DISCRETE DYNAMICAL SYSTEM – CELLULAR AUTOMATA AND ITS APPLICATIONS.

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ABSTRACT

The world of mathematics has been confined to the linear world. That is to say, mathematicians and physicists have overlooked dynamical systems as random and unpredictable. However, the problem arises that we humans do not live in an even remotely linear world; in fact our world should indeed be categorized as non-linear.

This dissertation has been designed to be a descriptive version of non-linear dynamical system through Cellular Automata and making it speculative and thought provoking. A brief view of some of the basic properties of Cellular Automata (CA) and its application is explained in the presented paper.

Keywords: Discrete Dynamical system, Cellular Automata, Additive Cellular Automata, Graphical representation.

I. INTRODUCTION

1.1DYNAMICAL SYSTEM

In mathematics, a dynamical system is a system in which a function describes the dependence of a point in a geometrical shape. It is a system which changes from time to time. Examples include the mathematical models that describe the swinging of a clock pendulum, the flow of water in a pipe, and the number fish each springtime in a lake.

This dynamical system is classified into two types, such as

1. Discrete dynamical system
2. Continuous dynamical system.

A discrete dynamical system is a dynamical system whose state evolves over state space in discrete time steps according to a fixed rule. In continuous dynamical system, quantity changes value over continuous time interval.

We are going to discuss discrete dynamical system in detail.

1.2DISCRETE DYNAMICAL SYSTEM.

A dynamical system is called to be a discrete if time is measured in discrete steps. To be more precise, a sequence of numbers that are defined recursively, that is, there is a rule relating each number in the sequence to the previous in the sequence. *[1]*

Thus in general, the equations that describe a relationship between one point in time and a previous point in time are called discrete dynamical system or difference equation.

EXAMPLES.

Consider the sequence 0, 1, 2, 3 … Denoting each of these numbers by a (k), for k= 0, 1, 2, 3… we note that the rule relating the numbers is A (n+1) = A (n) +1.

In Genetics, the genetic characteristics change from generation to generation and the variable representing a generation is a discrete variation.

In Economics, the price changes are considered from one period to another, say year, month, and week of day. Here the time variable is discretized. *[1]*

In Population Dynamics, there are changes in population from one group to another and the variable representing the age group is a discrete variable.

1.2.1TYPES OF DISCRETE DYNAMICAL SYSTEM.

1) *LINEAR DYNAMICAL SYSTEM.*

When the function is linear ad goes through the origin, such dynamical system is called linear dynamical system. The linear dynamical system is further subdivided into

a.) Linear Homogeneous

b.) Linear Non-Homogeneous.

2) *AFFINE.*

When the function is linear and does not go through the origin is called affine dynamical system.

3) *NON-LINEAR DYNAMICAL SYSTEM.*

Consider A (n + 1) = 3 A (n) [1 – A (n)]. If the function is not linear then such a dynamical system is called non-linear dynamical system. The dynamical system is further divided into

a.) Non-linear Homogeneous

b.) Non-linear Non-Homogeneous.

4) *AUTONOMOUS DYNAMICAL SYSTEM.*

Consider the dynamical system A (n + 1) = f [A (n)]. When the Co-efficient of A(n) does not depend on ‘n’ we call dynamical system as Autonomous Dynamical system. The Autonomous Dynamical system is further divided into

a.) Liner Autonomous

b.) Non-Linear Autonomous.

5) NON-AUTONOMOUS DYNAMICAL SYSTEM.

Consider the dynamical system A (n + 1) = f [n, A (n)] when the Co-efficient of A (n) depends on 'n' we call the dynamical system Non-Autonomous dynamical system. This Non-Autonomous dynamical system is divided into

a.) Linear Non-Autonomous b.) Non-linear Autonomous.

1.3FIXED POINT

Consider a first order dynamical system A (n + 1) = f [A (n)]. A number 'a' is called an equilibrium value or fixed point for this dynamical system if A (k) = a, for all values of k.

1.4PERIODIC SOLUTION

A solution A (k) is periodic if A (k + m) = A (k). For some fixed integer m and all k. the smallest integer 'm' for which this holds is called the period of the solution.

1.5COBWEB

Suppose we have a dynamical system A (n + 1) = f [A (n)] with A (0) given. Draw a graph of the curve Y = f(x) and the line Y = x. Pick the first x value A (0) and go vertically to a point on curve. Then go horizontally to a point on line. Then go horizontally to a point on line The Co-ordinate of the point on the line is A (1). Repeat these steps to get A (2), A (3)… The resulting figure is called a cobweb for the given dynamical system.

1.6CYCLE

Two numbers a_1 and a_2 form a cycle for a first order dynamical system if when A (n) = a_1 then a (n + 1) = a_2 and A (n+1) = a_1 and so on.

1.7NON-LINEAR MAP AND ITS CHARACTERISTIC.

There are different types of Non-linear maps as follows;

- Tent map
- Logistic map
- Martin map
- Henon map
- Lorenz map
- Rossler map

1.Tent map

Tent map is defined by S (n + 1) = 2 S (n), S (n) Є (0, 0.5)

= 2 [1-S (n)], S (n) Є (0.5, 1)

2.Logistic map

Logistic map is defined by X (n + 1) = r X (n) [1-X (n)]

3.Henon map

Henon map is defined by $X(n+1) = 1 + Y(n) - aX^2(n)$

$Y(n+1) = bx(n).$

4.Lorenz map

Lorenz map is defined by X (n + 1) = X (n) – a X (n) dt

+a Y (n) dt Y(n +1) = Y(n) = bX(n)dt – Y(n)dt –

Z(n)X(n)dt

Z(n +1) = Z(n) – cZ(n)dt + X(n)Y(n)dt.

5. Rossler Attractor

The attractor is formed with another bunch of Navier Strokes

equations namely X (n+1) = X (n) – Y (n) dt + Z (n) dt

Y (n+1) = Y (n) + X (n) dt + Y (n) dt

Z(n+1) = Z(n) + bdt + X(n)Z(n)dt –

cZ(n)dt.

1.8CHAOTIC DYNAMICAL SYSTEM

A Dynamical system is said to be chaotic under the following condition.

1) A dynamical system is transitive if a_0 is close to S, then A(k) gets closer to every point in S.
2) A dynamical system has sensitive dependence if whenever we take a_0, b_0 close to each other as initial values, then A(k) and B(k) eventually get apart. *[4]*
3) A dynamical system has a continuous broad band Fourier power spectrum.
4) Ergodicity: (X_t+ N, Y_t+ N) is obtained from a map. Sooner or later a new value [X_t + N, Y_t + N] arbitrarity close to [X_t, Y_t] will be found. This is a Chaotic Cignal.
5) Chaotic Dynamical system has at least one positive Lyapunov exponent.

II. REVIEW OF LITERATURE.

Many Authors have discussed about cellular automata and its applications. Some Authors have analysed cellular Automata and partial differential equations. Some authors have studied cellular automata through Fractal dimension. Some Authors have studied cellular automata and graph theory. Here we have discussed about some of the authors who have studied the cellular automata through its application.

***1.** May R. M.*

May R.M[1976] in his article titled “Simple Mathematical Models with very complicated Dynamics “ in the Journal “Nature” ; gives complete picture about discrete Mathematical Models which are Non-Linear and its Chaotic behaviour with regard to complicated Dynamics.

***2.** James T.Sandefur*

James T.Sandefur [1998] in his book on “Discrete Dynamical System, Theory & Application” study in detail various Linear & Non-Linear models leading to Chaotic Behaviour.

***3.** Kraft R.L.*

Kraft R.L. [1999] in his article “Chaos Cantor Set and Hyperbolicity for the Logistic Map”- American Mathematical monthly studied in detail about logistic map with reference to cantor set.

4. *Tommaso Toffoli*

Tommaso Toffoli, MIT Laboratory for Computer Science, Cambridge, U.S.A., made a study under the topic "Cellular Automata as an Alternative to Differential Equations in Modeling Physics", which was published in the journal "Physica" 10D in 1984.

5. *Grassberger*

P. Grassberger, in 1984, made a study under the topic "Chaos and Diffusion in deterministic Cellular Automata", which was published in the journal "Physica" 10D. In this paper it is shown that the deterministic one-dimensional CA studied recently by Wolfram exhibit a kind of spontaneous symmetry breaking.

6. *Stephen J. Willson*

Stephen J.Willson, Department of Mathematics, Lown State University, U.S.A., made a study under the topic, "Growth Rates and fractional Dimensions in CA", which was published in the journal "Physica" 10D in 1984. This paper concerns some different ways by which a fractional dimension may be associated to a CA whose transition rule is linear modulus 'p'.

7. *George T. Yurkon*

George T. Yurkon, in the year 1997, discussed the scientific meaning of the word chaos in his article "Introduction to Chaos and Its Real world Application". This article discusses on how understanding chaos may be of great benefit to mankind. He talks about the literal explosion of scientific interest in chaos and how to control it.

8. *A.R.Kansal*

In the year 20, A.R.Kansal, Department of chemical Engineering, Department of Chemistry, presented a paper in "Stimulated Brain Tumor Growth Dynamics Using a three dimensional Cellular Automaton" which was published in the journal Theortical Biology, Volume: 203. This paper also predicts the composition and the dynamics of the tumor at selected time points in agreement with medical literature.

9. *Yousef Al- Assaf, Reyad El-Khazali, Wajdi Ahmed:*

Yousef Al- Assaf, American University of Sharjah, U.A.E., Reyad El-Khazali, Etisalat college of Engineering, U.A.E.; Wajdi Ahmed, University of Sharjah, U.A.E., presented a paper on the topic " Identification of fractional Chaotic system parameters" which was published in the journal "Chaos, Solitons and Fractals – the interdisciplinary journal for Non – linear Science, Nano and Quantum technology ", Volume:22, in the year 2004.

III. CELLULAR AUTOMATA

The increasing prominence of computers has led to a new way of viewing nature as a form of computation.

That is, we treat objects as simple ROBOTS, Each obeying its own set of laws.

- Cellular Automata (CA) were introduced by John Von Neumann, in the 1940's and described by Arthur Burks, in 1970.
- During the 1970's and 1980's Cellular Automata had a strong revival through the work of Stephen Wolfram, who published an interesting survey.
- Today CA had become a very important modeling and simulation tool in science and technology, from physics, chemistry and biology to computational fluid dynamics in airplane and ship design, to philosophy and sociology.

3.1*OVERVIEW:*

- CA is a branch of automata, which is a branch of computer science.
- A cellular automaton is an array of identically programmed automata or "cells" Which interact one another.
- It is a dynamical system in which cells are generated according to some law.
- The arrays usually form either a 1-dimensional string of cells, a 2-dimensional grid, or a 3-dimensional solid.

3.2*DEFINITION:*

- Discrete in both space and time,
- Homogeneous in space and time (same update rule at all cells at all times),
- Local in their interactions.

A cellular automaton is a model of a system of "cell" objects with the following characteristics.

1) *Grid*. The simplest grid would be one-dimensional: a line of cells.
2) *S*

tates									

. The simplest set of states (beyond having only one state) would be two states: 0 or 1.

1	0	1	0	1	1	1	0	0	1

3) *Neighbourhood*. The simplest neighbourhood in one dimension for any given cell would be the cell itself and its two adjacent neighbours: one to the left and one to the right.

1	0	1	0	1	1	1	0	0	1

The automaton can be 1 – dimensional where its cells are simply linked up like a chain or 2- dimensional where cells are arranged in an array covering the plane. To run a cellular automaton we need two entities of information:

a) An initial state of its cells (i.e. an initial layer)
b) A set of rules of laws

These rules describe how the state of a cell in a new layer. (In the next step) is determined from the states of a group of cells from the preceding layer. The rules should not depend on the position of the group within the layer.

***3.3** <u>ATTRACTING AND REPELLING FIXED POINTS:</u>*

Cellular Automata evolve after a finite number of times steps from almost all initial states to unique homogenous state, in which all sites have the same value. Such CA may be considered to involve to simple “attracting fixed points”.

If the sites repel away from a fixed point, then those points are known as repelling fixed points.

EXAMPLE 1: [a]

F(a b c) = (b + c) (mod2),A = {0,1}

LOCAL RULE TABLE:

A	B	c	F(a,b, c)
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0

Table 1: local rule table for example 1

EXAMPLE 2:

F(a,b,c) = max {a,b}, A={0,1,2}

LOCAL RULE TABLE:

A	b	c	F(a,b, c)
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1

Table 2: local rule table for example 2

For different initial configuration, different CA can be generated.

3.3.1 THE TECHNICAL TERMS USED:

- Trajectory
- Periodic point
- Eventually periodic point
- Right most primitive
- Transitive
- Sensitive

3.4 *SIMPLE CELLULAR AUTOMATA:*

In some sense we might say that Pascal triangle is the first CA. It sets a good example for CA. Pascal triangle is a mathematical triangle made up of staggered rows of numbers.

The initial conditions used are:

$$A_{i1} = 1, A_{ii} = 1$$

Where i-the row number,
The cells in the Pascal triangle are generated using the law,

$$Aij=a(i-1) + a(i-1)j$$

Where, i = the row number j = the column number

3.4.1 *PASCAL TRIANGLE*

1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
1	5	10	10	5	1

3.4.2 PASCAL TRIANGLE MODULO:

1

1 1

1 0 1

1 1 1 1

1 0 0 0 1

1 1 0 0 1 1

On replacing is by dot and 0's by blank space, takes from the above diagram

3.4.3 PASCAL TRIANGLE – A MODIFICATION

0

0 0

0 0

0 0 0 0

0 0

0 0 0 0

A beautiful aspect of Pascal triangle modulo 2 is that the pattern inside any triangle of points is similar in design to that of any Sub-triangle though larger in size. If we extend Pascal triangle to infinitely many rows and reduce the scale of our picture in halt each time that we double the number of rows, then the resulting design is self-similar known as FRACTALS, Cellular Automata is a fractal type dynamical system.

3.5 WILFRAM CLASSIFICATION.

Before we move on to looking at CA in two dimensions, it's worth taking a brief look at Wolfram's classification for cellular automata. As we noted earlier, the vast majority of elementary CA rule sets produce uninspiring results, while some result in wondrously complex patterns like those found in nature. Wolfram has divided up the range of outcomes into four classes:

1: Uniformity. Class 1 CAs end up, after some number of generations, with every cell constant. This is not terribly exciting to watch. Rule 222 is a class 1 CA; if you run it for enough generations, every cell will eventually become and remain black.

2: Repetition. Like class 1 CAs, class 2 CAs remain stable, but the cell states are not constant. Rather, they oscillate in some regular pattern back and forth from 0 to 1 to 0 to 1 and so on. In rule 190, each cell follows the sequence 11101110111011101110.

Class 3: Random. Class 3 CAs appear random and have no easily discernible pattern. In fact, rule 30 is used as a random number generator in Wolfram's Mathematical software. Again, this is a moment where we can feel amazed that such a simple system with simple rules can descend into a chaotic and random pattern

Class 4: Complexity. Class 4 CAs can be thought of as a mix between class 2 and class 3. One can find repetitive, oscillating patterns inside the CA, but where and wh en these patterns appear is unpredictable and seemingly random. Class 4 CAs exhibit the properties of complex systems that we described earlier in this chapter and in Chapter 6. If a class 3 CA wowed you, then a class 4 like Rule 110 above should really blow your mind.

3.5.1 *Game of life.*

We start with a well-known example, Game-of-life, invented
by John Conway in 1970. It is a cellular automaton that consists of an infinite grid of square cells — like an infinite graph paper — where each square is coloured white or black. The colour is called the state of the cell. We say that a black cell is alive while a white cell is not. A colouring of the entire grid is called a configuration of Game-of-life.

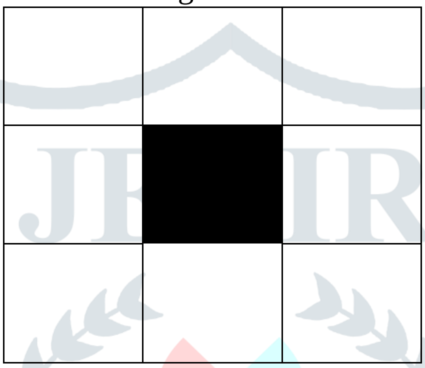

Figure 1: Example for game of life.

There is a simple local update rule according to which the cells change their states. The new state of a cell only depends on the current states of the cell itself and its eight nearest neighbours:
Here black = 1, and white = 0.

1. *Death.* If a cell is alive (= 1) it will die (becomes 0) under the following circumstances.
   - *Overpopulation:* If the cell has four or more alive neighbours, it dies.
   - *Loneliness:* If the cell has one or fewer alive neighbours, it dies.
2. *Birth.* If a cell is dead (= 0) it will come to life (becomes 1) if it has exactly three alive neighbours (no more, no less).
3. *Stasis.* In all other cases, the cell state does not change. To be thorough, let's describe those scenarios.
   - *Staying Alive:* If a cell is alive and has exactly two or three live neighbours, it stays alive.
   - *Staying Dead:* If a cell is dead and has anything other than three live neighbours, it stays dead.

All cells use the same update rule, and all cells change their states simultaneously. This changes the colouring of the grid, i.e. the configuration changes into a new one. The process is then repeated over and over again, which creates a time evolution of the system.

IV. APPLICATIONS

4.1 ADDITIVE CELLULAR AUTOMATA.

4.1.1RULE 250:

Rule 250 specifies the next colour in a cell, depending on its colour and its immediate neighbours. Its rule outcomes are encoded in the binary representation $250 = 11\,111\,010_2$. This rule is illustrated with the evolution of a single black cell it produces after 15 steps *(OEIS A071028; Wolfram 2002, p. 55)*. For initial conditions of a single black cell, this rule produces identical evolution to rules 50, 58, 114, 122, 178, 186, and 242, which are precisely those with binary representation $x\,x\,11\,x\,010_2$.
Rule 250 is a generalized additive elementary cellular automata under the operation OR( ) *(Wolfram 2002, p. 952)*, where is the value of the neighbouring cell to the left and is the value of the neighbouring cell to the right.

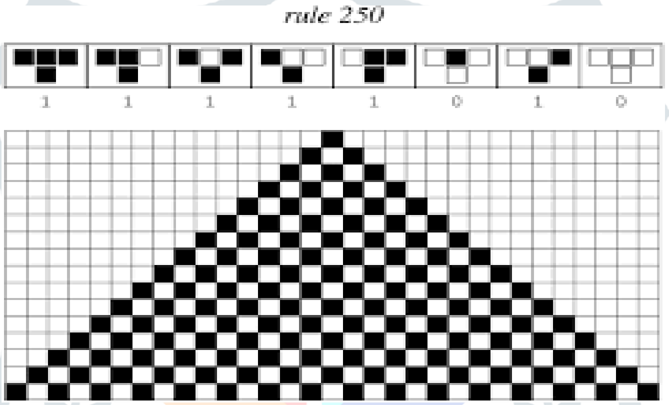

Figure 2: Diagrammatic representation of rule 250.

4.1.2RULE 60:

Rule 60 specifies the next colour in a cell, depending on its colour and its immediate neighbours. Its rule outcomes are encoded in the binary representation $60 = 00\,111\,100_2$. This rule is illustrated with the evolution of a single black cell it produces after 15 steps *(OEIS A075438; Wolfram 2002, p. 55)*.

Starting with a single black cell, successive generations are given by interpreting the numbers 1, 3, 5, 15, 17, 51, 85, 255, 257, 771, 1285, ... *(OEIS A001317)* in binary (where left n cells in step n of the triangle are always 0), namely 1, 11, 101, 1111, 10001; ... *(OEIS A047999)*.

The mirror image is rule 102, the complement is rule 195, and the mirrored complement is rule 153. Rule 60 is one of the eight additive elementary cellular automata.

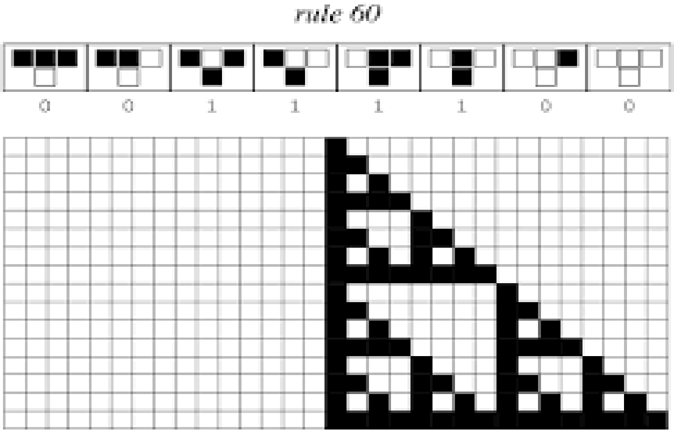

Figure 3: Diagrammatic representation of rule 60.

4.1.3RULE 90

A simple example of an additive cellular automata on is provided by the rule 90. Elementary cellular automata on as can be seen from the graphical representation of this rule. The rule has a function of left central and right neighbours. Is simply given by the sum of the rules for the left and the right neighbours taken modulo to, where white cells are assigned the value 0 and the black cells are assigned the value1 (this is equivalent to the XOR operation and means that “adding” two white cells or two blacks cells gives a white cell, while adding one white cell and one black cell gives a black cell).

For example, the rule for (1,1,1) is 1 + 1 = 0(mod 2),

The rule for (1,1,0) is 1 + 0 = 1(mod 2)

The rule for (1,0,1) is 1 + 1 = 0(mod2) and so on. Repeating this for each of the 2 x 2 x 2 =8 possible states of neighbours gives the binary representation $90 = 01011010_2$.

Rule 90 is amphichiral, and its complement is rule 165.

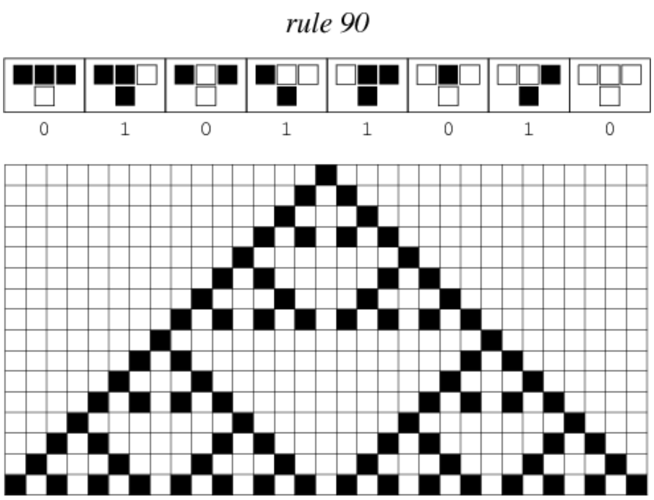

Figure 4: diagrammatic representation of rule 90.

Similarly we have rules from 0 to 255. Some of them are….

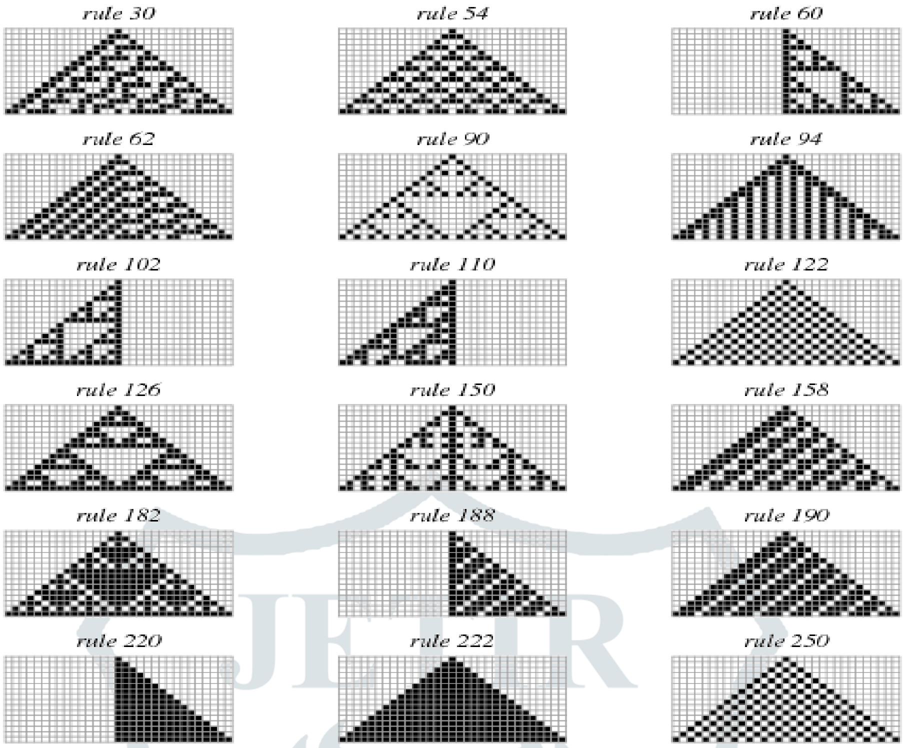

Figure 5: diagrammatic representation of some other rules in CA.

4.2 IN LINEAR PARTIAL DIFFERENTIAL EQUATIONS.

A. DOESCHL*, M. DAVISON, H. RASMUSSEN AND G. REID, Department of Applied Mathematics, University of Western Ontario London had done a paper on "Assessing Cellular Automata Based Models Using Partial Differential Equations." The main goal of this study was to demonstrate how partial differential equations can be used to assess numerical models arising from a cellular automata approach. The study focused on two rival numerical models for morphological processes in river beds.

The relationship between the numerical cellular automata based models and partial differential equations was established by viewing the models as numerical solution schemes of partial differential equations with initial and boundary values derived from the governing equations of the cellular models.

But in our paper as a result of additivity, the evaluations of additive cellular automata behave similarly to the solutions of linear partial differential equations. The resulting evolution can be found by convolving the evolution of a single cell with the initial condition. *(Wolf ram 2002, P.952) [7] & [10]*

Cellular automata can be applied to get the behaviour of the solution of a linear partial differential equation. A Linear partial differential equation can be solved by any method. If S_1, & S_2 be two solution of a linear partial differential equation, then $S_1 + S_2$ is also a solution of the linear partial differential equation. This concept is very much displayed by additive cellular automata. A linear partial differential equation is discretized. Then the solution can be obtained by any one method available. A suitable cellular automata can also be formed for the solution of a linear partial differential equation. Cellular automata formed – gives the behaviour of the solution of a linear partial differential equation. If two solutions are obtained. Then linear combination of two solution is also a solution of the linear partial differential equation. This is very much illustrated using the concepts of additive cellular automata.

4.3 APPLICATIONS IN BIOMEDICAL SCIENCES

In the early 1980's scientists began to apply chaos theory to physiological systems. Many organs and systems proved to be fractal in nature. Intuitively, it seems logical that chaos would be more apparent in pathological or disease states.

Fractal techniques have already been applied to a variety of disciplines in medicine. *[14]*. Growth of tumour might to influenced by the fractal structure of their tissues of origin, Tumour boundaries and chromatin texture have been studied by fractal analysis and this may prove useful in discriminating between benign and malignant cells. Publications on the application of chaos and fractals are increasingly seen in a variety of disciplines including ophthalmology, neuropathology and urology. Perhaps the most existing prospects for the application of chaos theory in medicine are those related to simulation of cancerous growths. *[13]*

The identification and characterization of common complex multifactorial human diseases remains a statistical and computational challenge. Mathematical tools viz., Cellular automata, L- system, etc., serve as a novel computational approach with which we demonstrate using simulated data that the approach has good power to identifying high-order, non-linear interactions.

4.4 CELLULAR AUTOMATA GENERATED INTO GRAPH.

A Cellular automata generated can be conveniently represented graphically. The concept is illustrated with the help of Sierpinski's triangle. Consider a Sierpinski's triangle.

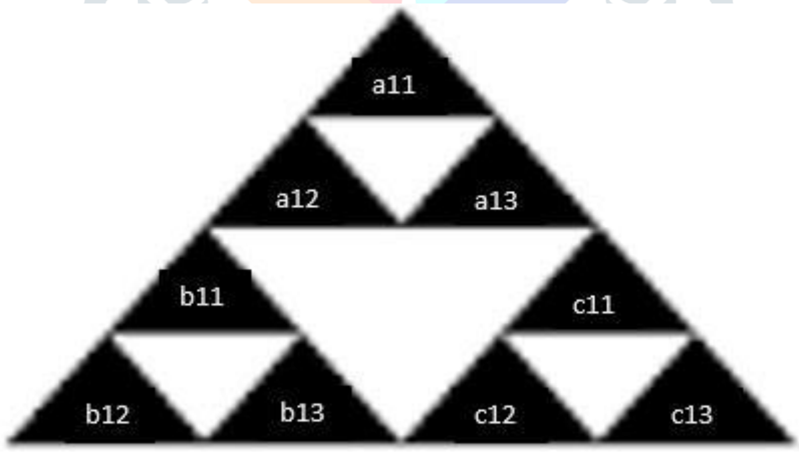

Figure 6: Sierpinski's triangle.

The corresponding tree topology is represented in.

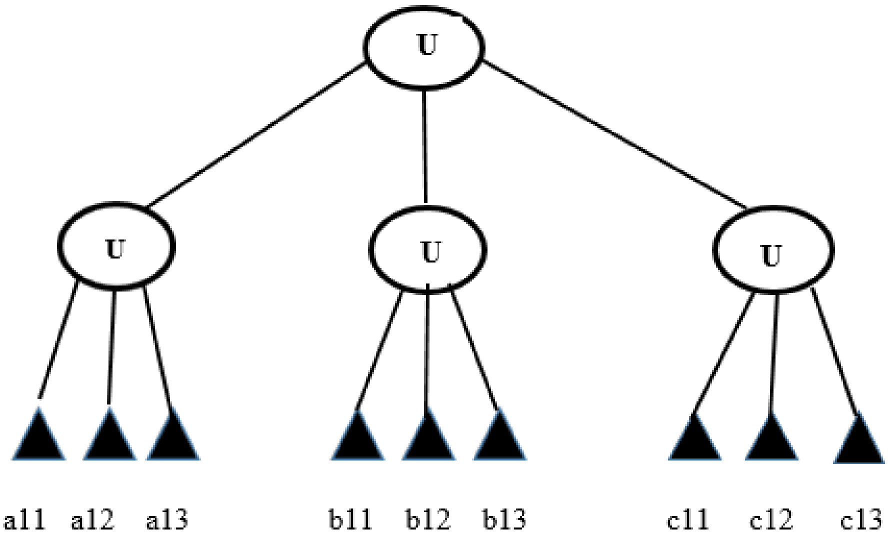

Figure 7: tree topology of Sierpinski's triangle.

The corresponding triangle is also represented graphically.

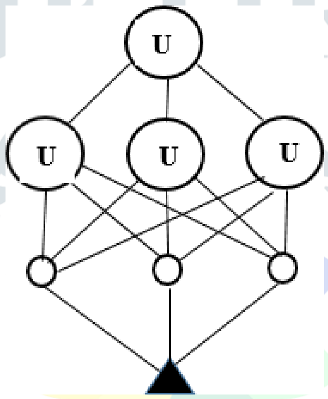

Figure 8: Graphical representation of Sierpinski's triangle.

If we generate more triangles in a Sierpinski's triangle such as

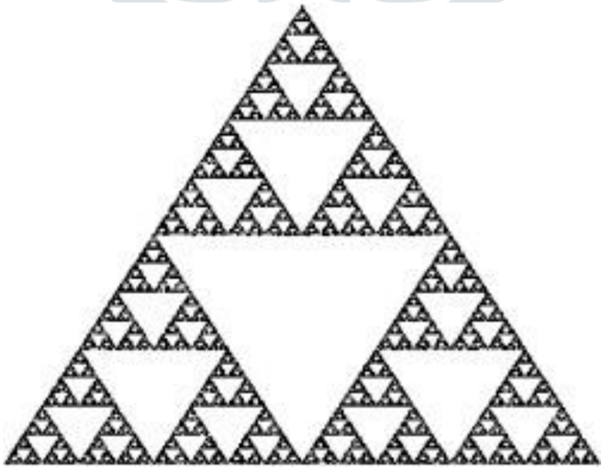

Figure 9: generation of more triangles in Sierpinski's triangle.

A corresponding graphical model is derived to represent the cellular automata.

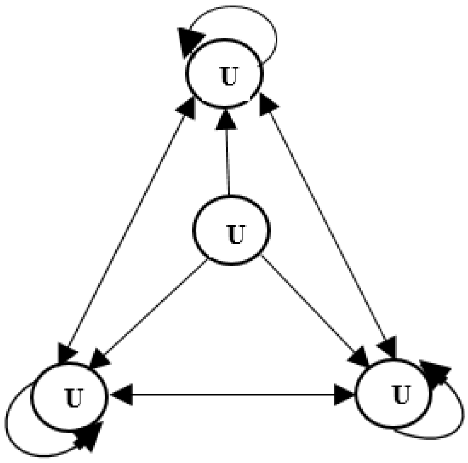

Figure 10: graphical model representing cellular automata.

Hence for any cellular automata generated we have a graphical representation. It is possible to represent graphically Sierpinski's square, Sierpinski's pentagon etc.,

V. INFERENCE AND CONCLUSION.

There are various tools used to gain to gain the comprehensive overview of the non-linear dynamics via cellular automata. The dynamical system we have studied are called cellular automata (CA) which is used as a basic thing to solve partial differential equation. Apart from this Cellular automata can be represented graphically using Graph Theoretic Concepts.

So by applying various tools such as L- system, CA, Non-linear Dynamical system, we spread its wings to different fields. Thus interdisciplinary study with the mixture of CA, L-system and Discrete Dynamical system, we have a good scope for its inter presentation and analysis of our cosmos.

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