Common Fixed Point's Theorem of Compatible Maps in Fuzzy Metric Space Using General Contractive Condition of Integral Type

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Abstract : **The aim of this paper is to obtain two common fixed points theorem of compatible maps in fuzzy metric space using the general contractive condition of integral type. Our main result improves and extends several known results.**

Key words: Fuzzy set, compatible mappings, fuzzy metric space, common fixed point, integral type.

1. INTRODUCTION:

The concept of fuzzy sets was initially investigated by Zadeh [5] as a new way to represent vagueness in everyday life. Subsequently, it was developed extensively by many authors of used in various fields. To use this concept in topology and analysis, several researchers have defired fuzzy metric space in various ways. In this paper we deal with the fuzzy metric space defined by Kramosil and Michalek [4] and modified by George and Veerarnani [1].

Jungck [2, 3] gave the more generalized concept compatibility than commutativity and weak commutativity in metric space and proved common fixed point theorems. Branciari [6] obtained a fixed point theorem for a single mapping satisfying an analogous of Banach's contraction principle for an integral type inequality. In this paper we introduce the concept of compatibility in fuzzy metric space and use in to prove common fixed point theorems for four compatible mappings using general contractive condition of integral type.

2. PRELIMINARIES:

Throughout this paper we use all symbols and basic definitions of George and Veeramani [1].

Definition 2.1: The 3-tuple (X, M, *) is said to be a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions:

(2.1.1) $M(x, y, t) > 0$,

(2.1.2) $M(x, y, t) = 1$ if and only if $x = y$,

(2.1.3) $M(x, y, t) = M(y, x, t)$,

(2.1.4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(2.1.5) $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous,

$x, y, z \in X$ and $t, s > 0$.

Definition 2.2: A sequence $\{x_n\}$ in a fuzzy metric space (X, M, *) is a Cauchy sequence if and only for each $\varepsilon > 0$, $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.

Definition 2.3: A sequence $\{x_n\}$ in a fuzzy metric space (X, M, *) is a said to converge to x if and only if for each $\varepsilon > 0$, $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x, t) > 1 - \varepsilon$ for all $n \geq n_0$.

Definition 2.4: Self mappings F and G of a fuzzy metric space (X, M, *) are said to be compatible if and only if $M(FGx_n, GFx_n, t) \to 1$, for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Fx_n, Gx_n \to y$ for some y in X.

B. Singh, M. S. Chauhan [7] establish theorems (2.1) and (2.2).

Theorem 2.1: Let A, B, S and T be self maps of a complete fuzzy metric space (X, M, *) with continuous t-norm * defined by $a*b = \min\{a, b\}$, $a, b \in [0, 1]$, satisfying the following conditions:

(2.1.1) $A(X) \subset T(X)$, $B(X) \subset S(X)$ (2.1.2) S and T are continuous,

(2.1.3) (A, S), (B, T) are compatible pairs of maps,

(2.1.4) for all x, y in X, $k \in (0,1)$, $t > 0$

$$M(Ax, By, kt) \geq \min\{M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(By, Sx, 2t), M(Ax, Ty, t)\},$$

(2.1.5) For all x, y in X, $\lim M(x, y, t) \to 1$, as $t \to \infty$

Then A, B, S and T have a unique common fixed point in X.

Theorem 2.2: Let A, B, S and T be self maps of complete fuzzy metric space (X, M, *) with t-norm defined by $a*b = \min\{a, b\}$, a, $b \in [0, 1]$, satisfying conditions (2.1.1) and (2.1.5) of Theorem (2.1) and

(2.2.1) AS = SA, TB = BT

(2.2.2) $A^a(X) \subset T^t(X)$, $B^b \subset S^s(X)$

(2.2.3) $M((A^a x, B^b x, kt) \geq \min\{M(S^s x, T^t y, t), M(A^a x, S^s x, t), M(B^b y, T^t y, t), M(B^b y, S^s x, 2t), M(A^a x, T^t y, t)\}$

for all x, y in X, a, b, s, $t \in N$

Then A, B, S, and T have a unique common fixed point in X.

3. MAIN RESULTS:

Theorem 3.1: Let A, B, S and T be self maps of a complete fuzzy metric space (X, M, *) with continuous t-norm * defined by $a*b = \min\{a, b\}$, a, $b \in [0, 1]$, satisfying the following conditions:

$A(X) \subset T(X)$, $B(X) \subset S(X)$ (3.1.1)

S and T are continuous (3.1.2)

(A, S), (B, T) are compatible pairs of maps, (3.1.3)

for all x, y in X, $k \in (0,1)$, $t > 0$

$$\int_0^{M(Ax, By, Kt)} \phi(t)\,dt \geq \left(\int_0^{m(x, y, t)} \phi(t)\,dt\right) \quad (3.1.4)$$

where $\phi : R^+ \to R^+$ is a legesque-integrable mapping which is summable, nonnegative, and such that

$$\int_0^{\varepsilon} \phi(t)\,dt > 0 \text{ for each } \varepsilon > 0,$$

where $m(x, y, t) = \min\{M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(By, Sx, 2t), M(Ax, Ty, t)\}$,

for all x, y in X, $\lim M(x, y, t) \to 1$, as $t \to \infty$ (3.1.5)

Then A, B, S and T have a unique common fixed point in X.

Proof: Let x_0 be an arbitrary point in X. we construct a sequence $\{y_n\}$ in X such that

$$y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}$$

$$y_{2n} = Sx_{2n} = Bx_{2n-1},\ n = 1, 2, 3, \ldots..$$

from (3.1.4) we have,

$$\int_0^{M(y_{2n+1}, y_{2n+2}, kt)} \phi(t)\, dt = \int_0^{M(Ax_{2n}, Bx_{2n+1}, kt)} \phi(t)\, dt \geq \int_0^{m(x_{2n}, x_{2n+1}, t)} \phi(t)\, dt .$$

$m(x_{2n}, x_{2n+1}, t) = \min\{M(Sx_{2n}, Ty_{2n+1}, t), M(Ax_{2n}, Sx_{2n}, t), M(Bx_{2n+1}, Tx_{2n+1}, t),$

$M(Bx_{2n+1}, Sx_{2n}, 2t), M(Ax_{2n}, Tx_{2n+1}, t)\}.$

$= \min\{M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n}, t), M(y_{2n+2}, y_{2n+1}, t),$

$M(y_{2n+2}, y_{2n}, 2t), M(y_{2n+1}, y_{2n+1}, t)\}.$

$\geq \min\{M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n}, t), M(y_{2n+2}, y_{2n+1}, t),$

$M(y_{2n+2}, y_{2n+1}, t), M(y_{2n+1}, y_{2n}, t), M(y_{2n+1}, y_{2n+1}, t)\}.$

$\geq \min\{M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, t), 1\}$

$\geq M(y_{2n}, y_{2n+1}, t)$

which implies

$$\int_0^{M(y_{2n+1}, y_{2n+2}, kt)} \phi(t)\, dt \geq \int_0^{M(y_{2n}, y_{2n+1}, t)} \phi(t)\, dt$$

In general

$$\int_0^{M(y_n, y_{n+1}, kt)} \phi(t)\, dt \geq \int_0^{M(y_{n-1}, y_n, t)} \phi(t)\, dt \qquad (3.1.6)$$

To prove that $\{y_n\}$ is a Cauchy sequence, we prove (3.1.7) is true for all $n \geq n_0$ and for every $m \in N$,

$$\int_0^{M(y_n, y_{n+m}, t)} \phi(t)\, dt > 1 - \lambda. \qquad (3.1.7)$$

From (3.1.6), we have, for every positive integer n,

$$\int_0^{M(y_n, y_{n+1}, kt)} \phi(t)\, dt \geq \int_0^{M(y_{n-1}, y_n, t/k)} \phi(t)\, dt$$

$$\geq \int_0^{M(y_{n-2}, y_{n-1}, t/k^2)} \phi(t)\, dt \geq .....$$

$$\geq \int_0^{M(y_0, y_1, t/k^n)} \phi(t)\, dt \to 1, \text{ as } n \to \infty$$

i.e. for $t > 0$, $\lambda \in (0, 1)$, we can choose $n_o \in N$, such that

$$\int_0^{M(y_n, y_{n+1}, t)} \phi(t)\, dt > 1 \square\square \lambda \qquad (3.1.8)$$

Thus (3.1.7) is true for m = 1. Suppose (3.1.7) is true for m then we shall show that it is also true for m + 1.

Using the definition of fuzzy metric space, (3.1.6) and (3.1.7), we have

$$\int_0^{M(y_n, y_{n+m+1}, t)} \phi(t)\,dt \geq \int_0^{\min\{M(y_n, y_{n+m}, t/2), M(y_{n+m}, y_{n+m+1}, t/2)} \phi(t)\,dt$$

$$> 1 - \lambda$$

Hence (3.1.7) is true for m+1. Thus $\{y_n\}$ is a Cauchy sequence. By completeness of (X, M, *), $\{y_n\}$ converges to some point z in X. Thus $\{Ax_{2n}\}$, $\{Sx_{2n}\}$, $\{Bx_{2n-1}\}$ and $\{Tx_{2n-1}\}$ also converges to z. Now $Ax_{2n} \to z$ and S is continuous hence $SAx_{2n} \to Sz$. Thus for tz_o, $\lambda \in (0, 1)$, there exists an $n_0 \in N$ such that

$$\int_0^{M(SAx_{2n}, Sz, t/2)} \phi(t)\,dt > 1 - \lambda, \text{ for all } n \geq n_0.$$

Using (3.1.3), we have

$$\int_0^{M(ASx_{2n}, SAx_{2n}, t/2)} \phi(t)\,dt \to 1.$$

$$\int_0^{M(ASx_{2n}, S_z, t)} \phi(t)\,dt \geq \int_0^{\min\{M(ASx_{2n}, SAx_{2n}, t/2), M(SAx_{2n}, S_z, t/2)\}} \phi(t)\,dt$$

$$> 1 - \lambda, \text{ for all } n \geq n_0$$

Hence $ASx_{2n} \to Sz$ (3.1.9)

Similarly, $BTx_{2n-1} \to Tz$ (3.1.10)

Using (3.1.4), we have

$$\int_0^{M(ASx_{2n}, BTx_{2n-1}, kt)} \phi(t)\,dt \geq \int_0^{\substack{\min\{M(S^2x_{2n}, T^2x_{2n-1}, t), M(ASx_{2n}, S^2x_{2n}, t), M(BTx_{2n-1}, T^2x_{2n-1}, t), \\ M(BTx_{2n-1}, S^2x_{2n}, 2t), M(ASx_{2n}, T^2x_{2n-1}, t)\}}} \phi(t)\,dt$$

Taking limit as $n \to \infty$ and using (3.1.9) and (3.1.10), we get

$$\int_0^{M(Sz, Tz, kt)} \phi(t)\,dt \geq \int_0^{M(Sz, Tz, t)} \phi(t)\,dt,$$

which implies $Sz = Tz$ (3.1.11)

Now
$$\int_0^{M(Ay, BTx_{2n-1}, kt)} \phi(t)\,dt \geq \int_0^{\substack{\min\{M(Sy, T^2x_{2n-1}, t), M(Ay, Sy, t), M(BTx_{2n-1}, T^2x_{2n-1}, t), \\ M(BTx_{2n-1}, Sy, 2t), M(Ay, T^2x_{2n-1}, t)\}}} \phi(t)\,dt$$

Taking the limit as $n \to \infty$ and using (3.1.9) (3.1.11),

we get $Az = Tz$ (3.1.12)

Now using (3.1.11) and (3.1.12)

$$\int_0^{M(Az, BTz, kt)} \phi(t)\,dt \geq \int_0^{\substack{\min\{M(Sz, Tz, t), M(Az, Sz, t), M(Bz, Tz, t), \\ M(Bz, Sz, 2t), M(Az, Tz, t)\}}} \phi(t)\,dt$$

$$= \int_0^{\min\{M(Tz, Tz, t), M(Az, Az, t), M(Az, Bz, t), M(Az, Bz, 2t), M(Az, Az, t)\}} \phi(t)\,dt$$

$$\geq \int_0^{\min\{M(Az, Bz, t)} \phi(t)\,dt.$$

which implies $Az = Bz$ (3.1.13)

Using (3.1.11) – (3.1.13)

$Az = Bz = Sz = Tz$ (3.1.14)

Now $$\int_0^{M(Ax_{2n}, Bz, kt)} \phi(t)\,dt \geq \int_0^{\min\{M(Sx_{2n}, Tz, t), M(Ax_{2n}, Sx_{2n}, t), M(Bz, Tz, t), M(Bz, Sx_{2n}, 2t), M(Ax_{2n}, Tz, t)\}} \phi(t)\,dt$$

Taking limit and using (3.1.14), we get

$z = Bz$ (3.1.15)

Thus z is a common fixed point of A, B, S and T.

Uniqueness: Let w be another common fixed point of A, B, S and T. Then we have

$$\int_0^{M(Az, Bw, kt)} \phi(t)\,dt \geq \int_0^{\min\{M(Sz, Tw, t), M(Az, Sz, t), M(Bw, Tw, t), M(Bw, Sw, 2t), M(Az, Tw, t)\}} \phi(t)\,dt$$

i.e. $$\int_0^{M(z, w, kt)} \phi(t)\,dt \geq \int_0^{M(z, w, kt)} \phi(t)\,dt$$

Hence $z = w$.

This completes the proof.

Theorem 3.2: Let A, B, S and T be self maps of complete fuzzy metric space (X, M, *) with t-norm defined by a*b = min{a, b}, a, b∈ [0, 1], satisfying conditions (3.1.1) and (3.1.5) of Theorem (3.1) and

$AS = SA, TB = BT$ (3.2.1)

$A^a(X) \subset T^t(X), B^b \subset S^s(X)$ (3.2.2)

$$\int_0^{M(A^a x, B^b y, kt)} \phi(t)\,dt \geq \int_0^{m(x, y, t)} \phi(t)\,dt \quad (3.2.3)$$

where $m(x, y, t) = \min\{M(S^s x, T^t y, t), M(A^a x, S^s x, t), M(B^b y, T^t y, t),$

$M(B^b y, S^s x, 2t), M(A^a x, T^t y, t)\}$

for all x, y in X, a, b, s, t ∈ N

Then A, B, S, and T have a unique common fixed point in X.

Proof: Since A and B commute with S and T so A^a and B^b also commute with S^s and T^t, respectively. Also commutativity implies compatibility; hence by Theorem 3.1, A^a, B^b, S^s and T^t have unique common fixed point say z i.e.

$z = A^a z = B^b z = S^s z = T^t z.$

Now $Az = A(A^a z) = A^a(Az)$ and $Az = A(S^s z) = S^s(Sz)$.

Hence Az is a common fixed point of A^a and S^s. Similarly it can be shown that Bz is also a common fixed point of B^b and T^t.

$$\int_0^{M(Az,Bz,kt)} \phi(t)\,dt = \int_0^{M(A^a(Az),B^b(Bz),kt)\}} \phi(t)\,dt$$

$$\geq \int_0^{\min\{M(S^s(Az),T^t(Bz),t),M(A^a(Az),S^s(Az),t),M(B^b(Bz),T^t(Bz),t),M(B^b(Bz),S^s(Az),2t),M(A^a(Az),T^t(Bz),t)\}} \phi(t)\,dt$$

$$\geq \int_0^{\min\{M(Az,Bz,t),M(Az,Az,t),M(Bz,Bz,t),M(Bz,Az,2t),M(Az,Bz,t)\}} \phi(t)\,dt$$

$$\geq \int_0^{M(Az,Bz,t)} \phi(t)\,dt$$

which implies that Az = Bz.
Similarly Sz = Tz
Since z is the unique common fixed point of A^a, B^b, S^s, T^t and Az(=Bz), Sz(=Tz) are common fixed points of A^a, S^s and B^b, T^t, respectively.
Hence z = Az (=Bz) = Sz (=Tz).
This complete the proof.

Example 3.3: Let (X, M, *) be a fuzzy metric space with X = [0, 1], t-norm*defined by a*b = min(a, b), a, b $\in$ [0, 1] and M is the fuzzy set on $X^2 \times (0, \infty)$, defined by

$$\int_0^{M(x,y,t)} \phi(t)\,dt = \int_0^{[\exp(x-y/t)]^{-1}} \phi(t)\,dt, \text{ for all } x, y \in X, t > 0$$

let us define self maps A, B, S, T of X such that $Ax = \frac{x}{16}$, $Bx = \frac{x}{8}$, $Tx = \frac{x}{2}$, $Sx = \frac{x}{4}$.

Then for $k \in [\frac{1}{4}, 1)$,

$$\int_0^{M(Ax,By,t)} \phi(t)\,dt = \int_0^{[\exp(\frac{x}{16}-\frac{y}{8}/kt)]^{-1}} \phi(t)\,dt$$

$$\geq \int_0^{[\exp(\frac{x}{4}-\frac{y}{2}/t)]^{-1}} \phi(t)\,dt$$

$$= \int_0^{M(Sx,Ty,t)} \phi(t)\,dt$$

$$\geq \int_0^{\text{Min}\{M(Sx,Ty,t),M(Ax,Sx,t),M(By,Ty,t)\,M(By,Sx,2t),M(Ax,Ty,t)\}} \phi(t)\,dt.$$

Here the conditions (3.1.1), (3.1.2), (3.1.3), (3.1.5) of Theorem (3.1) are also satisfied and zero is the unique common fixed point of A, B, S and T.

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