A study of Lightlike Submanifold equipped with Quarter-symmetric non-metric Connection

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Abstract

The object of the present paper is to study A study of Lightlike Submanifold equipped with Quarter-symmetric non-metric Connection and some properties of quarter-symmetric non-metric connection.

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1 Introduction

In this paper we study lightlike submanifolds of a semi-Riemannian manifold with respect to quarter-symmetric non-metric connection. In lightlike manifolds the semi-symmetric non-metric connection has been studied by A. Yusesan and E. Yasar [?]. We have studied the quarter-symmetric non-metric connection on the lightlike submanifolds.

In the present paper we have proved some results on lightlike submanifolds equipped with quarter-symmetric non-metric connection for *screen distribution* $S(TM)$, *complementary vector bundle* $S(TM^{\perp})$ and *lightlike transversal vector bundle* $ltr(TM)$. We have shown that the Ricci tensor of lightlike submanifold of Riemannian space form is not parallel with respect to quarter-symmetric non-metric connection.We have obtained integrability condition for screen distribution with respect to quarter-symmetric nonmetric connection.

2 Preliminaries

Let $(\tilde{M},\tilde{g})$ be a real n-dimensional semi-Riemannian manifold of constant index such that $1 \le r \le n-1$ and (M,g) be an m-dim submanifold of $\tilde{M}$. Thus there exists nondegenerate screen distribution $S(TM)$ which is complementary vector subbundle to Rad TM in the tangent bundle TM. Therefore

$$TM = \text{Rad}\, TM \oplus_{orth} S(TM) \qquad (2.1)$$

where $\oplus_{orth}$ denotes orthogonal direct sum. $S(TM)$ is not unique but it is isomorphic to factor vector bundle $TM/\text{Rad}(TM)$. We denote r-lightlike submanifold by $(M,g,S(TM),S(TM^{\perp}))$ where $S(TM^{\perp})$ is complementary vector bundle of Rad(TM) in $TM^{\perp}$ and $S(TM^{\perp})$ is non-degenerate with respect to $\tilde{g}$. Let the complementary vector bundle to TM in $T\tilde{M}_{|M}$ is $tr(TM)$ and

$$trTM = ltr(TM) \oplus_{orth} S(TM^{\perp}) \qquad (2.2)$$

where $ltr(TM)$ is arbitrary *lightlike transversal vector bundle* of M. Let $T\tilde{M}$ is tangent bundle of the manifold M then

$$T\tilde{M}_{|M} = TM \oplus tr(TM) \qquad (2.3)$$

Thus by equation (**??**), (**??**) and (**??**) we have

$$T\tilde{M}_{|M} = [\mathrm{Rad}TM \oplus_{orth} S(TM)] \oplus [ltr(TM) \oplus_{orth} S(TM^{\perp})]$$

$$= [\mathrm{Rad}(TM) \oplus ltr(TM)] \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}) \quad (2.4)$$

Now we assume that U is local coordinate neighbourhood of M. We have the local quasi-orthonormal field of frame on $\tilde{M}$ along M is

$$\{\xi_1,\xi_2,...,\xi_r,X_{r+1},X_{r+2},...,X_m,N_1,N_2,...,N_r,W_{r+1},W_{r+2},...,W_{n-m}\}$$

where $\{\xi_1,\xi_2,...,\xi_r\}$ is lightlike basis of $\Gamma(Rad(TM)_{|U})$, $\{X_{r+1},X_{r+2},...,X_m\}$ is orthonormal basis of $\Gamma(S(TM)_{|U})$, $\{N_1,N_2,...,N_r\}$ is lightlike basis of $\Gamma(ltr(TM)_{|U})$ and

$\{W_{r+1},W_{r+2},...,W_{n-m}\}$ is ortonormal basis of $\Gamma(S(TM^{\perp})_{|U})$.

We have the following conditions are satisfied $\tilde{g}(N_i,\xi_j) = \delta_j^i$, for $1 \le i, j \le r$ $\tilde{g}(N_i,N_j) = \tilde{g}(N_i,X_k) = 0$, for $X_k \in \Gamma(S(TM))_{|M}$ & $N_i \in \Gamma(ltr(TM))_{|M}$. and $r + 1 \le k \le m$.

Example 1. Let us consider a surface M in R_2^4 given by equations [**?**]

$$x_3 = \frac{1}{\sqrt{2}}(x_1 + x_2)$$
$$x_4 = \frac{1}{2}\log(1 + (x_1 - x_2)^2)$$

Then we have TM =span (U_1,U_2) and $TM^{\perp}$ =Span (H_1,H_2) Take

$$U_1 = \sqrt{2}(1 + (x_1 - x_2)^2)\frac{\partial}{\partial x_1} + (1 + (x_1 - x_2)^2)\frac{\partial}{\partial x_3} + \sqrt{2}(x_1 - x_2)\frac{\partial}{\partial x_4}$$
$$U_2 = \sqrt{2}(1 + (x_1 - x_2)^2)\frac{\partial}{\partial x_2} + (1 + (x_1 - x_2)^2)\frac{\partial}{\partial x_3} - \sqrt{2}(x_1 - x_2)\frac{\partial}{\partial x_4}$$

and

$$H_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \sqrt{2}\frac{\partial}{\partial x_3}$$
$$H_2 = 2(x_2 - x_1)\frac{\partial}{\partial x_2} + \sqrt{2}(x_2 - x_1)\frac{\partial}{\partial x_3} + (1 + (x_1 - x_2)^2)\frac{\partial}{\partial x_4}$$

It is obvious that $Rad(TM)$ is a distribution of rank 1 on M spanned by $\xi = H_1$. Hence M is a 1-lightlike submanifold of R_2^4. Let $S(TM)$ spanned by U_2 and $S(TM^{\perp})$ is spanned by H_2 which are timelike and space like respectively. The lightlike transversal vector bundle is

$$ltr(TM) = \text{Span}\left\{N = -\frac{1}{2}\frac{\partial}{\partial x_1} + \frac{1}{2}\frac{\partial}{\partial x_2} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_3}\right\}$$

and the transversal vector bundle is

$$tr(TM) = \text{Span}\{N, H_2\}.$$

3 Quarter-symmetric non-metric Connection

Let $\widetilde{M}$ be an n-dimensional semi-Riemannian manifold with a semi-Riemannian metric $\tilde{g}$ with index $1 \le \nu \le n-1$ and $\widetilde{\nabla}$ denotes the linear connection in $\widetilde{M}$. The torsion tensor

$\widetilde{T}$ of $\widetilde{\nabla}$ is given by

$$\widetilde{T}(\widetilde{X},\widetilde{Y}) = \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} - \widetilde{\nabla}_{\widetilde{Y}}\widetilde{X} - [\widetilde{X},\widetilde{Y}] \quad (3.1)$$
$$\forall \widetilde{X},\widetilde{Y} \in \Gamma(T\widetilde{M})$$

and have type (1, 2). A linear connection $\widetilde{\nabla}$ on $\widetilde{M}$ is said to be quarter-symmetric non-metric connection if its torsion tensor $\widetilde{T}$ satisfies

$$\widetilde{T}(\widetilde{X},\widetilde{Y}) = \tilde{\pi}(\widetilde{Y})F\widetilde{X} - \tilde{\pi}(\widetilde{X})F\widetilde{Y} \quad (3.2)$$

for every $\widetilde{X}$ and $\widetilde{Y} \in \Gamma(T\widetilde{M})$, where $\tilde{\pi}$ is a 1-form and F is (1, 1) tensor.

Also

$$\widetilde{\nabla}_{\widetilde{X}}\tilde{g}(\widetilde{Y},\widetilde{Z}) \neq 0 \quad (3.3)$$

We suppose that there exists a quarter-symmetric non-metric connection $\widetilde{\nabla}$ on $\widetilde{M}$ given by

$$\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} = \widetilde{\nabla}^0_{\widetilde{X}}\widetilde{Y} + \tilde{\pi}(\widetilde{Y})F\widetilde{X}, \quad \forall \widetilde{X},\widetilde{Y} \in \Gamma(T\widetilde{M}) \quad (3.4)$$

where $\widetilde{\nabla}^0$ is Levi-Civita connection with respect to $\tilde{g}$ and $\tilde{\pi}$ is a 1-form associated with vector field $\widetilde{Q}$ on $\widetilde{M}$ given by

$$\tilde{\pi}(\widetilde{X}) = \tilde{g}(\widetilde{X},\widetilde{Q}).$$

From (**??**) the vector field $\widetilde{Q}$ on $\widetilde{M}$ is decomposed as

$$\widetilde{Q} = Q + \sum_{i=1}^{r}\lambda_i N_i + \sum_{\alpha=r+1}^{n-m}\lambda_\alpha W_\alpha \quad (3.5)$$

$\lambda_i, \lambda_\alpha$ is a real valued function on M and Q is vector a field.

Let ∇^0 is symmetric linear connection induced connection on submanifold M from $\widetilde{\nabla}^o$ on $\widetilde{M}$. Then Gauss formula with respect to ∇^o is given by

r $n-m$

$$\tilde{\nabla}^0_X Y = \nabla^0_X Y + \sum_{i=1}^{r} h^{0l}_i(X,Y)N_i + \sum_{\alpha=r+1}^{n-m} h^{0s}_\alpha(X,Y)W_\alpha \quad (3.6)$$

for $X,Y \in \Gamma(TM)$ and $N_i \in \Gamma ltr(TM)$ and $W_\alpha \in \Gamma S(TM^\perp)$ and h^{0l}_i are local lightlike second fundamental forms and h^{0s}_α are local screen second fundamental forms of M.

Let the connection ∇ on M which is induced from the quarter-symmetric non-metric connection $\tilde{\nabla}$ on $\tilde{M}$ is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h^l_i(X,Y)N_i + \sum_{\alpha=r+1}^{n-m} h^s_\alpha W_\alpha \quad (3.7)$$

$\forall X,Y \in \Gamma(TM)$ and $N_i \in ltr(TM)$ and $W_\alpha \in \Gamma S(TM^\perp)$, where h^l_i and h^s_α are *local lightlike second fundamental form* and *local screen second fundamental forms* of M which are tensors of type (0, 2) on M. The above equation (**??**) is called the Gauss formula with respect to $\tilde{\nabla}$.

Thus in view of (**??**) we have

$$\tilde{\nabla}_X Y = \tilde{\nabla}^0_X Y + \tilde{\pi}(Y)FX \quad (3.8)$$

Thus using (**??**),(**??**) into (**??**), we get

$$\begin{aligned}\nabla_X Y + \sum_{i=1}^{r} h^l_i(X,Y)N_i + \sum_{\alpha=r+1}^{n-m} h^s_\alpha(X,Y)W_\alpha &= \nabla^0_X Y + \sum_{i=1}^{r} h^{0l}_i(X,Y)N_i \\ &+ \sum_{\alpha=r+1}^{n-m} h^{0s}_\alpha(X,Y)W_\alpha + \tilde{\pi}(Y)FX\end{aligned} \quad (3.9)$$

from which we have

$$\nabla_X Y = \nabla^o_X Y + \pi(Y)FX. \quad (3.10)$$

where $\pi(Y) = \tilde{\pi}(Y)$ and we also have $h^{0l}_i = h^l_i$ and $h^{0s}_\alpha = h^s_\alpha$ for any X and $Y \in \Gamma(TM)$ for $1 \le i \le r$ and $r + 1 \le \alpha \le n - m$. The induced connection on lightlike submanifold from Levi Civita connection is non-metric, we get

$$\begin{aligned}\nabla_X g(Y,Z) &= \nabla^0_X g(Y,Z) + \pi(Y)g(FX,Z) \\ &= \sum_{i=1}^{r}\{h^l_i(X,Y)\eta_i(Z) + h^l_i(X,Z)\eta_i(Y) \\ &\quad - \pi(Y)g(FX,Z) - \pi(Z)g(FX,Y)\}\end{aligned} \quad (3.11)$$

where

$$\eta_i(Z) = g(N_i,Z), \qquad 1 \le i \le r, \quad (3.12)$$

for any X,Y and $Z \in \Gamma(TM)$, $N_i \in \Gamma(ltr(TM))$.

Also from (**??**), the torsion tensor of the connection ∇ is

$$T(X,Y) = \pi(Y)FX - \pi(X)FY \quad (3.13)$$

then from (**??**) and (**??**), we have:

Thus we state the following theorem:

Theorem 3.1. The induced connection on a screen distribution of a lightlike submanifold of semi-Riemannian manifold equipped with quarter-symmetric non-metric connection, is quarter-symmetric non-metric connection.

Theorem 3.2. Let $(M,g,S(TM),S(TM^{\perp}))$be a lightlike submanifold of a semi-Riemannian manifold $(\tilde{M},\tilde{g})$ admitting quarter-symmetric non-metric connection then the screen distribution $S(TM)$ is integrable if and only if the second fundamental form of the screen distributionh_i^* is symmetric.

Proof. Since the torsion tensor T of ∇ does not vanish and by using

$$X = PX + \sum_{i=1}^{r} \eta_i(X)\xi_i \qquad (3.14)$$

We can get

$$\begin{aligned}[X,Y] =&\nabla^*_X PY - \nabla^*_Y PX + \sum_{i=1}^{r} \eta_i(X)A^*_{\xi_i}Y - \eta_i(Y)A^*_{\xi_i}X \\ &+ \sum_{i=1}^{r}\{h_i^*(X,PY) - h_i^*(Y,PX) + X(\eta_i(Y)) - Y(\eta_i(X))\}\xi_i \\ &+ \sum_{i,j=1}^{n}\{\eta_i(Y)u_{ij}(X) - \eta_i(X)u_{ij}(Y)\}\xi_j \\ &+ \{\pi(PY) + \sum_{i=1}^{r}\eta_i(X)\lambda_i\}PY - \{\pi(PX) + \sum_{i=1}^{r}\eta_i(Y)\lambda_i\}PX \\ &+ \sum_{i=1}^{r}\{\pi(PX)\eta_i(Y) - \pi(PY)\eta_i(X)\}\xi_i \qquad (3.15)\end{aligned}$$

Taking the scalar product of the above equation with N_i, $i \le i \le r$, we have

$$\begin{aligned}\tilde{g}([X,Y],N_i) =& h_i^*(Y,PX) - h_i^*(X,PY) + X(\eta_i(Y)) - Y(\eta_i(X)) \\ &+ \sum_{j=1}^{r}\eta_i(Y)u_{ij} - \eta_i(X)u_{ij}(Y) \\ &+ \pi(PX)\eta_i(Y) - \pi(PY)\eta_i(X) \qquad (3.16)\end{aligned}$$

By help of previous two equation

$$\begin{aligned}2d\eta_i(X,Y) =& h_i^{0i}(Y,PX) - h_i^{0i}(X,PY) + \sum_{j=1}^{r}\{\eta_i(X)\{u_{ij}(Y) \\ &+ \pi(PY)\} - \eta_i(Y)\{u_{ij}(X) + \pi(PX)\} \qquad (3.17)\end{aligned}$$

$$2d\eta_i(PX,PY) = h_i^*(PY,PX) - h_i^*(PX,PY) \qquad (3.18)$$

proves the theorem. □

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