STUDY THE NEW CLASS OF HYPERGEOMETRIC FUNCTIONS OF SECOND KIND AND DIRICHLET AVERAGE

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ABSTRACT

This article investigates the Dirichlet averages or the product of two elementary functions including their fractional integral representation. Certain results are obtained involving Gaussian hypergeometric functions ${}_2F_1^{(1)}$ and ${}_2F_1^{(2)}$ which illustrate their applications and partial derivatives in terms of Dirichlet averages, with respect to various parameters are also reported.

KEYWORD :dirichlet, hypergeometric, fractional integral operator, partial derivatives

1. Introduction

Carlson [3, 4] introduced the concept of Dirichlet average (also called as weighed average), which denotes a certain kind of integral average with respect to a complex measure. Dirichlet averages satisfy certain special class or partial differential equations (cf. Carlson [3]) and are employed to evaluate certain integrals involving classical orthogonal polynomials and various elliptical integrals (cf. Carlson [3, 4]). Hidden symmetries of special functions in the context of Dirichlet averages were studied by Carlson [3] and Prabhakar [10]. One may refer to the works of Deora [6], Vyas [11], Vyas and Banerji [11], Banerji and Deora [7], Deora and Banerji [6,7], Deora, Banerji and Saigo [9], and Vyas, Banerji and Saigo [12]. It may not be out of place to contributions by Banerji. Vyas and Deora, as cited in the references. It still requires rather intensive study and finding its applications.

In the present paper we compute the Dirichlet average or the product $e^{\lambda j}t^{-\tau}$ with respect to a complex measure μ_b and express it in terms of fractional integral operator and confluent hypergeometric function of two variable. For various terminology's and definitions of Dirichlet averages and fractional calculus, one may refer to Banerji [1] and Vyas [11].

Following the terminology of Carlson [3], the Dirichlet average of the product $e^{\lambda j}t^{-\tau}$ is expressed as :

$$W_{\tau,\lambda}(b,z) = \int_E (u,z)^{-\tau} e^{\lambda(u,z)} d\mu_b(u) \quad \ldots\ldots \qquad (1.1)$$

where $z=(z_1,\ldots..,z_k)$, $b=(b_1,\ldots..,b_k)$

and $z = \sum_{i=1}^{k-1} u_1 z_1$, $u_k = 1 - \sum_{i=1}^{k-1} u_1$

2. Main Results

For convenience, this section is divided into three subsections.

$$W_{\tau,\lambda}(\alpha,\beta;x,y) = \Gamma\begin{bmatrix}\alpha+\beta \\ \alpha,\beta\end{bmatrix} \int_0^1 [ux+(1-u)y]^{-\tau} e^{\lambda[ux+(1-u)y]}$$

$$\times u^{\alpha-1}(1-u)^{\beta-1}du, \,.. \qquad (2.1)$$

where

$$d\mu_b(u) = \Gamma\begin{bmatrix}\alpha+\beta \\ \alpha,\beta\end{bmatrix} u^{\alpha-1}(1-u)^{\beta-1}du$$

and

$$\Gamma\begin{bmatrix}\alpha+\beta\\ \alpha,\beta\end{bmatrix}=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}.$$

Rearranging the terms in (2.1), we have the modified version of the result as :

$$W_{\tau,\lambda}(\alpha,\beta;x,y)=\Gamma\begin{bmatrix}\alpha+\beta\\ \alpha,\beta\end{bmatrix}\int_0^1[y-u(y-x)]^{-\tau}e^{\lambda[y-u(y-x)]}$$

$$\times u^{\alpha-1}(1-u)^{\beta-1}du, \quad .. \quad (2.2)$$

which can be written as

$$W_{\tau,\lambda}(\alpha,\beta;x,y)=\Gamma\begin{bmatrix}\alpha+\beta\\ \alpha,\beta\end{bmatrix}y^{-\tau}e^{\lambda y}\int_0^1\left[1-u(1-\frac{x}{y})\right]^{-\tau}e^{\lambda\mu(x-y)}$$

$$\times u^{\alpha-1}(1-u)^{\beta-1}du, \ldots. \quad (2.3)$$

An appeal to the integral representation of confluent hypergeometric function of two variables permits us to write (cf. Erdelyi et al. [37]).

$$W_{\tau,\lambda}(\alpha,\beta;x,y)=e^{\lambda y}y^{-\tau}\phi_1\left[\tau,\alpha;\alpha+\beta;1-\frac{x}{y},\lambda(x-y)\right], \quad \ldots \quad (2.4)$$

where $\phi_1(.)$ is the confluent hypergeometric function of two variables. An alternate form of (2.4) is

$$W_{\tau,\lambda}(\alpha,\beta;x,y)=e^{\lambda y}y^{-\tau}\phi_1\left[\tau,\beta;\alpha+\beta;1-\frac{x}{y},\lambda(x-y)\right],\ldots \quad (2.5)$$

And is justified by the hidden symmetry of $\phi_1(.)$ which accounts for their transformation properties (cf. Prabhakar [10] p.166).

3. Fractional Integral Operator and Dirichlet Average :

Setting u=t/(y–x) in (2.2), we obtain the following integral

$$W_{\tau,\lambda}(\alpha,\beta;x,y)=\Gamma\begin{bmatrix}\alpha+\beta\\ \alpha,\beta\end{bmatrix}(y-x)^{1-\alpha-\beta}e^{\lambda y}$$

$$\times\int_0^{y-x}\{(y-x)-t\}^{\beta-1}t^{\alpha-1}e^{-\lambda\tau}(y-t)^{-\tau}dt\ldots.\quad(3.1)$$

which upon using the definition of fractional integral operator, assumes the following form

$$W_{\tau,\lambda}(\alpha,\beta;x,y)=\Gamma\begin{bmatrix}\alpha+\beta\\ \alpha,\beta\end{bmatrix}(y-x)^{1-\alpha-\beta}e^{\lambda y}{}_oD_{y-x}^{-\beta}[(y-x)^{\alpha-1}e^{\lambda(x-y)}x^{-\tau}],$$

$$\ldots\ldots\quad(3.2)$$

where $D^{-\beta}$ stands for the fractional integral operator of order β.

Particular Cases of (2.4) :

(i) If we replace β by β-α the explicit form for $\phi_1(.)$ is obtained as

$$W_{\tau,\lambda}(\alpha,\beta-\alpha;x,y)=e^{\lambda y}y^{-\tau}\phi_1[\beta,\tau;\alpha;1-\frac{x}{y},\lambda(x-y)]\ldots..\quad(3.3)$$

(ii) If we set λ=0 in (4.3.3), we get (cf. (58)]

$$\left.\begin{aligned}W_{\tau,\lambda}(\alpha,\beta-\alpha;x,y)&=y^{-\tau}{}_2F_1\left(\alpha,\tau;\beta;1-\frac{x}{y}\right)\\&=R_{-\tau}(\beta,\beta-\alpha;x,y)\end{aligned}\right\}\quad\ldots..\quad(3.4)$$

(iii) Setting λ=0 and x↔y in (3.3), and using the hidden symmetry of R-function, we obtain (cf. Carlson[3]).

$$\left.\begin{aligned}W_{\tau,\lambda}(\alpha,\beta-\alpha;y,x)&=x^{-\tau}{}_2F_1\left(\alpha,\tau;\beta;1-\frac{x}{y}\right)\\&=R_{-\tau}(\beta,\beta-\alpha;x,y)\end{aligned}\right\}$$

$$\ldots..\quad(3.5)$$

4. Dirichlet Average of $e^{(\lambda+1)^t}$

Taking $\tau=-n$ in (4.2.1), we have

$$W_{n,\lambda}(\alpha,\beta;x,y)=\Gamma\begin{bmatrix}\alpha+\beta\\ \alpha,\beta\end{bmatrix}\int_0^1 [ux+(1-u)y]^n e^{\lambda[ux+(1-u)y]}$$

$$\times u^{\alpha-1}(1-u)^{\beta-1}du, \quad .... \quad (4.1)$$

Multiplying both the sides of (4.1) by 1/n! and taking summation, we get

$$P_{,\lambda}(\alpha,\beta;x,y)=\sum_{n=1}^{\infty}\frac{1}{n!}W_{-n\lambda}(\alpha,\beta;x,y)$$

$$=\Gamma\begin{bmatrix}\alpha+\beta\\ \alpha,\beta\end{bmatrix}\int_0^1 e^{(\lambda+1)[ux+(1-u)y]}(1-u)^{\beta-1}du, ... \quad (4.2)$$

$\lambda\neq-1$, $Re(\alpha)$, $Re(\beta) > 0$.

which denotes the Dirichlet average of $e^{(\lambda+1)t}$ due to change of order of summation and integration. An alternate form of (4.2) is given by

$$P_{\lambda}(\alpha,\beta;x,y)=e^{(\lambda+1)y}\ {}_1F_1[\alpha,\beta+\alpha;(\lambda+1)(x-y)], ... \quad (4.3)$$

where ${}_1F_1$ denotes the Kummer's confluent hypergeometric function (cf. Salter [14]. For $\lambda=-1$, (4.2) and (4.3) reduce to unity, i.e. $P_{-1}(\alpha,\beta;x,y)=1$. Also for $\lambda=0$, (4.2) and (4.3) reduce to Gupta and Agarwal [58].

$$P_o(\alpha,\beta;x,y)=S(\alpha,\beta;x,y)=e^y\ {}_1F_1[\alpha,\beta+\alpha;(x-y)], ... \quad (4.4)$$

S(.) being the single Dirichlet average e^1 (cf. Gupta and Agrawal [58]).

5. Derivatives of Dirichlet Average

In what follows are certain results incorporating the Dirichlet average, associated hypergeometric function and digamma function.

Following formulae are used in the analysis of results.

$$ {}_1F_1^{(1)}(\alpha,\beta,\gamma;z) = \frac{i\Gamma(\gamma)\exp[i\pi(\beta-\gamma)]}{i\Gamma(\beta)\Gamma(\gamma-\beta)2\sin[\pi(\gamma-\beta)]} $$

$$ \times\int_0^1 \frac{t^{\beta-1}(1-t)^{\gamma-\beta-1}}{(1-tz)^{\alpha}}\log(1-tz)dt \quad \ldots.. \quad (5.1) $$

$$ {}_2F_1^{(2)}(\alpha,\beta,\gamma;z) = \frac{i\Gamma(\gamma)\exp[i\pi(\beta-\gamma)]}{i\Gamma(\beta)\Gamma(\gamma-\beta)2\sin[\pi(\gamma-\beta)]} $$

$$ \times\int_0^1 \frac{t^{\beta-1}(1-t)^{\gamma-\beta-1}}{(1-tz)^{\alpha}}\log t\, dt \quad \ldots \quad (5.2) $$

The validity for both (5.1) & (5.2) are Re $(\beta)>0$, $|\arg(1-z)|<\pi$, $(\gamma-\beta)\neq 1,2,3\ldots$

6. Partially Differentiating with respect to τ

Differentiation of (2.1) with respect to τ as parameter leads to a class of functions which is supposed to be new and contains associated confluent hypergeometric function of two variables. The associated Gauss hypergeometric function follows as particular cases of these functions.

Differentiating (2.1), for $\tau \to -\tau$, we obtain

$$ Q_{\tau,\lambda}(\alpha,\beta,x,y) = \frac{\partial}{\partial\tau}\{W_{-\tau,\lambda}(\alpha,\beta,x,y)\} $$

$$e^{\lambda y} y^{\tau} \log y \phi_1^{(1)} W_{\tau,\lambda}(\beta, -\tau; \alpha+\beta; 1-\frac{x}{y}, \lambda(x-y)), \quad .... \quad (6.1)$$

where $\phi_1^{(1)}$ stands for associated confluent hypergeometric polynomial of the first kind. Further for $\lambda=0$ and $\tau=n$ (6.1) reduces to

$$Q_{n,0}(\alpha,\beta,x,y) = \frac{-2\sin[\pi(\beta-\alpha)]\partial}{i\exp[i,\pi(\beta-\alpha)]}\, {}_2F_1^{(1)}\left(\begin{matrix}\beta,-n \\ \alpha\end{matrix}; 1-\frac{x}{y}\right), ... \quad (6.2)$$

which is believed to be another interesting result for Dirichlet averages, (6.2) can also be interpreted in terms of known result (cf. Gupta and Agarwal [58] as

$$Q_{n,0}(\alpha,\beta,x,y) = L_n(\alpha,\beta,x,y) \quad .... \quad (6.3)$$

Here $L_n(.)$ is the single Dirichelt average of $x^n \log x$ (cf. Carlson[4]).

Differentiating (2.1) r-times with respect to τ, we obtain

$$\frac{\partial}{\partial\tau'}\{W_{-\tau,\lambda}(\alpha,\beta,x,y)\} = L_\tau^{(\tau)}(\alpha,\beta,x,y) \quad .... \quad (6.4)$$

7.Partially Differentiating with respect to α:

Here we deal with partial derivatives of $W_{-\tau,\lambda}[.]$ with respect to α i.e. one of the parameters of hypergeometric function. Now, let

$$I_i = \int_0^1 [ux+(1-u)y]^{-\tau} e^{\lambda[ux+(1-u)y]} u^{\alpha-1}(1-u)^{\beta-1} du. \quad .... \quad (7.1)$$

Rewriting (2.1) as

$$W_{-\tau,\lambda}(\alpha,\beta;x,y)\} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\alpha)} I_i \quad ...... \quad (7.2)$$

Differentiating (7.2) partially with respect to α, one can write

$$\frac{\partial}{\partial\alpha}W_{\tau,\lambda}(\alpha,\beta;x,y)= \Gamma\begin{bmatrix}\alpha+\beta\\ \alpha,\beta\end{bmatrix}\frac{\partial I_i}{\partial\alpha}+I_i\frac{\partial I_i}{\partial\alpha}\left\{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\alpha)}\right\}\ldots \qquad (7.3)$$

and we get the following relationship involving associated confluent hypergeometric function of the second kind :

$$\frac{\partial}{\partial\alpha}\{W_{\tau,\lambda}(\alpha,\beta;x,y)\}=e^{\lambda y}y^{\tau}\phi_1^{(2)}\left(\beta,\tau;\beta+\alpha;1-\frac{x}{y},\lambda(x-y)\right)$$

$$+\{\psi(\alpha+\beta)-\psi(\alpha)\}W_{\tau,\lambda}(\alpha,\beta;x,y). \qquad \ldots.. \quad (7.4)$$

where $\phi_1^{(2)}(.)$ stands for associated confluent hypergeometric function of the second kind while $\psi(\alpha)$ denotes the digamma function.

For λ=0, (4.7.4) implies the following :

$$\frac{\partial}{\partial\alpha}W_{\tau,o}(\alpha,\beta,-\alpha;x,y)=\frac{2\sin[\pi(\beta-\alpha)]}{i\exp[i\,\pi(\beta-\alpha)]}y^{-\tau}\;{}_2F_1^{(1)}\left(\begin{matrix}\beta,\tau;\\ \alpha;\end{matrix}\;1-\frac{x}{y}\right)$$

$$+\{\psi(\alpha+\beta)-\psi(\alpha)\}\,R_{-\tau}(\alpha,\beta;x,y), \qquad \ldots.. \quad (7.5)$$

here $R_{-\tau}(.)$ is given in (4.3.4) (cf. Vyas and Banerji (11]) and ${}_2F_1^{(1)}$ is the associated Gauss hypergeometric function of the second kind.

8. Partially Differentiating with respect to λ:

We have two differential recurrence relations for $W_{\tau,\lambda}[.]$ are given.

Differentiating (2.1) partially with respect to λ, we get.

$$\frac{\partial}{\partial\lambda}W_{\tau,\lambda}(\alpha,\beta;x,y)=W_{\tau+1,\lambda}(\alpha,\beta;x,y) \qquad \ldots.. \quad (8.1)$$

The r-times partially differentiation with respect to λ, gives rise to the following recurrence relation

$$\frac{\partial^{/}}{\partial\lambda^{/}}W_{\tau,\lambda}(\alpha,\beta;x,y)=W_{\tau+\tau,\lambda}(\alpha,\beta;x,y)$$

$$e^{\lambda y}y^{-\tau}\phi_1\left(\beta,\tau+\tau;\beta+\alpha;1-\frac{x}{y},\lambda(x-y)\right) \quad ..... \quad (8.2)$$

Conclusion

(i) If we take λ=0 throughout the text, the results of Gupta and Agrawal [58] follow as particular cases of our results.

(ii) If we take τ=0 and λ=1 in the main result simultaneously, further results of [4] follow as the special cases.

(iii) For λ=τ=0, the integrals are trivial.

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