framework of $f(R, L_m)$ gravity and constrained both the background and perturbed dynamics.

The early time evolution of the universe has become an important area of cosmological research in recent years. It is suggested that the universe evolved during different stages through its development. In the early phase of transition of the universe, structures known as vacuum domain walls may have been generated, and some of them could still remain today. Such domain walls are examples of topological defects, that are evaluated important in understanding the formation of large-scale structures in the universe. By virtue of their feasible role in cosmic evolution, domain walls have attracted significant focus from researchers. Various studies have examined domain walls under different theoretical frameworks. Katore et al. [35–37] studied domain walls in various cosmological scenarios. Reddy et al. [38] explored the nature of domain walls and cosmic strings within bimetric theory. Press et al. [39] studied the dynamical evolution of domain walls in an expanding universe, while Tiwari [40] investigated the phase transition of a five dimensional perfect fluid universe in modified $f(R, T)$ gravity theory.

The above discussion motivate us to discuss a cosmological model in $f(R, L_m)$ gravity with Friedmann-Lemaître - Robertson - Walker (FLRW) line element in presence of domain walls. The paper is organized as follows, section 2 deals with metric and field equation. In section 3, 4 and 5, we obtained solution of field equations. In section 6, we discussed energy conditions. Section 7 concludes the remark of the paper.

2 METRIC AND FIELD EQUATION:

Here we consider the expression for the action of $f(R, L_m)$ gravity theory as [30]

$$S = \int f(R, L_m) \sqrt{-g} d^4x \quad (1)$$

where $f(R, L_m)$ is an arbitrary function of the Ricci scalar R and the matter Lagrangian L_m , g is the determinant of the metric tensor.

The gravitational field equation derived by taking the variation of the action (1) with respect to the metric tensor $g_{\mu\nu}$ is

$$f_R R_{\mu\nu} + (g_{\mu\nu} \square - \nabla_\mu \nabla_\nu) f_R - \frac{1}{2} (f - f_{L_m} L_m) g_{\mu\nu} = \frac{1}{2} f_{L_m} T_{\mu\nu} \quad (2)$$

Here, we introduce the notations $f_R = \frac{\partial f}{\partial R}$ and $f_{L_m} = \frac{\partial f}{\partial L_m}$. Furthermore, $T_{\mu\nu}$ represents the energy momentum tensor for a perfect fluid, which is defined as

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta \sqrt{-g} L_m}{\delta g^{\mu\nu}} \quad (3)$$

we can establish a relationship between energy momentum tensor T_{ij} , Ricci scalar R , and matter Lagrangian density L_m by using the contraction on the field equation (2). The required relation is exhibited as

$$Rf_R + 3\Box f_R - 2(f - f_{L_m}L_m) = \frac{1}{2}f_{L_m}T, \quad (4)$$

where $\Box f_R$ denotes the d'Alembertian operator applied to a function f_R , defined as $\Box f_R = \frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g} g^{\alpha\beta} \partial_\beta f_R)$.

Moreover, upon taking the covariant derivative of Eq. (2), we obtain following result,

$$\nabla^\mu T_{\mu\nu} = 2\nabla^\mu \ln(f_{L_m}) \frac{\partial L_m}{\partial g^{\mu\nu}}, \quad (5)$$

where, $\frac{\partial L_m}{\partial g^{\mu\nu}} = -\frac{1}{2}(g_{\mu\nu}L_m - T_{\mu\nu})$.

We consider the Friedmann-Robertson-Walker (FRW) cosmological model in the following form;

$$ds^2 = dt^2 - a^2 \left(\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi) \right) \quad (6)$$

In FRW space time, the angles $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq 2\pi$ are the usual azimuthal and polar angles of spherical coordinates. The k represents the curvature of the space. The curvature of space has three different values. When $k = 1$ the radius is finite and the universe is closed. When $k = 0$, the universe is flat and when $k = -1$ the radius is infinite or imaginary corresponding to open universe [41]. Pradhan et al. [42] have presented general solutions for domain walls in Lyra geometry. Khadekar et al. [43] have studied Kaluza-Klein type FRW cosmological model with domain wall.

The Ricci scalar R associated with the line element (6) can be derived as follows

$$R = -6 \left[\frac{k}{a^2} + \left(\frac{\dot{a}}{a} \right)^2 + \frac{\ddot{a}}{a} \right] \quad (7)$$

where a is scale factor

The energy momentum tensor of domain walls [44] is given as:

$$T_{ij} = \rho(g_{ij} + \mu_i\mu_j) + p\mu_i\mu_j \quad (8)$$

here ρ, p denotes energy density and pressure of domain wall respectively, and μ_i is the unit space like vector in the same direction with $\mu_i\mu^i = -1$. Based on the default model of particle physics, it is assumed that when the hot early universe cooled and expanded, the field would have settled to single values during

extended regions. The boundaries of those different regions would be the domain walls. The energy momentum tensor T_{ij} of domain walls include normal matter ρ_m and pressure p_m as well as tension σ_d with the relation $p = p_m - \sigma_d$, $\rho = \rho_m + \sigma_d$ satisfying $p_m = (\gamma - 1)\rho_m$, $1 \leq \gamma \leq 2$.

The field equation for line element (6) using equation (1) and (8)

$$f_R \left[\frac{-2k}{a^2} - 2 \left(\frac{\dot{a}}{a} \right)^2 - \frac{\ddot{a}}{a} \right] + 3 \frac{\dot{a}}{a} \dot{f}_R + \ddot{f}_R - \frac{1}{2} (f - f_{L_m} L_m) = \frac{1}{2} f_{L_m} \rho \quad (9)$$

$$3 \frac{\ddot{a}}{a} f_R - 3 \frac{\dot{a}}{a} \dot{f}_R + \frac{1}{2} (f - f_{L_m} L_m) = \frac{1}{2} f_{L_m} P \quad (10)$$

where overdot (.) represent differentiation with respect to cosmic time t . We have two equations (9) and (10) in three unknown a, ρ and p .

To obtain exact solution of field equation we consider $f(R, L_m)$ [45] and deceleration parameter q [46] as follows

$$f(R, L_m) = \frac{R}{2} + L_m^\alpha \quad (11)$$

The parameter α initiates matter–geometry coupling and $L_m = \rho$, offering further flexibility to explain cosmic evolution. This model is useful for describing the late-time accelerated expansion of the universe excluding explicitly introducing dark energy and helps in studying various from standard cosmology [45].

$$q = -1 + \frac{d}{dt} \left(\frac{1}{H} \right) \quad (12)$$

Integrating equation (12) to give scale factor $a(t)$ as

$$a(t) = e^\delta \exp \left\{ \int \frac{dt}{f(1+q)dt + \eta} \right\} \quad (13)$$

where η and δ are arbitrary constants of integration. For a feasible integration of (13), we examine that (i) $q = \text{constant}$ is an easy choice that provides $a(t)$ as an explicit function of t , it is investigated by Berman [47]. (ii) when q is taken to vary with time, an explicit determination of $a(t)$ gives to a possible choice of q as

$$q = -\frac{\mu}{t^2} + (\beta - 1) \quad (14)$$

which is the main ansatz of the paper. Here, $\mu > 0$ is a parameter having the dimension of square of time and $\beta > 1$ is a dimensionless constant. Obviously, the different values of α and β will give rise to different models.

Using equations (13) and (14) we obtain the scale factor as

$$a(t) = e^\delta \exp \left\{ \frac{1}{\beta} \int \frac{t}{t^2 + \frac{\eta}{\beta} t + \frac{\mu}{\beta}} dt \right\} \quad (15)$$

The equation (15) can be integrated for $\eta \neq 2\sqrt{\mu\beta}$, $\eta = 2\sqrt{\mu\beta}$ and $\eta = 0$. It gives us following three different forms of the scale factor as

for $\eta = 0$

$$a(t) = e^{\delta} \left(t^2 + \frac{\mu}{\beta} \right)^{\frac{1}{2\beta}} \quad (16)$$

for $\eta = 2\sqrt{\mu\beta}$

$$a(t) = e^{\delta} \left(t + \sqrt{\frac{\mu}{\beta}} \right)^{\frac{1}{\beta}} \exp \left\{ \frac{\sqrt{\mu}}{\beta^{\frac{3}{2}}} \left(\frac{1}{t + \sqrt{\frac{\mu}{\beta}}} \right) \right\} \quad (17)$$

for $\eta \neq 2\sqrt{\mu\beta}$

$$a(t) = e^{\delta} \left(t^2 + \frac{\eta}{\beta} t + \frac{\mu}{\beta} \right)^{\frac{1}{2\beta}} \times \left(\frac{t + \frac{\eta - \sqrt{\eta^2 - 4\mu\beta}}{2\beta}}{t + \frac{\eta + \sqrt{\eta^2 - 4\mu\beta}}{2\beta}} \right)^{-\frac{\eta}{2\beta\sqrt{\eta^2 - 4\mu\beta}}} \quad (18)$$

we set initial point of the time coordinate at the bounce of these bouncing models and concentrate only on the expanding part of the solutions that represent the observable universe. It is then easy to see from (16), (17) and (18) that at $t = 0$, $a = a_0$ (say) $\neq 0$ but $\dot{a} = 0$ and $\ddot{a} = \frac{a_0}{\alpha} = \text{constant}$, here the suffix 'zero' represent the value of the parameter at $t = 0$. This shows that in all the three cases the models are free from origin singularity and start with a finite acceleration. One can choose the parameters early to give a high initial value $\ddot{a}$ so that the majority of the thermal history of the model is akin to the successful thermal history of the standard cosmology (except in the close vicinity of the singularity). Equation (14) indicates that the deceleration parameter $q \rightarrow -\infty$ at $t = 0$ and decreases to zero at time $t = \sqrt{\frac{\mu}{\beta-1}}$. Clearly, the period of accelerated expansion also depends on the values of μ and β . This puts a restriction on β as $1 \leq \beta \leq 2$. If we take $\beta = 1$ the universe has an accelerated expansion throughout the evolution as is clear from (14).

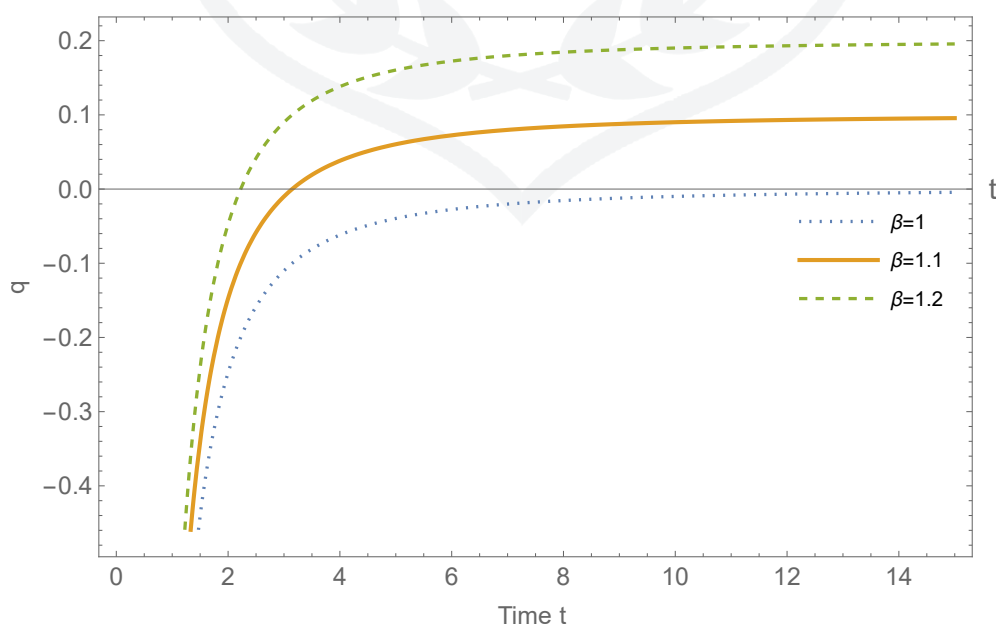


Figure 1: Plot of Deceleration Parameter q vs. cosmic time t for $\mu = 0.99$

From figure 1, it is clear that deceleration parameter (DP) is negative for all values of β i.e. at early stage of evolution, the universe is accelerating. For $\beta = 1.1$ and $\beta = 1.2$, as $t \rightarrow \infty$ q tends to positive values i.e. early accelerating universe tends to late deceleration phase which is not as per the expectation of standard cosmological model. For $\beta = 1$, DP remains negative throughout the evolution of the universe.

3 Model I

We put for $\eta = 0$ and we obtain from (15),

$$e^\delta = a_0 \left(\frac{\beta}{\mu} \right)^{\frac{1}{2\beta}} \text{ and consequently}$$

$$a(t) = a_0 \left(\frac{\beta}{\mu} \right)^{\frac{1}{2\beta}} \left(t^2 + \frac{\mu}{\beta} \right)^{\frac{1}{2\beta}} \quad (19)$$

Expansion Scalar found to be

$$\theta = 3H = 3 \frac{\dot{a}}{a} = \frac{3t}{\beta \left(t^2 + \frac{\mu}{\beta} \right)} \quad (20)$$

Using equations (7),(9),(10),(11) and (19) ρ and p obtained as

$$\rho = \left[\frac{k}{(a_0)^2 \left(\frac{\beta}{\mu} \right)^{\frac{1}{\beta}} \left(t^2 + \frac{\mu}{\beta} \right)^{\frac{1}{\beta}}} + \frac{t^2}{\beta^2 \left(t^2 + \frac{\mu}{\beta} \right)^2} + \frac{2\frac{\mu}{\beta} + 2\left(\frac{1}{\beta} - 1\right)t^2}{\beta \left(t^2 + \frac{\mu}{\beta} \right)^2} \right]^{\frac{1}{\alpha}} \quad (21)$$

$$p = \frac{\frac{2(1-\alpha)\frac{\mu}{\beta} + 2(1-\alpha)\left(\frac{1}{\beta} - 1\right)t^2}{\beta \left(t^2 + \frac{\mu}{\beta} \right)^2} - \frac{(2+\alpha)k}{(a_0)^2 \left(\frac{\beta}{\mu} \right)^{\frac{1}{\beta}} \left(t^2 + \frac{\mu}{\beta} \right)^{\frac{1}{\beta}}} - \frac{(2+\alpha)t^2}{\beta^2 \left(t^2 + \frac{\mu}{\beta} \right)^2}}{\alpha \left[\frac{k}{(a_0)^2 \left(\frac{\beta}{\mu} \right)^{\frac{1}{\beta}} \left(t^2 + \frac{\mu}{\beta} \right)^{\frac{1}{\beta}}} + \frac{t^2}{\beta^2 \left(t^2 + \frac{\mu}{\beta} \right)^2} + \frac{2\frac{\mu}{\beta} + 2\left(\frac{1}{\beta} - 1\right)t^2}{\beta \left(t^2 + \frac{\mu}{\beta} \right)^2} \right]} \left(\frac{\alpha-1}{\alpha} \right) \quad (22)$$

From figure 2 it is observed that energy density is positive for parameter $k = 1, k = -1$ and $k = 0$. Plot shows that at early time of the universe energy density was high and at late time of universe energy density is decreases.

Also, the energy density of normal matter ρ_m is found to be

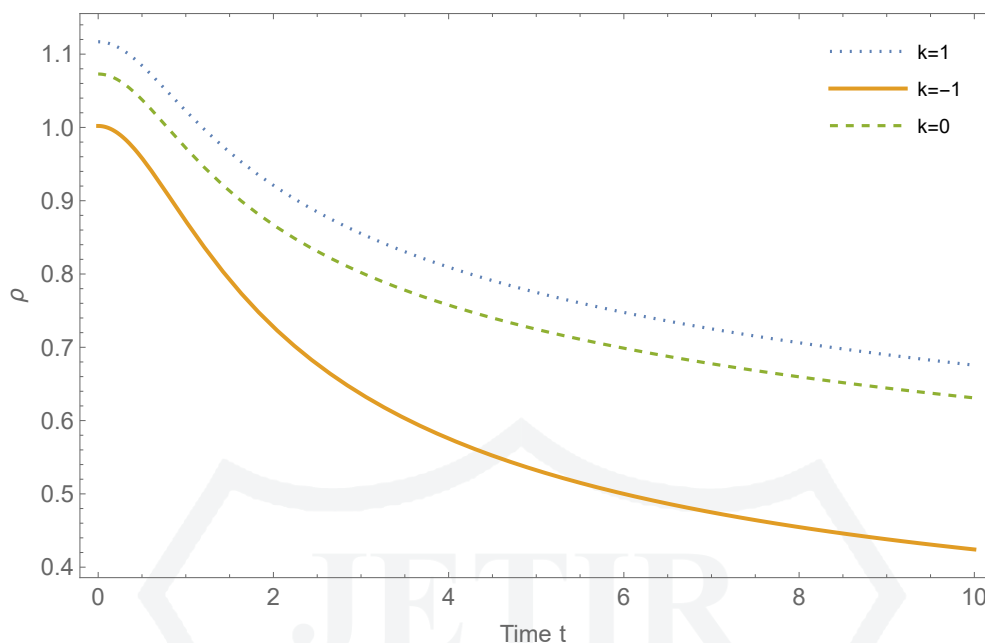


Figure 2: Plot of energy density ρ vs. cosmic time t of Model I for $\mu = 0.99$, $\beta = 1$, $a_0 = 1$, $\alpha = 10$

$$\begin{aligned}
 \rho_m = & \frac{2(1-\alpha)\frac{\mu}{\beta} + 2(1-\alpha)\left(\frac{1}{\beta}-1\right)t^2}{\gamma\beta\left(t^2+\frac{\mu}{\beta}\right)^2} - \frac{(2+\alpha)k}{\gamma(a_0)^2\left(\frac{\beta}{\mu}\right)^{\frac{1}{\beta}}\left(t^2+\frac{\mu}{\beta}\right)^{\frac{1}{\beta}}} - \frac{(2+\alpha)t^2}{\gamma\beta^2\left(t^2+\frac{\mu}{\beta}\right)^2} \\
 & \alpha \left[\frac{k}{(a_0)^2\left(\frac{\beta}{\mu}\right)^{\frac{1}{\beta}}\left(t^2+\frac{\mu}{\beta}\right)^{\frac{1}{\beta}}} + \frac{t^2}{\beta^2\left(t^2+\frac{\mu}{\beta}\right)^2} + \frac{2\frac{\mu}{\beta}+2\left(\frac{1}{\beta}-1\right)t^2}{\beta\left(t^2+\frac{\mu}{\beta}\right)^2} \right]^{\left(\frac{\alpha-1}{\alpha}\right)} \\
 & + \frac{1}{\gamma} \left[\frac{k}{(a_0)^2\left(\frac{\beta}{\mu}\right)^{\frac{1}{\beta}}\left(t^2+\frac{\mu}{\beta}\right)^{\frac{1}{\beta}}} + \frac{t^2}{\beta^2\left(t^2+\frac{\mu}{\beta}\right)^2} + \frac{2\frac{\mu}{\beta}+2\left(\frac{1}{\beta}-1\right)t^2}{\beta\left(t^2+\frac{\mu}{\beta}\right)^2} \right]^{\frac{1}{\alpha}}
 \end{aligned} \tag{23}$$

Furthermore, the tension of domain walls σ_d is obtained as

$$\sigma_d = \left(\frac{\gamma-1}{\gamma} \right) \left[\frac{k}{(a_0)^2 \left(\frac{\beta}{\mu} \right)^{\frac{1}{\beta}} \left(t^2 + \frac{\mu}{\beta} \right)^{\frac{1}{\beta}}} + \frac{t^2}{\beta^2 \left(t^2 + \frac{\mu}{\beta} \right)^2} + \frac{2\frac{\mu}{\beta} + 2\left(\frac{1}{\beta} - 1 \right) t^2}{\beta \left(t^2 + \frac{\mu}{\beta} \right)^2} \right]^{\frac{1}{\alpha}}$$

$$- \frac{\frac{2(1-\alpha)\frac{\mu}{\beta} + 2(1-\alpha)\left(\frac{1}{\beta} - 1 \right) t^2}{\gamma \beta \left(t^2 + \frac{\mu}{\beta} \right)^2} - \frac{(2+\alpha)k}{\gamma (a_0)^2 \left(\frac{\beta}{\mu} \right)^{\frac{1}{\beta}} \left(t^2 + \frac{\mu}{\beta} \right)^{\frac{1}{\beta}}} - \frac{(2+\alpha)t^2}{\gamma \beta^2 \left(t^2 + \frac{\mu}{\beta} \right)^2}}{\alpha \left[\frac{k}{(a_0)^2 \left(\frac{\beta}{\mu} \right)^{\frac{1}{\beta}} \left(t^2 + \frac{\mu}{\beta} \right)^{\frac{1}{\beta}}} + \frac{t^2}{\beta^2 \left(t^2 + \frac{\mu}{\beta} \right)^2} + \frac{2\frac{\mu}{\beta} + 2\left(\frac{1}{\beta} - 1 \right) t^2}{\beta \left(t^2 + \frac{\mu}{\beta} \right)^2} \right]^{\left(\frac{\alpha-1}{\alpha} \right)}} \quad (24)$$

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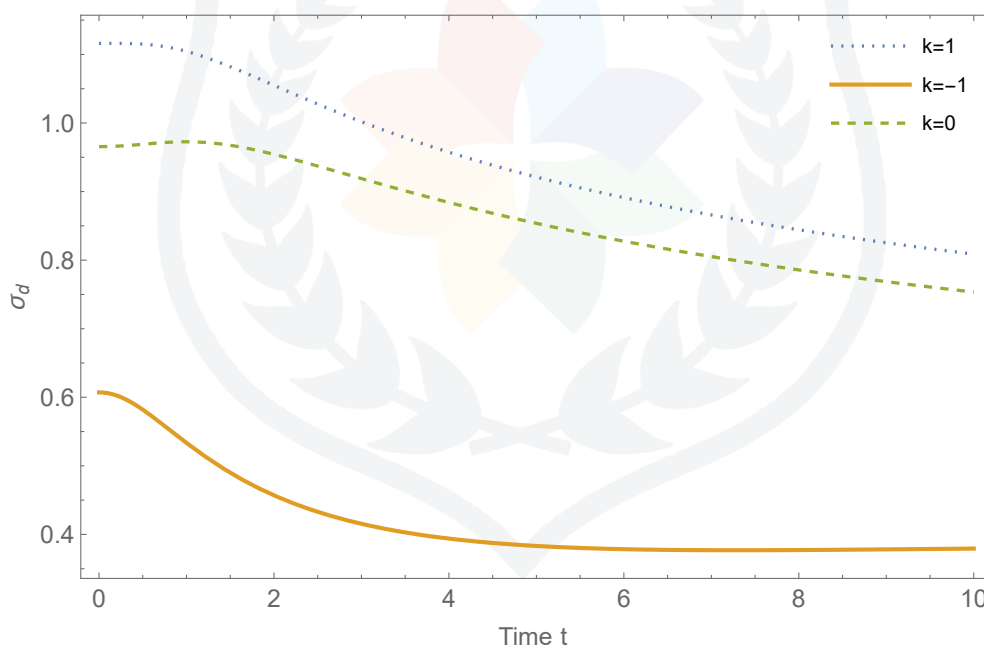


Figure 3: Plot of Tension of Domain Walls σ_d vs. cosmic time t of Model I for $\mu = 0.99$, $\beta = 1$, $a_0 = 1$, $\alpha = 10$, $\gamma = 1$

Figure 3 shows that Tension of domain walls is positive, it means domain walls have massive amount of energy. The domain walls dominates over the matter energy density during expansion of the universe. Due to positive tension, walls naturally contract to minimize their surface area. Domain walls oscillate, intersect and slicing the universe until decay into gravitational waves and radiation [48]

Equation of state (EoS) parameter is obtained as

$$\omega = \frac{p}{\rho} = - \frac{\frac{2(1-\alpha)\frac{\mu}{\beta} + 2(1-\alpha)\left(\frac{1}{\beta}-1\right)t^2}{\beta\left(t^2 + \frac{\mu}{\beta}\right)^2} - \frac{(2+\alpha)k}{(a_0)^2\left(\frac{\beta}{\mu}\right)^{\frac{1}{\beta}}\left(t^2 + \frac{\mu}{\beta}\right)^{\frac{1}{\beta}}} - \frac{(2+\alpha)t^2}{\beta^2\left(t^2 + \frac{\mu}{\beta}\right)^2}}{\alpha \left[\frac{k}{(a_0)^2\left(\frac{\beta}{\mu}\right)^{\frac{1}{\beta}}\left(t^2 + \frac{\mu}{\beta}\right)^{\frac{1}{\beta}}} + \frac{t^2}{\beta^2\left(t^2 + \frac{\mu}{\beta}\right)^2} + \frac{2\frac{\mu}{\beta} + 2\left(\frac{1}{\beta}-1\right)t^2}{\beta\left(t^2 + \frac{\mu}{\beta}\right)^2} \right]^{\left(\frac{\alpha-2}{\alpha}\right)}} \quad (25)$$

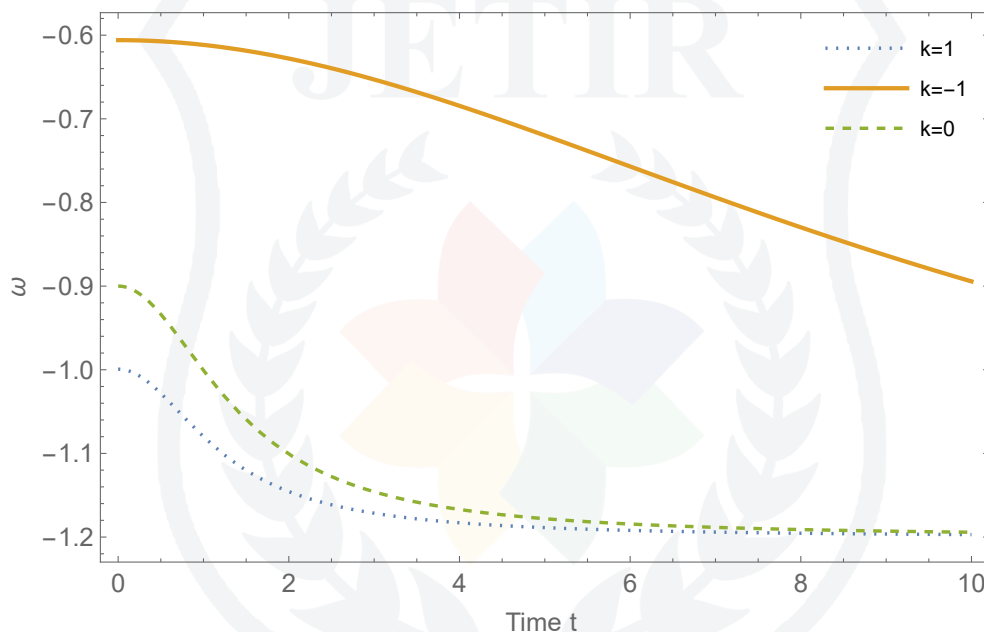


Figure 4: Plot of Equation of State ω vs. cosmic time t of Model I for $\mu = 0.99$, $\beta = 1$, $a_0 = 1$, $\alpha = 10$, $\gamma = 1$

Figure 4 shows the evolution of the equation of state (EoS) parameter with cosmic time t for different values of the curvature parameter k . It is examined that the EoS parameter continues negative throughout the entire evolution, suggesting that the accelerated expansion of the universe. For $k = -1$, the EoS parameter decreases slowly from approximately -0.6 and approaches -0.9 , remaining in the quintessence region $-1 < \omega < -\frac{1}{3}$. For $k = 0$ and $k = 1$, the EoS parameter decreases below -1 and tends toward -1.2 at late times, showing phantom-like behavior $\omega < -1$. This behavior of the curves indicates that the model evolves toward a stable dark-energy-dominated phase in the late-time universe. Therefore, the obtained results indicate that the cosmological dynamics depend significantly on the curvature parameter and support an accelerated expansion scenario. It is important to note here that domain walls behave like dark energy.

4 Model II

Using equation (17) for $\eta = 2\sqrt{\mu\beta}$, we obtain.

$$e^\delta = \frac{a_0}{\left(\sqrt{\frac{\mu}{\beta}}\right)^{\frac{1}{\beta}} \times \exp\left(\frac{1}{\beta}\right)}$$

The average scale factor obtained as

$$a(t) = \frac{a_0}{\left(\sqrt{\frac{\mu}{\beta}}\right)^{\frac{1}{\beta}} \times \exp\left(\frac{1}{\beta}\right)} \times \left(t + \sqrt{\frac{\mu}{\beta}}\right)^{\frac{1}{\beta}} \times \exp\left\{\frac{\sqrt{\mu}}{\beta^{\frac{3}{2}}}\left(\frac{1}{t + \sqrt{\frac{\mu}{\beta}}}\right)\right\} \quad (26)$$

Expansion scalar is obtained as

$$\theta = 3H = 3\frac{\dot{a}}{a} = \frac{3}{\beta\left(t + \sqrt{\frac{\mu}{\beta}}\right)} - \frac{3\sqrt{\mu}}{\beta^{\frac{3}{2}}\left(t + \sqrt{\frac{\mu}{\beta}}\right)^2} \quad (27)$$

Using equations (7),(9),(10),(11) and (26) ρ and p obtained as

$$\rho = (l_1 + l_2 + l_3)^{\frac{1}{\alpha}} \quad (28)$$

$$p = \frac{(1 - \alpha)l_1 - (2 + \alpha)l_2 - (2 + \alpha)l_3}{\alpha\left(l_1 + l_2 + l_3\right)^{\left(\frac{\alpha-1}{\alpha}\right)}} \quad (29)$$

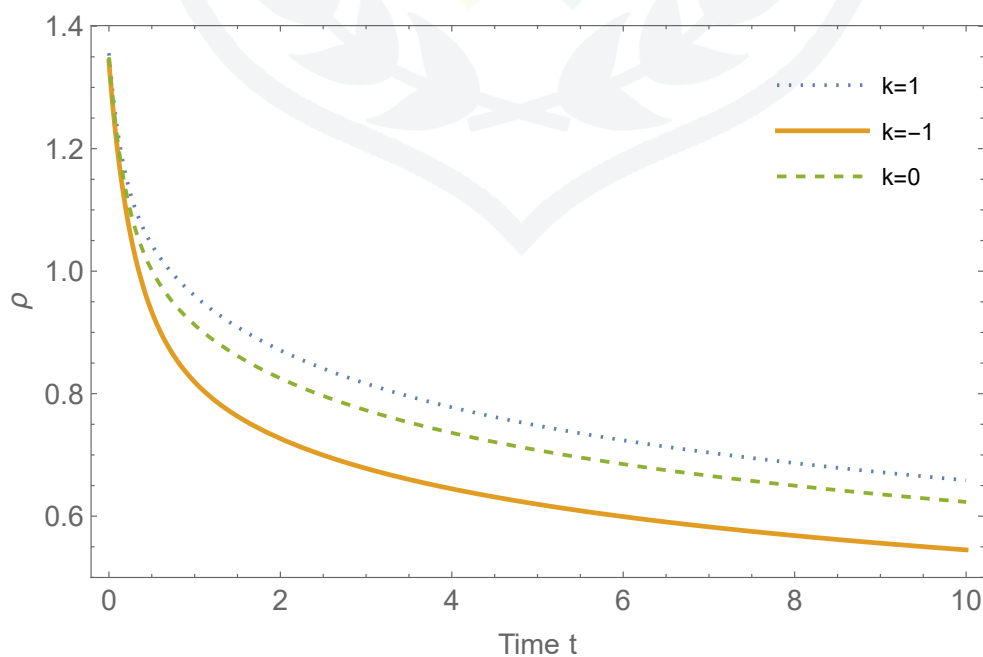


Figure 5: Plot of energy density ρ vs. cosmic time t of Model II for $\mu = 0.1$, $\beta = 1.0001$, $a_0 = 1$, $\alpha = 10$

$$\text{where, } l_1 = \frac{2(1-\beta)}{\beta^2 \left(t + \sqrt{\frac{\mu}{\beta}}\right)^2} + \frac{4(\beta-1)\sqrt{\mu}}{\beta^{\frac{5}{2}} \left(t + \sqrt{\frac{\mu}{\beta}}\right)^3} + \frac{2\mu}{\beta^3 \left(t + \sqrt{\frac{\mu}{\beta}}\right)^4},$$

$$l_2 = \frac{k}{\left(\frac{a_0}{\left(\sqrt{\frac{\mu}{\beta}}\right)^{\frac{1}{\beta}} \times \exp\left(\frac{1}{\beta}\right)}\right)^2 \times \left(t + \sqrt{\frac{\mu}{\beta}}\right)^{\frac{2}{\beta}} \times \exp\left\{\frac{2\sqrt{\mu}}{\beta^{\frac{3}{2}} \left(t + \sqrt{\frac{\mu}{\beta}}\right)}\right\}}$$

$$l_3 = \left[\frac{1}{\beta \left(t + \sqrt{\frac{\mu}{\beta}}\right)} - \frac{\sqrt{\mu}}{\beta^{\frac{3}{2}} \left(t + \sqrt{\frac{\mu}{\beta}}\right)^2} \right]^2$$

the energy density of normal matter ρ_m is obtained as

$$\rho_m = \frac{\frac{(1-\alpha)}{\gamma} l_1 - \frac{(2+\alpha)}{\gamma} l_2 - \frac{(2+\alpha)}{\gamma} l_3}{\alpha \left(l_1 + l_2 + l_3\right)^{\left(\frac{\alpha-1}{\alpha}\right)}} + \frac{(l_1 + l_2 + l_3)^{\frac{1}{\alpha}}}{\gamma} \quad (30)$$

Furthermore, the tension of domain walls found to be

$$\sigma_d = \left(\frac{\gamma-1}{\gamma}\right) (l_1 + l_2 + l_3)^{\frac{1}{\alpha}} - \frac{\frac{(1-\alpha)}{\gamma} l_1 - \frac{(2+\alpha)}{\gamma} l_2 - \frac{(2+\alpha)}{\gamma} l_3}{\alpha \left(l_1 + l_2 + l_3\right)^{\left(\frac{\alpha-1}{\alpha}\right)}} \quad (31)$$

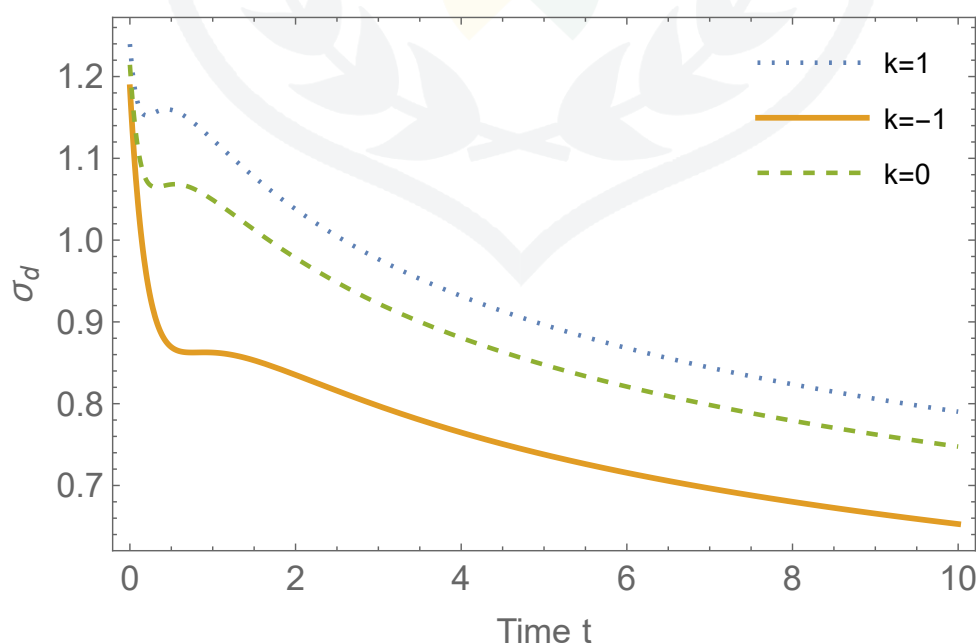


Figure 6: Plot of Tension of Domain Walls σ_d vs. cosmic time t of Model II for $\mu = 0.1$, $\beta = 1.0001$, $a_0 = 1$, $\alpha = 10$, $\gamma = 1$

From figure 5 and 6, it shows that energy density is positive for different values of curvature parameter k . Initially, energy density was high, as cosmic time increases energy density decreases and it tends towards 0, as $t \rightarrow \infty$. Tension of domain walls is positive for all curvature parameter k .

Equation of state (EoS) parameter is obtained as

$$\omega = \frac{p}{\rho} = \frac{(1-\alpha)l_1 - (2+\alpha)l_2 - (2+\alpha)l_3}{\alpha(l_1 + l_2 + l_3)^{\left(\frac{\alpha-2}{\alpha}\right)}} \quad (32)$$

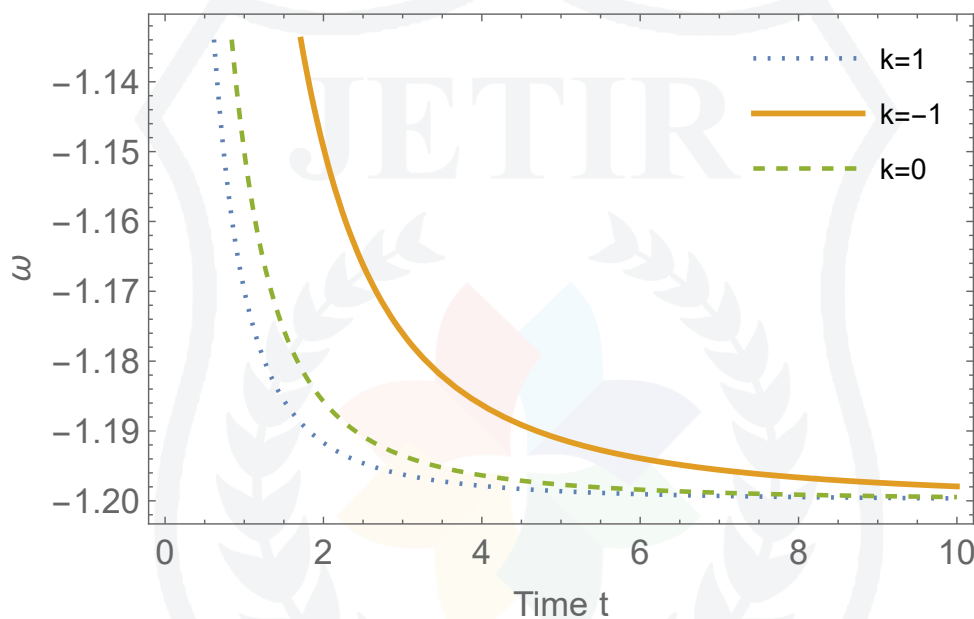


Figure 7: Plot of Equation of State parameter ω vs. cosmic time t of Model II for $\mu = 0.1$, $\beta = 1.0001$, $a_0 = 1$, $\alpha = 10$, $\gamma = 1$

Figure 7 shows the evolution of the equation of state (EoS) parameter ω with cosmic time t for different curvature parameters $k = 1$, $k = 0$, and $k = -1$. It is showed that for all three cases, the EoS parameter remains less than -1 throughout the cosmic evolution, i.e., $\omega < -1$, which indicates a phantom dark energy dominated universe. At early time, the values of ω differ for different curvature parameters, but at late time, all the trajectories converge toward $\omega = -1.2$. This behavior shows that the model dominated to phantom like dark energy. Here, the domain walls behave like dark energy.

5 Model III

Using equation (18) for $\eta \neq 2\sqrt{\mu\beta}$ we get $e^\delta = a_0 \left(\frac{\beta}{\mu}\right)^{\frac{1}{2\beta}} \left(\frac{\eta - \sqrt{\eta^2 - 4\mu\beta}}{2\beta}\right)^{\frac{\eta}{2\beta\sqrt{\eta^2 - 4\mu\beta}}} \left(\frac{\eta + \sqrt{\eta^2 - 4\mu\beta}}{2\beta}\right)^{\frac{\eta}{2\beta\sqrt{\eta^2 - 4\mu\beta}}}$

and consequently we obtain scale factor as

$$a(t) = a_0 \left(\frac{\beta}{\mu} \right)^{\frac{1}{2\beta}} \left(\frac{\eta - \sqrt{\eta^2 - 4\mu\beta}}{2\beta} \right)^{\frac{\eta}{2\beta\sqrt{\eta^2 - 4\mu\beta}}} \left(t^2 + \frac{\eta}{\beta}t + \frac{\mu}{\beta} \right)^{\frac{1}{2\beta}} \times \left(\frac{t + \frac{\eta - \sqrt{\eta^2 - 4\mu\beta}}{2\beta}}{t + \frac{\eta + \sqrt{\eta^2 - 4\mu\beta}}{2\beta}} \right)^{-\frac{\eta}{2\beta\sqrt{\eta^2 - 4\mu\beta}}} \quad (33)$$

Expansion scalar is obtained as

$$\theta = 3 \frac{\dot{a}}{a} = \frac{3 \left(t + \frac{\eta}{2\beta} \right)}{\beta t^2 + \eta t + \mu} - \frac{3\eta}{2\beta\sqrt{\eta^2 - 4\mu\beta}} \left[\frac{1}{t + \frac{\eta - \sqrt{\eta^2 - 4\mu\beta}}{2\beta}} - \frac{1}{t + \frac{\eta + \sqrt{\eta^2 - 4\mu\beta}}{2\beta}} \right] \quad (34)$$

Using equations (7),(9),(10),(11) and (33) ρ and p obtained as

$$\rho = (l_4 + l_5 + 2l_6)^{\frac{1}{\alpha}} \quad (35)$$

$$p = \frac{2(1 - \alpha)l_6 - (2 + \alpha)l_4 - (2 + \alpha)l_5}{\alpha \left(l_4 + l_5 + 2l_6 \right)^{\left(\frac{\alpha - 1}{\alpha} \right)}} \quad (36)$$

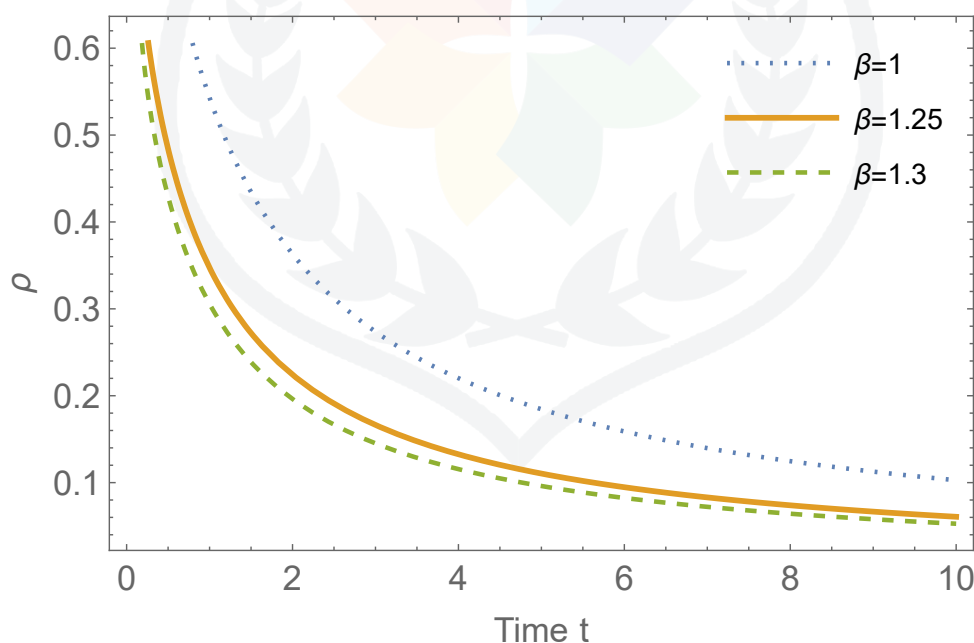


Figure 8: Plot of energy density ρ vs. cosmic time t of Model III for $\mu = 0.001$, $a_0 = 1$, $\alpha = 2.1$, $\eta = 0.9$

Also, the energy density of normal matter ρ_m is found to be

$$\rho_m = \frac{(l_4 + l_5 + 2l_6)^{\frac{1}{\alpha}}}{\gamma} + \frac{2\frac{(1-\alpha)}{\gamma}l_6 - \frac{(2+\alpha)}{\gamma}l_4 - \frac{(2+\alpha)}{\gamma}l_5}{\alpha(l_4 + l_5 + 2l_6)^{\left(\frac{\alpha-1}{\alpha}\right)}} \quad (37)$$

Furthermore, the tension of domain walls

$$\sigma_d = \left(\frac{\gamma-1}{\gamma}\right)(l_4 + l_5 + 2l_6)^{\frac{1}{\alpha}} - \frac{2\frac{(1-\alpha)}{\gamma}l_6 - \frac{(2+\alpha)}{\gamma}l_4 - \frac{(2+\alpha)}{\gamma}l_5}{\alpha(l_4 + l_5 + 2l_6)^{\left(\frac{\alpha-1}{\alpha}\right)}} \quad (38)$$

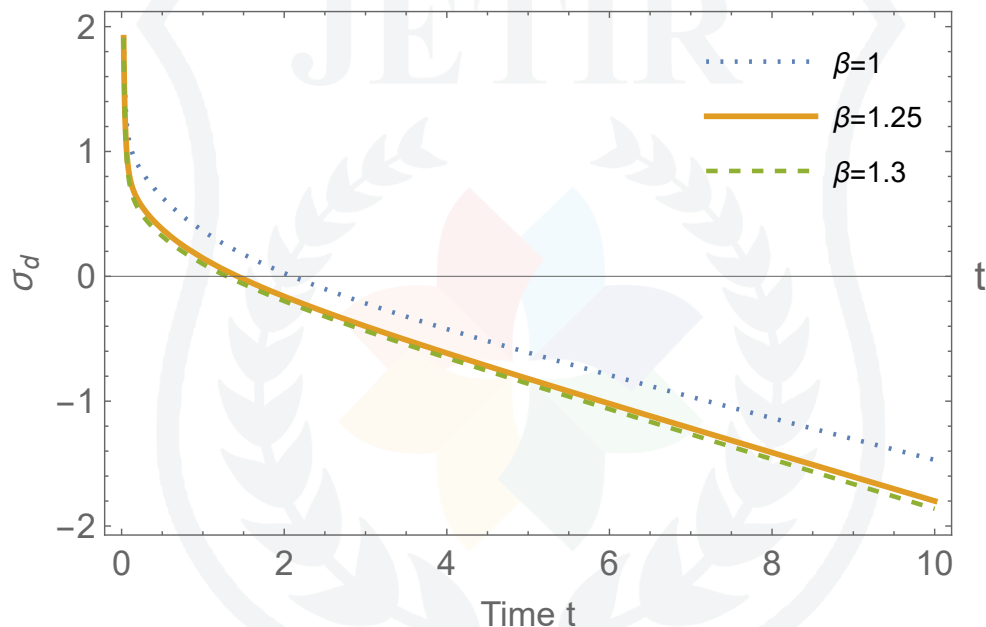


Figure 9: Plot of Tension of Domain Walls σ_d vs. cosmic time t of Model III for $\mu = 0.001, a_0 = 1, \alpha = 1, \eta = 0.9, \gamma = 1.5$

Figure 8 and 9 show that energy density and Tension of domain wall vs. cosmic time t . Energy density is positive for different values of β . At early time of the universe energy density was high, as cosmic time increases energy density decreases and tends to zero, as $t \rightarrow \infty$. At early time of the universe domain wall is positive and at late time of the universe it is negative. Near $t = 0$, the tension of domain wall is positive. Therefore at early stage of evolution of the universe, domain walls have massive amount of energy. Not only mass is source of gravity but also pressure and tension is also source of gravity. As time increases, tension of domain walls tends to negative i.e. domain walls behave like negative pressure. In such a case domain walls act like repulse force which causes accelerating and expansion of the universe. It confirms our earlier work in $f(Q)$ theory of gravitation [49].

Equation of state (EoS) parameter is given as

$$\omega = \frac{p}{\rho} = \frac{2(1-\alpha)l_6 - (2+\alpha)l_4 - (2+\alpha)l_5}{\alpha(l_4 + l_5 + 2l_6)} \left(\frac{\alpha-2}{\alpha} \right) \quad (39)$$

where, $l_4 = \frac{k}{(a_0)^2 \left(\frac{\beta}{\mu} \right)^{\frac{1}{\beta}} \left(\frac{\eta - \sqrt{\eta^2 - 4\mu\beta}}{2\beta} \right)^{\frac{\eta}{\beta\sqrt{\eta^2 - 4\mu\beta}}} \left(t^2 + \frac{\eta}{\beta}t + \frac{\mu}{\beta} \right)^{\frac{1}{\beta}} \times \left(\frac{\eta - \sqrt{\eta^2 - 4\mu\beta}}{2\beta} \right)^{-\frac{\eta}{\beta\sqrt{\eta^2 - 4\mu\beta}}}}$

$$l_5 = \left[\frac{t + \frac{\eta}{2\beta}}{\beta t^2 + \eta t + \mu} - \frac{\eta}{2\beta\sqrt{\eta^2 - 4\mu\beta}} \left[\frac{1}{t + \frac{\eta - \sqrt{\eta^2 - 4\mu\beta}}{2\beta}} - \frac{1}{t + \frac{\eta + \sqrt{\eta^2 - 4\mu\beta}}{2\beta}} \right] \right]^2$$

$$l_6 = \frac{(\beta t^2 + \eta t + \mu) - 2\beta \left(t + \frac{\eta}{2\beta} \right)^2}{(\beta t^2 + \eta t + \mu)^2} - \frac{\eta}{2\beta\sqrt{\eta^2 - 4\mu\beta}} \left[\frac{-1}{\left(t + \frac{\eta - \sqrt{\eta^2 - 4\mu\beta}}{2\beta} \right)^2} + \frac{1}{\left(t + \frac{\eta + \sqrt{\eta^2 - 4\mu\beta}}{2\beta} \right)^2} \right] + \left[\frac{t + \frac{\eta}{2\beta}}{\beta t^2 + \eta t + \mu} - \frac{\eta}{2\beta\sqrt{\eta^2 - 4\mu\beta}} \left[\frac{1}{t + \frac{\eta - \sqrt{\eta^2 - 4\mu\beta}}{2\beta}} - \frac{1}{t + \frac{\eta + \sqrt{\eta^2 - 4\mu\beta}}{2\beta}} \right] \right]^2$$

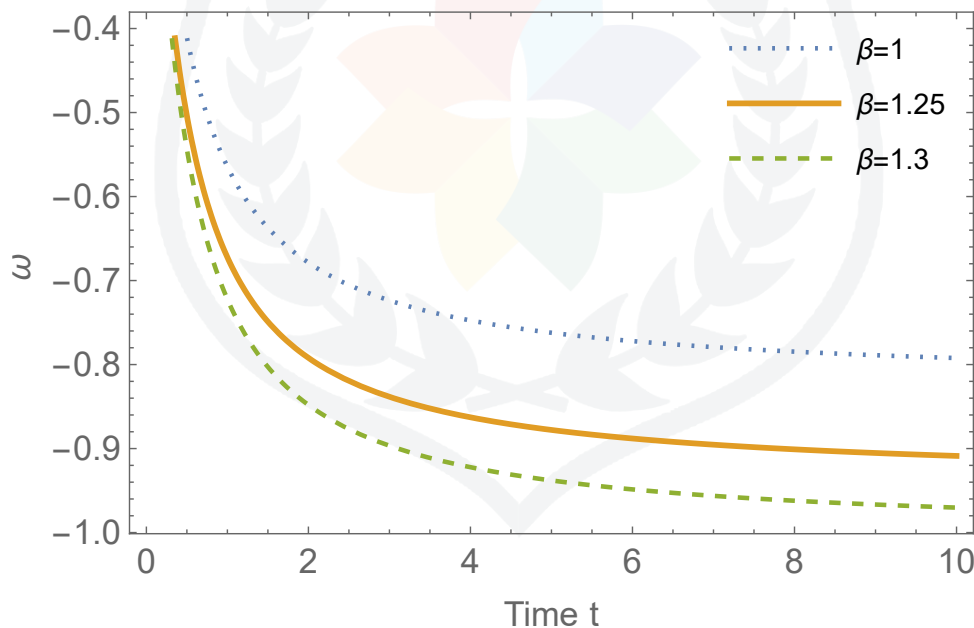


Figure 10: Plot of Equation of State parameter ω vs. cosmic time t of Model III for $\mu = 0.001, a_0 = 1, \alpha = 2.1, \eta = 0.9, \gamma = 1.5$

Figure 10 shows the variation of the equation of state (EoS) parameter ω vs. cosmic time t for different values of β . It is indicated that the EoS parameter decreases continuously with the evolution of the universe for all chosen values of β . Initially, the trajectories start from $\omega = -0.4$, which corresponds to a quintessence-type dark energy region $-1 < \omega < -\frac{1}{3}$. As cosmic time increases, the EoS parameter slowly tends toward more negative values, indicating the shift of the universe from a weak dark energy

phase to a stronger accelerated expansion phase. For $\beta = 1$ and $\beta = 1.25$, the EoS parameters remains in the quintessence region throughout the evolution. In the case $\beta = 1.3$, the trajectory evolves closer to the cosmological constant $\omega = -1$, showing behavior similar to the Λ CDM model at late times. which corresponds to accelerating universe. Thus, the graph demonstrates that the parameter β plays an important role in determining the dynamical nature of dark energy. Smaller values of β favor quintessence behavior, while larger values lead the universe toward the Λ CDM model. Overall, the negative values of the EoS parameter confirm the late-time accelerated expansion of the universe. It is important to note that domain walls behave like quintessence type dark energy.

6 Energy Condition

In general relativity and its extended theories, energy conditions are important mathematical conditions that help to explain the physical evolution of matter and the structure of spacetime. Energy conditions are mathematical state based on the energy-momentum tensor that help determine whether a gravitational theory is physically acceptable. Sahoo et al. [50] studied the significance of energy conditions in non-minimally coupled $f(R, T)$ gravity and introduced their cosmological implications. The main energy conditions considered are the null energy condition (NEC), weak energy condition (WEC), strong energy condition (SEC), and dominant energy condition (DEC), which are defined as follows:

- i) Null Energy Condition (NEC) $\iff \rho + p \geq 0$
- ii) Weak Energy Condition (WEC) $\iff \rho + p \geq 0$ and $\rho \geq 0$
- iii) Strong Energy Condition (SEC) $\iff \rho + p \geq 0$ and $\rho + 3p \geq 0$
- iv) Dominant Energy Condition (DEC) $\iff \rho \pm p \geq 0$ and $\rho \geq 0$

6.1 Energy Condition of Model I

Expression of NEC is obtained as

$$\rho + p = \left[\frac{k}{(a_0)^2 \left(\frac{\beta}{\mu}\right)^{\frac{1}{\beta}} \left(t^2 + \frac{\mu}{\beta}\right)^{\frac{1}{\beta}}} + \frac{t^2}{\beta^2 \left(t^2 + \frac{\mu}{\beta}\right)^2} + \frac{2\frac{\mu}{\beta} + 2\left(\frac{1}{\beta} - 1\right)t^2}{\beta \left(t^2 + \frac{\mu}{\beta}\right)^2} \right]^{\frac{1}{\alpha}}$$

$$+ \frac{\frac{2(1-\alpha)\frac{\mu}{\beta} + 2(1-\alpha)\left(\frac{1}{\beta} - 1\right)t^2}{\beta \left(t^2 + \frac{\mu}{\beta}\right)^2} - \frac{(2+\alpha)k}{(a_0)^2 \left(\frac{\beta}{\mu}\right)^{\frac{1}{\beta}} \left(t^2 + \frac{\mu}{\beta}\right)^{\frac{1}{\beta}}} - \frac{(2+\alpha)t^2}{\beta^2 \left(t^2 + \frac{\mu}{\beta}\right)^2}}{\alpha \left[\frac{k}{(a_0)^2 \left(\frac{\beta}{\mu}\right)^{\frac{1}{\beta}} \left(t^2 + \frac{\mu}{\beta}\right)^{\frac{1}{\beta}}} + \frac{t^2}{\beta^2 \left(t^2 + \frac{\mu}{\beta}\right)^2} + \frac{2\frac{\mu}{\beta} + 2\left(\frac{1}{\beta} - 1\right)t^2}{\beta \left(t^2 + \frac{\mu}{\beta}\right)^2} \right]^{\left(\frac{\alpha-1}{\alpha}\right)}} \quad (40)$$

Expression of WEC is as follows,

$$\rho + p = \left[\frac{k}{(a_0)^2 \left(\frac{\beta}{\mu}\right)^{\frac{1}{\beta}} \left(t^2 + \frac{\mu}{\beta}\right)^{\frac{1}{\beta}}} + \frac{t^2}{\beta^2 \left(t^2 + \frac{\mu}{\beta}\right)^2} + \frac{2\frac{\mu}{\beta} + 2\left(\frac{1}{\beta} - 1\right)t^2}{\beta \left(t^2 + \frac{\mu}{\beta}\right)^2} \right]^{\frac{1}{\alpha}}$$

$$+ \frac{\frac{2(1-\alpha)\frac{\mu}{\beta} + 2(1-\alpha)\left(\frac{1}{\beta} - 1\right)t^2}{\beta \left(t^2 + \frac{\mu}{\beta}\right)^2} - \frac{(2+\alpha)k}{(a_0)^2 \left(\frac{\beta}{\mu}\right)^{\frac{1}{\beta}} \left(t^2 + \frac{\mu}{\beta}\right)^{\frac{1}{\beta}}} - \frac{(2+\alpha)t^2}{\beta^2 \left(t^2 + \frac{\mu}{\beta}\right)^2}}{\alpha \left[\frac{k}{(a_0)^2 \left(\frac{\beta}{\mu}\right)^{\frac{1}{\beta}} \left(t^2 + \frac{\mu}{\beta}\right)^{\frac{1}{\beta}}} + \frac{t^2}{\beta^2 \left(t^2 + \frac{\mu}{\beta}\right)^2} + \frac{2\frac{\mu}{\beta} + 2\left(\frac{1}{\beta} - 1\right)t^2}{\beta \left(t^2 + \frac{\mu}{\beta}\right)^2} \right]^{\left(\frac{\alpha-1}{\alpha}\right)}} \quad (41)$$

and

$$\rho = \left[\frac{k}{(a_0)^2 \left(\frac{\beta}{\mu}\right)^{\frac{1}{\beta}} \left(t^2 + \frac{\mu}{\beta}\right)^{\frac{1}{\beta}}} + \frac{t^2}{\beta^2 \left(t^2 + \frac{\mu}{\beta}\right)^2} + \frac{2\frac{\mu}{\beta} + 2\left(\frac{1}{\beta} - 1\right)t^2}{\beta \left(t^2 + \frac{\mu}{\beta}\right)^2} \right]^{\frac{1}{\alpha}}$$

Expression of SEC is obtained as follows,

$$\rho + p = \left[\frac{k}{(a_0)^2 \left(\frac{\beta}{\mu} \right)^{\frac{1}{\beta}} \left(t^2 + \frac{\mu}{\beta} \right)^{\frac{1}{\beta}}} + \frac{t^2}{\beta^2 \left(t^2 + \frac{\mu}{\beta} \right)^2} + \frac{2\frac{\mu}{\beta} + 2\left(\frac{1}{\beta} - 1 \right) t^2}{\beta \left(t^2 + \frac{\mu}{\beta} \right)^2} \right]^{\frac{1}{\alpha}}$$

$$+ \frac{\frac{2(1-\alpha)\frac{\mu}{\beta} + 2(1-\alpha)\left(\frac{1}{\beta} - 1 \right) t^2}{\beta \left(t^2 + \frac{\mu}{\beta} \right)^2} - \frac{(2+\alpha)k}{(a_0)^2 \left(\frac{\beta}{\mu} \right)^{\frac{1}{\beta}} \left(t^2 + \frac{\mu}{\beta} \right)^{\frac{1}{\beta}}} - \frac{(2+\alpha)t^2}{\beta^2 \left(t^2 + \frac{\mu}{\beta} \right)^2}}{\alpha \left[\frac{k}{(a_0)^2 \left(\frac{\beta}{\mu} \right)^{\frac{1}{\beta}} \left(t^2 + \frac{\mu}{\beta} \right)^{\frac{1}{\beta}}} + \frac{t^2}{\beta^2 \left(t^2 + \frac{\mu}{\beta} \right)^2} + \frac{2\frac{\mu}{\beta} + 2\left(\frac{1}{\beta} - 1 \right) t^2}{\beta \left(t^2 + \frac{\mu}{\beta} \right)^2} \right]^{\left(\frac{\alpha-1}{\alpha} \right)}}$$

and

(42)

$$\rho + 3p = \left[\frac{k}{(a_0)^2 \left(\frac{\beta}{\mu} \right)^{\frac{1}{\beta}} \left(t^2 + \frac{\mu}{\beta} \right)^{\frac{1}{\beta}}} + \frac{t^2}{\beta^2 \left(t^2 + \frac{\mu}{\beta} \right)^2} + \frac{2\frac{\mu}{\beta} + 2\left(\frac{1}{\beta} - 1 \right) t^2}{\beta \left(t^2 + \frac{\mu}{\beta} \right)^2} \right]^{\frac{1}{\alpha}}$$

$$+ 3 \left[\frac{\frac{2(1-\alpha)\frac{\mu}{\beta} + 2(1-\alpha)\left(\frac{1}{\beta} - 1 \right) t^2}{\beta \left(t^2 + \frac{\mu}{\beta} \right)^2} - \frac{(2+\alpha)k}{(a_0)^2 \left(\frac{\beta}{\mu} \right)^{\frac{1}{\beta}} \left(t^2 + \frac{\mu}{\beta} \right)^{\frac{1}{\beta}}} - \frac{(2+\alpha)t^2}{\beta^2 \left(t^2 + \frac{\mu}{\beta} \right)^2}}{\alpha \left[\frac{k}{(a_0)^2 \left(\frac{\beta}{\mu} \right)^{\frac{1}{\beta}} \left(t^2 + \frac{\mu}{\beta} \right)^{\frac{1}{\beta}}} + \frac{t^2}{\beta^2 \left(t^2 + \frac{\mu}{\beta} \right)^2} + \frac{2\frac{\mu}{\beta} + 2\left(\frac{1}{\beta} - 1 \right) t^2}{\beta \left(t^2 + \frac{\mu}{\beta} \right)^2} \right]^{\left(\frac{\alpha-1}{\alpha} \right)}}$$

Expression of DEC is obtained as

$$\begin{aligned}
\rho + p &= \left[\frac{k}{(a_0)^2 \left(\frac{\beta}{\mu}\right)^{\frac{1}{\beta}} \left(t^2 + \frac{\mu}{\beta}\right)^{\frac{1}{\beta}}} + \frac{t^2}{\beta^2 \left(t^2 + \frac{\mu}{\beta}\right)^2} + \frac{2\frac{\mu}{\beta} + 2\left(\frac{1}{\beta} - 1\right)t^2}{\beta \left(t^2 + \frac{\mu}{\beta}\right)^2} \right]^{\frac{1}{\alpha}} \\
&\quad + \frac{\frac{2(1-\alpha)\frac{\mu}{\beta} + 2(1-\alpha)\left(\frac{1}{\beta} - 1\right)t^2}{\beta \left(t^2 + \frac{\mu}{\beta}\right)^2} - \frac{(2+\alpha)k}{(a_0)^2 \left(\frac{\beta}{\mu}\right)^{\frac{1}{\beta}} \left(t^2 + \frac{\mu}{\beta}\right)^{\frac{1}{\beta}}} - \frac{(2+\alpha)t^2}{\beta^2 \left(t^2 + \frac{\mu}{\beta}\right)^2}}{\left(\frac{\alpha-1}{\alpha}\right)}, \\
&\quad \alpha \left[\frac{k}{(a_0)^2 \left(\frac{\beta}{\mu}\right)^{\frac{1}{\beta}} \left(t^2 + \frac{\mu}{\beta}\right)^{\frac{1}{\beta}}} + \frac{t^2}{\beta^2 \left(t^2 + \frac{\mu}{\beta}\right)^2} + \frac{2\frac{\mu}{\beta} + 2\left(\frac{1}{\beta} - 1\right)t^2}{\beta \left(t^2 + \frac{\mu}{\beta}\right)^2} \right] \\
\rho - p &= \left[\frac{k}{(a_0)^2 \left(\frac{\beta}{\mu}\right)^{\frac{1}{\beta}} \left(t^2 + \frac{\mu}{\beta}\right)^{\frac{1}{\beta}}} + \frac{t^2}{\beta^2 \left(t^2 + \frac{\mu}{\beta}\right)^2} + \frac{2\frac{\mu}{\beta} + 2\left(\frac{1}{\beta} - 1\right)t^2}{\beta \left(t^2 + \frac{\mu}{\beta}\right)^2} \right]^{\frac{1}{\alpha}} \\
&\quad - \left[\frac{\frac{2(1-\alpha)\frac{\mu}{\beta} + 2(1-\alpha)\left(\frac{1}{\beta} - 1\right)t^2}{\beta \left(t^2 + \frac{\mu}{\beta}\right)^2} - \frac{(2+\alpha)k}{(a_0)^2 \left(\frac{\beta}{\mu}\right)^{\frac{1}{\beta}} \left(t^2 + \frac{\mu}{\beta}\right)^{\frac{1}{\beta}}} - \frac{(2+\alpha)t^2}{\beta^2 \left(t^2 + \frac{\mu}{\beta}\right)^2}}{\left(\frac{\alpha-1}{\alpha}\right)} \right] \\
&\quad \alpha \left[\frac{k}{(a_0)^2 \left(\frac{\beta}{\mu}\right)^{\frac{1}{\beta}} \left(t^2 + \frac{\mu}{\beta}\right)^{\frac{1}{\beta}}} + \frac{t^2}{\beta^2 \left(t^2 + \frac{\mu}{\beta}\right)^2} + \frac{2\frac{\mu}{\beta} + 2\left(\frac{1}{\beta} - 1\right)t^2}{\beta \left(t^2 + \frac{\mu}{\beta}\right)^2} \right] \\
&\quad \text{and} \\
\rho &= \left[\frac{k}{(a_0)^2 \left(\frac{\beta}{\mu}\right)^{\frac{1}{\beta}} \left(t^2 + \frac{\mu}{\beta}\right)^{\frac{1}{\beta}}} + \frac{t^2}{\beta^2 \left(t^2 + \frac{\mu}{\beta}\right)^2} + \frac{2\frac{\mu}{\beta} + 2\left(\frac{1}{\beta} - 1\right)t^2}{\beta \left(t^2 + \frac{\mu}{\beta}\right)^2} \right]^{\frac{1}{\alpha}}
\end{aligned} \tag{43}$$

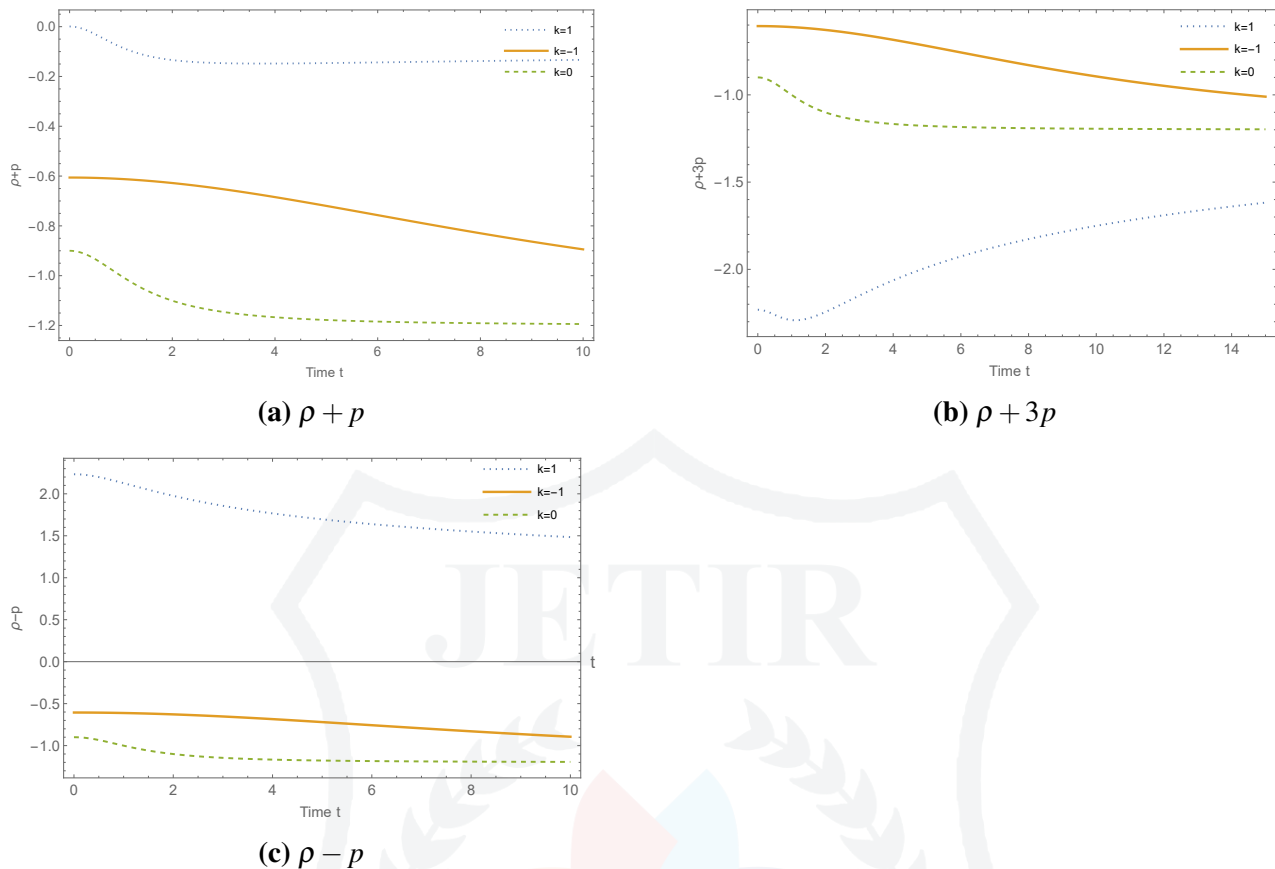


Figure 11: The graphical representation of Energy conditions of figure (a) ,(b) and (c) $\mu = 0.99$, $\beta = 1$, $a_0 = 1$, $\alpha = 10$, $\gamma = 1$,

Figure 11 shows that the evolution of the energy condition parameters with cosmic time t for different values of the curvature parameter k . In subfigure (a), NEC is negative throughout the evolution, indicating violation of the null energy condition (NEC) and supporting the accelerated expansion of the universe. In subfigure (b), SEC also remains negative, which indicates violation of the strong energy condition (SEC), which further supports the accelerating phase of cosmic evolution. In subfigure (c), the behavior of the dominant energy condition (DEC), represented by $(\rho - p)$, depends on the value of k . For $k = 1$, the parameter remains positive and satisfies the condition, whereas for $k = 0$ and $k = -1$, it remains negative, indicating violation of the condition. Overall, the results demonstrate that the cosmological behavior of the model is strongly influenced by the curvature parameter and supports accelerated cosmic expansion.

6.2 Energy Condition of Model II

Expression of NEC is obtained as

$$\rho + p = (l_1 + l_2 + l_3)^{\frac{1}{\alpha}} + \frac{(1 - \alpha)l_1 - (2 + \alpha)l_2 - (2 + \alpha)l_3}{\alpha \left(l_1 + l_2 + l_3 \right)^{\left(\frac{\alpha-1}{\alpha} \right)}} \quad (44)$$

Expression of WEC is as follows

$$\rho + p = (l_1 + l_2 + l_3)^{\frac{1}{\alpha}} + \frac{(1 - \alpha)l_1 - (2 + \alpha)l_2 - (2 + \alpha)l_3}{\alpha \left(l_1 + l_2 + l_3 \right)^{\left(\frac{\alpha-1}{\alpha} \right)}} \quad (45)$$

and

$$\rho = (l_1 + l_2 + l_3)^{\frac{1}{\alpha}}$$

Expression of SEC is obtained as

$$\rho + p = (l_1 + l_2 + l_3)^{\frac{1}{\alpha}} + \frac{(1 - \alpha)l_1 - (2 + \alpha)l_2 - (2 + \alpha)l_3}{\alpha \left(l_1 + l_2 + l_3 \right)^{\left(\frac{\alpha-1}{\alpha} \right)}} \quad (46)$$

and

$$\rho + 3p = (l_1 + l_2 + l_3)^{\frac{1}{\alpha}} + 3 \left[\frac{(1 - \alpha)l_1 - (2 + \alpha)l_2 - (2 + \alpha)l_3}{\alpha \left(l_1 + l_2 + l_3 \right)^{\left(\frac{\alpha-1}{\alpha} \right)}} \right]$$

Expression of DEC is as follows

$$\rho + p = (l_1 + l_2 + l_3)^{\frac{1}{\alpha}} + \frac{(1 - \alpha)l_1 - (2 + \alpha)l_2 - (2 + \alpha)l_3}{\alpha(l_1 + l_2 + l_3)^{\left(\frac{\alpha-1}{\alpha}\right)}},$$

$$\rho - p = (l_1 + l_2 + l_3)^{\frac{1}{\alpha}} - \left[\frac{(1 - \alpha)l_1 - (2 + \alpha)l_2 - (2 + \alpha)l_3}{\alpha(l_1 + l_2 + l_3)^{\left(\frac{\alpha-1}{\alpha}\right)}} \right] \quad (47)$$

and

$$\rho = (l_1 + l_2 + l_3)^{\frac{1}{\alpha}}$$

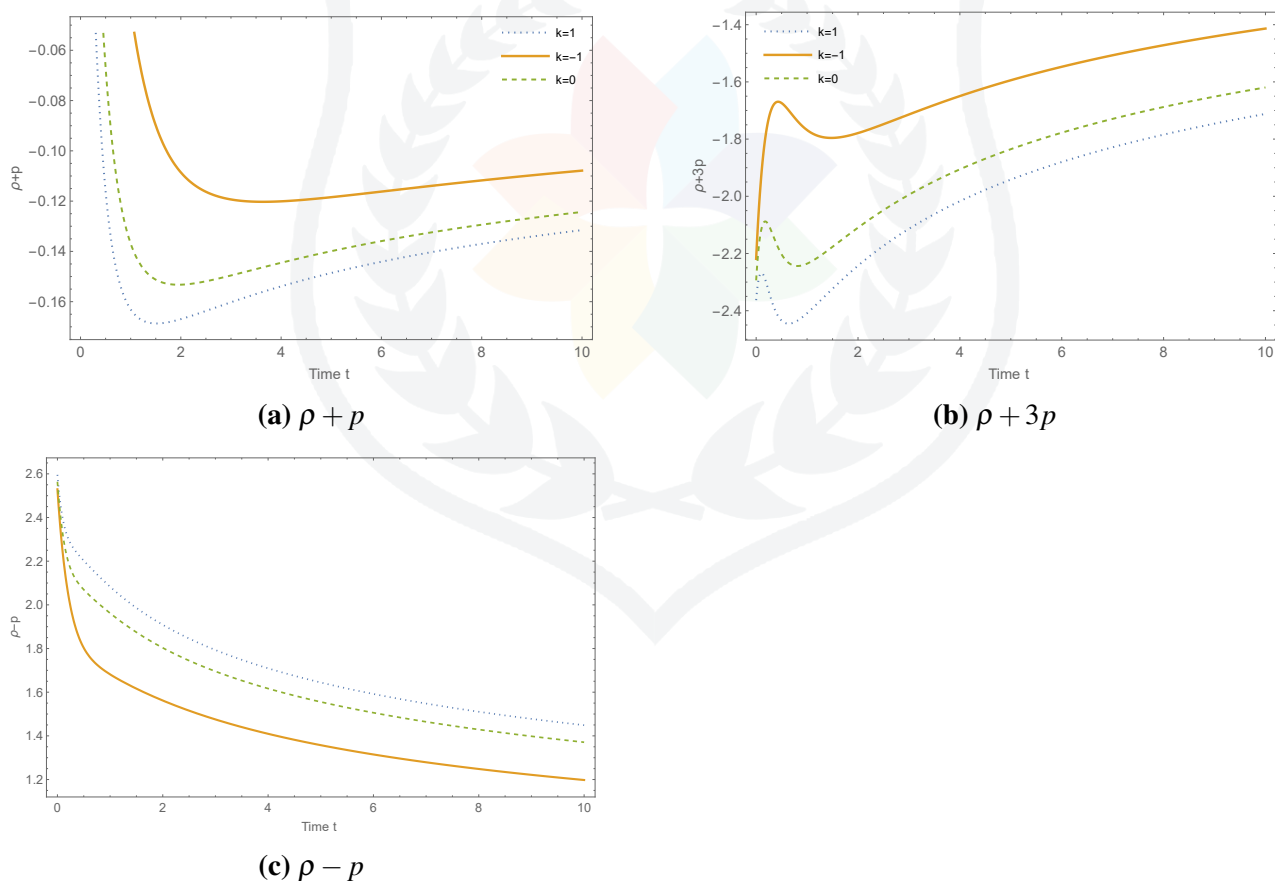


Figure 12: The graphical representation of Energy conditions of figure (a) ,(b) and (c) $\mu = 0.1$, $\beta = 1.0001$, $a_0 = 1$, $\alpha = 10$, $\gamma = 1$,

Figure 12 shows that the evolution of the energy condition parameter vs. cosmic time t . In subfigure (a)

, the NEC is negative for all parameter curvature k . It shows that the NEC is violated. In subfigure (b), SEC is negative throughout the cosmic evolution of the universe with all parameter curvature of k , which shows that SEC is violated. For accelerated expansion of the universe violation of SEC is necessary. In subfigure (c) DEC is positive and tends to zero as cosmic time increases. In this model DEC is satisfied. Chatterjee et al. [51] studied that NEC is violated implies the violation of other standard energy conditions.

6.3 Energy Condition of Model III

Expression of NEC is expressed as follows

$$\rho + p = (l_4 + l_5 + 2l_6)^{\frac{1}{\alpha}} + \frac{2(1-\alpha)l_6 - (2+\alpha)l_4 - (2+\alpha)l_5}{\alpha(l_4 + l_5 + 2l_6)^{\left(\frac{\alpha-1}{\alpha}\right)}} \quad (48)$$

WEC is obtained as follows

$$\rho + p = (l_4 + l_5 + 2l_6)^{\frac{1}{\alpha}} + \frac{2(1-\alpha)l_6 - (2+\alpha)l_4 - (2+\alpha)l_5}{\alpha(l_4 + l_5 + 2l_6)^{\left(\frac{\alpha-1}{\alpha}\right)}} \quad (49)$$

and

$$\rho = (l_4 + l_5 + 2l_6)^{\frac{1}{\alpha}}$$

Expression of SEC is as follows

$$\rho + p = (l_4 + l_5 + 2l_6)^{\frac{1}{\alpha}} + \frac{2(1-\alpha)l_6 - (2+\alpha)l_4 - (2+\alpha)l_5}{\alpha(l_4 + l_5 + 2l_6)^{\left(\frac{\alpha-1}{\alpha}\right)}} \quad (50)$$

and

$$\rho + 3p = (l_4 + l_5 + 2l_6)^{\frac{1}{\alpha}} + 3 \left[\frac{2(1-\alpha)l_6 - (2+\alpha)l_4 - (2+\alpha)l_5}{\alpha(l_4 + l_5 + 2l_6)^{\left(\frac{\alpha-1}{\alpha}\right)}} \right]$$

Expression of DEC is obtained as

$$\rho + p = (l_4 + l_5 + 2l_6)^{\frac{1}{\alpha}} + \frac{2(1-\alpha)l_6 - (2+\alpha)l_4 - (2+\alpha)l_5}{\alpha(l_4 + l_5 + 2l_6)^{\left(\frac{\alpha-1}{\alpha}\right)}},$$

$$\rho - p = (l_4 + l_5 + 2l_6)^{\frac{1}{\alpha}} - \left[\frac{2(1-\alpha)l_6 - (2+\alpha)l_4 - (2+\alpha)l_5}{\alpha(l_4 + l_5 + 2l_6)^{\left(\frac{\alpha-1}{\alpha}\right)}} \right] \quad (51)$$

and

$$\rho = (l_4 + l_5 + 2l_6)^{\frac{1}{\alpha}}$$

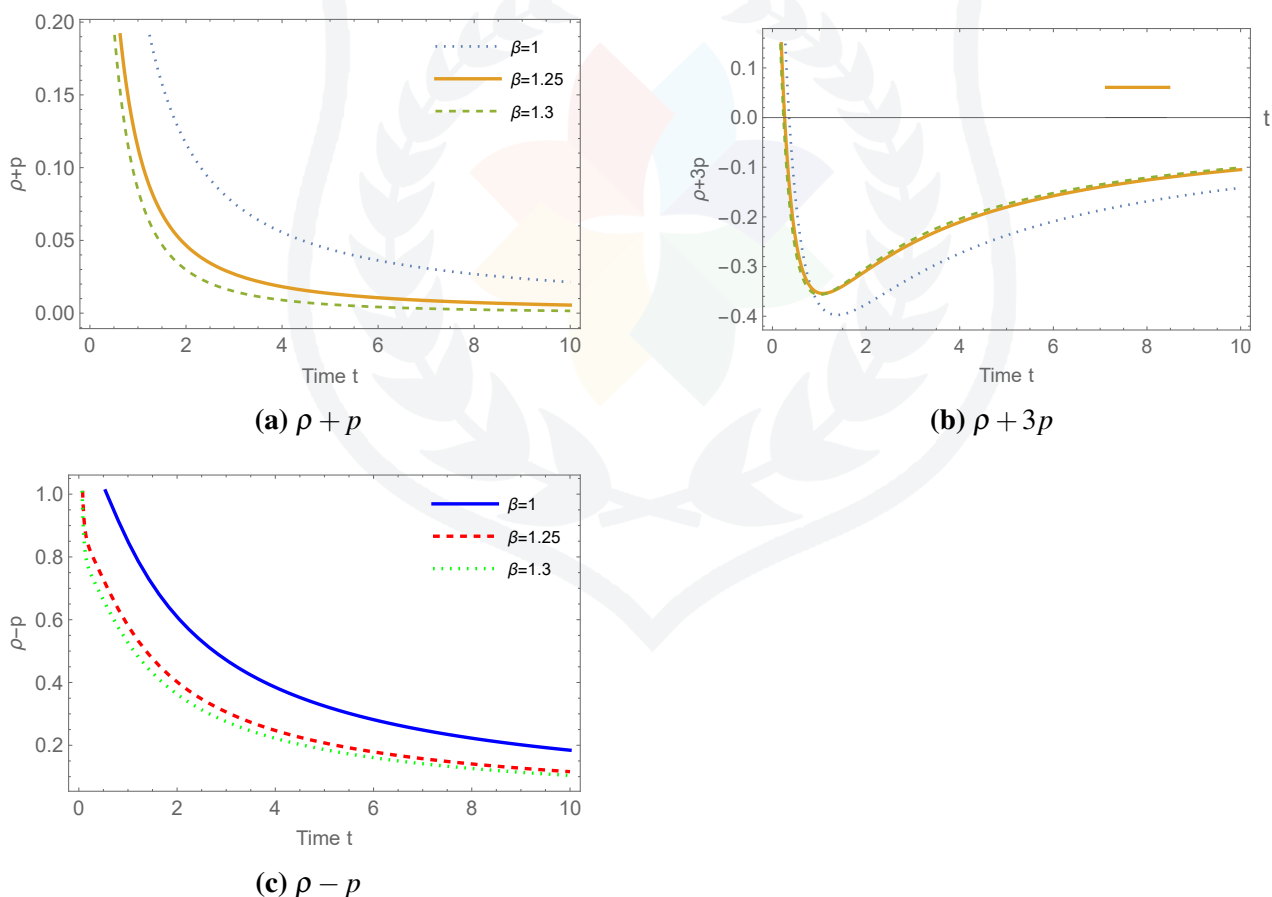


Figure 13: The graphical representation of Energy conditions of figure (a) ,(b) and (c) $\mu = 0.001$, $a_0 = 1$, $\alpha = 2.1$, $\eta = 0.9$, $\gamma = 1.5$,

Figure 13, shows that the evolution of energy condition vs. cosmic time t In subfigure (a) , NEC is

positive for all curvature parameter k , as cosmic time increases NEC is decreases and tends to zero. This shows that NEC is satisfied by this model. In subfigure (b), as cosmic time increases SEC is decreases and in late time of the universe SEC is lies in the negative region which shows that SEC is violated at late time. Violation of SEC is very important for acceleration of the universes. In subfigure (c), DEC is positive throughout the cosmic evolution of the universe which indicates that DEC is satisfied by this model. Sahoo et al.[50] studied the NEC and DEC along with the violation of SEC suggested an accelerated expansion the universe.

7 Conclusion

In this work, we investigated domain wall with the deceleration parameter in the context of $f(R, L_m)$ theory. To solve field equation we have considered three different models of deceleration parameter and $f(R, L_m)$ theory. Deceleration parameter is negative throughout the cosmic evolution of universe that shows the universe is an accelerating and expansion.

Further we have the following observations :

- In model I energy density and tension of domain wall is positive for all parameter of k during cosmic evolution of the universe. The EoS parameter is negative throughout cosmic evolution of the universe. For parameter $k = -1$ the trajectory of EoS lies in the range $-0.6 < \omega < -0.9$, which shows that model is dominated by quintessence dark energy, while for $k = 1$ and $k = 0$ the trajectory of EoS parameter lies in the region $\omega < -1$ at late time of the universe, which shows that model is phantom dominated. While NEC and SEC is violated for different values of k . For $k = 1$, DEC is positive and for $k = -1$ and $k = 0$, DEC is negative which shows that DEC is satisfied for parameter $k = 1$ and it is violated for $k = -1$ and $k = 0$.
- In model II, energy density and tension of domain wall is positive for all curvature parameter k during cosmic evolution of universe. EoS parameter is negative for different curvature parameter of k . Trajectory of EoS lies in the region $\omega < -1$, which indicates that EoS is dominated by phantom like dark energy. In energy condition NEC and SEC is negative and DEC is positive during evolution of cosmic universe. This shows that NEC and SEC is violated and DEC is satisfied for different curvature parameter of k .
- In model III, energy density is positive in the range of cosmic time for different values of β . At early time of the universe, tension of the domain wall is positive and at late time of the universe it is negative. At early time of the universe, trajectory of EoS lies in the region $-1 < \omega < -\frac{1}{3}$, i.e. the universe is quintessence type dark energy. At late time of the universe, for $\beta = 1$ and $\beta = 1.25$ the EoS parameter lies in quintessence region, and for $\beta = 1.3$ the trajectory of the EoS lies near to cosmological constant. NEC and DEC is positive for different values of β . It shows that NEC and DEC is satisfied by this model, while at early time of the universe SEC is positive and at late time of the universe it is negative, which indicates that SEC is violated at late time of the universe. Violation of SEC is necessary for acceleration of the universe.

- Novelty of the present work is that, domain walls behave like either quintessence type or phantom type dark energy.

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