Two Fixed Point Theorems in Complete and Compact Metric Spaces

[1]Pardeep Kumar Rana, [2]Sunil Kumar,
[1]Department of Mathematics, Moradabad Institute of Technology, Moradabad, India
[2]Research Scholar, Department of Mathematics, Gurukula Kangari Vishwavidyalaya, Haridwar, India

Abstract : The main object of this paper is to establish common fixed point theorems for eight mappings on a complete and compact metric space. Our result generalizes several previously known results due to Lal et. al. [1], Jungck [4], Sunil Kumar[19] and M. Imdad [15] and others.

Keywords: Fixed point, complete metric space, compact metric space, commuting maps, self mappings.

1. INTRODUCTION

Let R^+ denote the set of nonnegative reals and let ψ be the family of mappings ϕ from $(R^+)^5$ in to R^+ such that

(i) ϕ is non decreasing ,

(ii) ϕ is upper semi – continuous in each co – ordinate variable,

(iii) $\gamma(t) = \phi(t, t, a_1 t, a_2 t, t) < t$, where
$\gamma : R^+ \to R^+$ is mapping with $\gamma_{(O)} = 0$
and $a_1 + a_2 = 2$

Theorem 3.2 of Lal et. al. [1] for commuting mappings can be stated as follows.

Theorem 1.1: Let A, S, I and J be self mappings of a complete metric space (X, d) such that the pairs (A, I) and (S, J) are commuting and $A(X) \subset J(X)$ and $S(X) \subset I(X)$ such that

$$\begin{aligned}[1+ pd(Ax, Sy)]\, d(Ix, Jy) \le\ & p \max\{d(Ix, Ax) . d(Sy, Jy),\ d(Ix, Sy). d(Jy, Ax)\} \\ & + \phi(d(Ax, Sy), d(Ix, Ax), d(Sy, Jy), d(Ix, Sy), d(Jy, Ax)), \qquad (1)\end{aligned}$$

Remark 1.2: Theorem 1.1 was originally proved for "weakly compatible mappings of type (A)" (cf. [1]) but for a more natural setting we have adopted it for commuting mappings. M. Imdad generalized and extend theorem 1.1 for six mappings see [15].

THEOREM 1.3: Let A, B, S, T, I and J be self mappings of a complete metric space (X, d) such that the pairs (A, B) , (A, I) , (B, I) , (S, T) , (S, J) , and (T, J) are commuting $AB(X) \subset J(X)$, $ST(X) \subset I(X)$ satisfying the inequality

$$\begin{aligned}[1+ pd(ABx, STy)]\, d(Ix, Jy) \le\ & p \max\{d(Ix, ABx) .d(STy, Jy),\ d(Ix, STy) .d(Jy, ABx)\} \\ & + \phi(d(ABx, STy), d(Ix, ABx), d(STy, Jy), d(Ix, STy), d(Jy, ABx)) \qquad (2)\end{aligned}$$

for all $x, y \in X$ where $p \ge 0$ and $\phi \in \psi$. Then A, B, S, T, I and J have unique common fixed point provided one of these four mappings AB, ST, I and J is continuous.

Next we wise to indicate a similar result in compact metric spaces. For this purpose one can adopt a general fixed point theorem for commuting mappings in compact metric spaces due to Jungck [4], which was originally proved for compatible mappings (a notation due to Jungck [2]).

THEOREM 1.4 [4]: Let A, S, I, and J be self mappings of a compact metric space (X, d) with $A(X) \subset J(X)$ and $S(X) \subset I(X)$, if the pairs (A, I) and (S, J) are commuting and

$$d(Ax, Sy) < M(x, y) \qquad (3)$$

for all $x, y \in X$ where

$M(x, y) = \max\{d(Ix, Jy), d(Ix, Ax), d(Jy, Sy),$

$$\frac{1}{2}[d(Ix, Sy) + d(Jy, Ax)]\} \quad (4)$$

with $M(x, y) > 0$, then A, S, I and J have unique common fixed point provided all four mappings A, S, I, and J are continuous.

As an application M. Imdad derive the theorem 1.3 compact metric spaces involving six mappings (see [15]).

THEOREM 1.5: Let A, B, S, T, I and J be self-mappings of a compact metric space (X, d) with $AB(X) \subset J(X)$ and $ST(X) \subset I(X)$, if the pairs (A, B) (A, I), (B, I), (S, T), (S, J) and (T, J) are commuting and

$$d(ABx, STy) < M(x, y) \quad (5)$$

for all $x, y \in X$ where

$$M(x, y) = \max\{d(Ix, Jy), d(Ix, ABx), d(Jy, STy), \frac{1}{2}[d(Ix, STy) + d(Jy, ABx)]\} \quad (6)$$

With $M(x, y) > 0$, then A, B, S, T, I and J have a unique common fixed point provided all four mappings AB, ST, I and J are continuous.

2. MAIN RESULT

Now we give our main theorems.

Theorem 2.1: Let A, B, C, R, S, T, I and J be self mapping of a complete metric space (X, d), such that the pairs, (A, B), (B, C), (A, C), (A, I), (B, I), (C,I), (R,S), (S,T), (R, T), (R, J), (S, J) and (T, J) are commuting and $ABC(X) \subset J(X)$, $RST(X) \subset I(X)$ satisfying the inequality

$$\begin{aligned}[1 + pd(ABCx, RSTy)]\, d(Ix, Jy) \le p \max\{&d(Ix, ABCx) .d(RSTy, Jy),\\ &d(Ix,RSTy) .d(Jy, ABCx)\}\\ + \phi\, (&d(ABCx,RSTy), d(Ix, ABCx),\\ &d(RSTy, Jy), d(Ix,RSTy), d(Jy, ABCx))\end{aligned}$$

for all $x, y \in X$ where $p \ge 0$ and $\phi \in \psi$. Then A, B, C, R, S, T, I and J have a unique common fixed point provided one of these four mapping ABC, RST, I and J is continuous.

PROOF: We begin by observing that of ABC (resp. RST) does not demand the continuities of the component maps A or B or C or all of them (resp. R or S or T or all of them).

Since the pairs (A, B), (B, C), (A, C), (A, I), (B, I), (C, I), (R, S), (S, T), (R, T), (R, J), (S, J) and (T, J) are commuting which force the pairs (ABC, I) and (RST, J) to be commuting. After observing this we note that all the conditions of theorem (1.3) for four mappings ABC, RST, I and J are satisfied , hence (in view of theorem1.3) ABC, RST, I and J have a unique common fixed point z.

Here one can note that z also remains the unique common fixed point of the pairs (ABC, I) and (RST, J) separately.

Now it remains to show that z is also a common fixed point of A, B, C, R, S, T, I, and J, For this let z be the unique common fixed point of the pair (ABC, I), then

$$Az = A(ABCz) = A(BACz) = A(BCAz) = ABC(Az)$$
$$Bz = B(ABCz) = B(ACBz) = BAC(Bz) = ABC(Bz)$$
$$Cz = C(ABCz) = CAB(Cz) = ACB(Cz) = ABC(Cz)$$
$$Az = A(Iz) = I(Az)$$
$$Bz = B(Iz) = I(Bz)$$
$$Cz = C(Iz) = I(Cz)$$

which shows that Az, Bz and Cz are other fixed points of the pair (ABC,I) yielding there by

$$Az = Bz = Cz = Iz = ABCz = z$$

in view of the uniqueness of the common fixed point of the pair (ABC, I)

Again, let z is a common fixed point of the pair (RST, J) , then

$$Rz = R(RSTz) = R(SRTz) = R(STRz) = RST(Rz)$$

$$Sz = S(RSTz) = S(RTSz) = SRT(Sz) = RST(Sz)$$
$$Tz = T(RSTz) = TRS(Tz) = RTS(Tz) = RST(Tz)$$
$$Rz = R(Jz) = J(Rz)$$
$$Sz = S(Jz) = J(Sz)$$
$$Tz = T(Jz) = J(Tz)$$

which shows that Rz, Sz and Tz are other fixed points of the pair (RST,J) .

This shows that z is a unique common fixed point of R, S, T, RST and J.

This is complete proof.

As an application of Theorem 1.5 one can derive the following theorem in compact metric spaces involving eight mapping.

Theorem 2.2: Let A, B, C, R, S, T, I and J be self mappings of a compact metric space (X, d) with $ABC(X) \subset J(X)$ and $RST(X) \subset I(X)$. If the pairs (A, B), (B, C), (A, C), (A, I), (B, I), (C, I), (R, S), (S, T), (R, T), (R, J), (S, J) and (T, J) are commuting and

$$d(ABCx, RSTy) < M(x, y)$$

for all $x, y \in X$ where

$$M(x\ y) = \max\{d(Ix, Jy), (Ix, ABCx), d(Jy, RSTy), \frac{1}{2}[d(Ix, RSTy) + d(Jy, ABCx)]\}$$

with $M(x, y) > 0$, then A, B, C, R, S, T, I and J have a unique common fixed point provided all four mappings ABC, RST, I and J are continuous .

PROOF: The proof is essentially the same as that of theorem (1.5), hence we omit the proof.

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