On a Half Lightlike Submanifold admitting aQuarter-symmetric non-metric Connection

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Abstract

The object of the present paper is to study On a Half Lightlike Submanifold admitting a Quarter-symmetric non-metric Connection and some properties of quarter-symmetric non-metric connection.

***2000 Mathematics Subject Classification:* 53C15, 53C25.**

Keywords Quarter-symmetric non-metric connection, Lightlike submanifold, Semi-Riemannian manifolds.

Half-lightlike Submanifolds of a semi-Riemannian Manifold with a Quarter-symmetric Non-metric Connection

In this chapter we study half-lightlike submanifolds of a semi-Riemannian manifold admitting a quarter-symmetric non-metric connection. In the present chapter we obtain the structure equations for Einstein half-lightlike submanifold with a quarter-symmetric non-metric connection. We prove some results on irrotational semi-Riemannian manifolds admitting quarter-symmetric non-metric connection.

1 Preliminaries

A submanifold (M, g) of the manifold $\tilde{M}, \tilde{g}$ is called half-lightlike submanifold if the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ is a subbundle of tangent bundle TM and normal bundle $TM^{\perp}$ of *rank* 1. Therefore there exists non-degenerate complementary distributions $S(TM)$ and $S(TM^{\perp})$ of Rad(TM) in TM and $TM^{\perp}$. These distributions are called screen and co-screen distribution of M, such that

$$TM = \mathrm{Rad(TM)} \oplus_{orth} \mathrm{S(TM)} \qquad (1.1)$$

and

$$TM^{\perp} = \mathrm{Rad(TM)} \oplus_{orth} \mathrm{S(TM^{\perp})} \qquad (1.2)$$

where $\oplus_{orth}$ is orthogonal direct sum. The half-lightlike submanifold is denoted by $(M, g, S(TM))$.

The algebra of smooth functions on the $S(TM)$ is denoted by $\Gamma(S(TM))$. Similarlythe algebra of

$$S(TM)^{\perp} = S(TM^{\perp}) \oplus_{orth} S(TM^{\perp})^{\perp}$$

where $S(TM^{\perp})^{\perp}$ is orthogonal complementary of $S(TM^{\perp})$. For any null section ξ of $Rad(TM)$ on a coordinate neighbourhood $U \subset M$, there exists uniquely defined lightlike vector bundle $(ltrTM)$ and a null vector field N of $ltr(TM)_{|U}$ such that

$$\tilde{g}(\xi, N) = 1,$$
$$\tilde{g}(N, N) = \tilde{g}(N, X) = \tilde{g}(N, L) = 0, \qquad \forall X \in \Gamma(S(TM)).$$

where N is *lightlike transversal vector field*, $ltr(TM)$ is *lightlike transversal vector bundle* and $tr(TM)$ is *transversal vector bundle* with respect to screen distribution [**?**]. Thus tangent vector field $T\tilde{M}$ can be expressed as

$$\begin{aligned} TM &= T\tilde{M} \oplus tr(TM) \\ &= [Rad(TM) \oplus_{orth} S(TM)] \oplus [S(TM^{\perp}) \oplus_{orth} ltr(TM)] \\ &= [Rad(TM) \oplus ltr(TM)] \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}) \qquad (1.3) \end{aligned}$$

Given a screen distribution $S(TM)$, there exists a unique complementary vector bundle $tr(TM)$ to TM in $T\tilde{M}_{|M}$. Using (1.1),(1.2) and (1.3), there exists a local quasi orthonormal frame field of $\tilde{M}$ along M given by

$$F = \{\xi, N, L, W_{\alpha}\}, \qquad \alpha \in \{1, 2, \ldots\ldots\ldots, m\}$$

where $\{W_{\alpha}\}$ is orthonormal frame field of $S(TM)_{|U}$. We shall assume that ξ is tangent to M and X, Y, Z & $W \in \Gamma(TM)$. Let P be the projection morphism of TM on $S(TM)$ with respect to decomposition of (1.1) and (1.2). Then the The local Gauss and Weingarten formula of M and $S(TM)$ are

$$\tilde{\nabla}_X Y = \nabla_X Y + D_1(X, Y)N + D_2(X, Y)L \qquad (1.4)$$

$$\tilde{\nabla}_X N = -A_N X + \rho_1(X)N + \rho_2(X)L \qquad (1.5)$$

$$\tilde{\nabla}_X L = -A_L X + \varphi(X)N \qquad (1.6)$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi \qquad (1.7)$$

$$\nabla_X \xi = -A^*_{\xi} X - \rho_1(X)\xi, \qquad (1.8)$$

where D_1, D_2 are local *lightlike* and *screen second fundamental forms* of M and C is *local second fundamental* form on $S(TM)$,A_N, A^*_{ξ} and A_L are called the *shape operators*, ρ_1, ρ_2 and φ are one forms on TM and ∇ and $\tilde{\nabla}^*$ are induced linear connection on TM and

$S(TM)$ We have

$$h(X, Y) = D_1(X, Y)N + D_2(X, Y)L \tag{1.9}$$

is second fundamental form tensor of M.

2 Structure Equations

Let $\widetilde{M}$ is manifold equipped with quarter-symmetric non-metric connection $\tilde{\nabla}$ and the induced connection on the submanifold M. Consider ∇, $\tilde{R}$, R and R^* are curvature tensors of semi-symmetric non-metric connection on $\widetilde{M}$, the induced connection on M and induced connection on ∇^* in $S(TM)$ respectively. Using Gauss-Weingarten formula on M and $S(TM)$ we get Gauss-Codazzi equation for TM and $S(TM)$: The structure equations for quarter-symmetric non-metric connection are defined as

$$\begin{aligned}
\tilde{R}(X, Y)Z &= \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z \\
&= \tilde{\nabla}_X \left[\tilde{\nabla}_Y Z + D_1(Y, Z)N + D_2(Y, Z)L\right] \\
&\quad - \nabla_Y [\nabla_X \tilde{Z} + D_1(X, Z)N + D_2(X, Z)L] \\
&\quad - \left\{\nabla_{[X,Y]}Z + D_1([X, Y], Z)N + D_2([X, Y], Z)L\right\}
\end{aligned}$$

by using equation (1.4)

$$\begin{aligned}
\Rightarrow \tilde{R}(X, Y)Z &= R(X, Y)Z + D_1(X, Z)A_N Y - D_1(Y, Z)A_N X \\
&\quad + D_2(X, Z)A_L Y - D_2(Y, Z)A_L X \\
&\quad + \{(\nabla_X D_1)(Y, Z) - (\nabla_Y D_1)(X, Z) + \rho_1(X)D_1(Y, Z) \\
&\quad - \pi(X)D_1(Y, Z) - \rho_1(Y)D_1(X, Z) + \varphi(X)D_1(X, Z) \\
&\quad - \varphi(Y)D_2(X, Z)\}N + \{(\nabla_X D_2)(Y, Z) \\
&\quad - (\nabla_Y D_2)(X, Z) + \rho_2(X)D_1(Y, Z) - \pi(Y)D_2(X, Z)\}L
\end{aligned} \tag{2.1}$$

is first *structure equation* for half-lightlike submanifold equipped with quarter-symmetricnon-metric connection. Similarly *other structure equations* are

$$\begin{aligned}
R(X, \tilde{Y})N &= -\nabla_X(A_N Y) + \nabla_Y(A_N X) + A_N[X, Y] \\
&\quad + \{\rho_1(X)A_N Y - \rho_1(Y)A_N X + \rho_2(X)A_L Y \\
&\quad - \rho_2(Y)A_L X + D_1(Y, A_N X) - D_1(X, A_N Y) \\
&\quad + 2d\rho_1(X, Y) + \varphi(X)\rho_2(Y) \\
&\quad - \varphi(Y)\rho_2(X)\}N + \{D_2(Y, A_N X) - D_2(X, A_N Y) \\
&\quad + 2d\rho_2(X, Y) + \rho_1(Y)\rho_2(X) - \rho_1(X)\rho_2(Y)\}L
\end{aligned} \tag{2.2}$$

$$R(X,\tilde{Y})L = -\nabla_X(A_L Y) + \nabla_Y(A_L X) + A_L[X,Y] + \varphi(X)A_N Y - \varphi(Y)A_N X$$
$$+ \{D_1(Y, A_L X) - D_1(X, A_L Y) + 2d\varphi(X,Y) + \rho_1(X)\varphi(Y) - \rho_1(Y)\varphi(X)\}N$$
$$+ \{D_2(Y, A_L X) - D_2(X, A_L Y) + \varphi(Y)\rho_2(X) - \varphi(X)\rho_2(Y)\}L \quad (2.3)$$

$$R(X,Y)PZ = R^*(X,Y)PZ + C(X,PZ)A^*_\xi Y - C(Y,PZ)A_\xi X$$
$$+ \{(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) + C(X,PZ)[\rho_1(Y) + \pi(Y)]$$
$$- C(Y,PZ)[\rho_1(X) + \pi(X)]\}\xi \quad (2.4)$$

and

$$R(X,Y)\xi = -\nabla^*_X(A^*_\xi Y) + \nabla^*_Y(A^*_\xi X) + A^*_\xi[X,Y]$$
$$+ \rho_1(Y)A^*_\xi X - \rho_1(X)A^*_\xi(Y) + \{C(Y, A^*_\xi X)$$
$$- C(X, A^*_\xi Y) - 2d\rho_1(X,Y)\}\xi \quad (2.5)$$

for any X, Y and $Z \in \Gamma(TM)$.

Let P is projection of TM on $S(TM)$ then

$$X = PX + \eta(X)\xi, \qquad \forall X \in \Gamma(TM)$$

Let $\{F_1, F_2, ...F_{m-1}\}$ is an orthogonal basis of $\Gamma(S(TM))$. Here $(\tilde{M}, \tilde{g})$ is m-dimensional semi-Riemannian manifold with index $q \geq 1$ and (M, g) is lightlike submanifold of codimension 2 of $\tilde{M}$. A semi-Riemannian manifold $\tilde{M}$ of constant curvature c is called a semi-Riemannian space form and it is denoted by $\tilde{M}(c)$.The curvature tensor $\tilde{R}$ of $\tilde{M}(c)$ is given by

$$\tilde{R}(X,Y)Z = c[\tilde{g}(Y,Z)X + \tilde{g}(X,Z)Y], \qquad \forall X,Y,Z \in \Gamma(T\tilde{M}) \quad (2.6)$$

Taking scalar product with L and ξ to equation (2.6) we obtain

$\tilde{g}(\tilde{R}(X,Y)Z, L) = 0$ and $g(\tilde{R}(X,Y)Z, \xi) = 0$, $\forall X, Y, Z \in \Gamma(TM)$.

By help of these equations and equation (1.4), we get

$$\tilde{R}(X,Y)Z = R(X,Y)Z + D_1(X,Z)A_N Y - D_1(Y,Z)A_N X$$
$$+ D_2(X,Z)A_L Y - D_2(Y,Z)A_L X, \qquad \forall X,Y,Z \in \Gamma(T\tilde{M}) \quad (2.7)$$

Definition 2.1. A half-lightlike submanifold M of a semi-Riemannian manifold $\tilde{M}$ is screen conformal if the second fundamental form D_1 and C satisfies

$$C(X,PY) = \varphi D_1(X,Y), \qquad \forall X, Y \in \Gamma(TM).$$

where φ is non vanishing function on a coordinate neighbourhood U in M.

Theorem 2.2. Let M be irrotational half-lightlike submanifold of a Lorentzian space form $\tilde{M}(c)$ admitting quarter-symmetric non-metric connection such that ζ is tangent to M. If M is screen conformal then $c = 0$.

Definition 2.3. A vector field X on a manifold is conformal killing if the Lie derivatives $L_X\tilde{g} = -2\delta\tilde{g}$, where δ is scalar function and the Lie derivative is defined as

$$(\tilde{L}_X\tilde{g})(Y,Z) = X(\tilde{g}(Y,Z)) - \tilde{g}([X,Y],Z) - \tilde{g}(Y,[X,Z]), \forall X,Z \in \Gamma(T\tilde{M})$$

If $\delta = 0$ then X is called killing vector field on M.

Theorem 2.4. Let M be half-lightlike submanifold of a semi-Riemannian manifold $\tilde{M}$ admitting quarter-symmetric non-metric connection if the canonical normal vector field L is a conformal killing one, then L is killing vector field.

Proof. By equation and we have

$$(L_X\tilde{g})(Y,Z) = \tilde{g}(\tilde{\nabla}_Y X, Z) + \tilde{g}(Y, \nabla_Y X) - 2\pi(X)\tilde{g}(Y,Z)$$
$$\tilde{\nabla}_X L = -A_L X + \varphi(X)N$$

Since L is conformal killing vector field, so by help of equation and we have

$$\tilde{g}(\tilde{\nabla}_X L, Z) = -\tilde{g}(A_L X, Y) + \varphi(X)\tilde{g}(N,Y)$$
$$= -D_2(X,Y) + \varphi(X)\eta(Y) + \varphi(X)\tilde{g}(N,Y)$$
$$= -D_2(X,Y) + \varphi(X)\eta(Y) + \varphi(X)\eta(Y)$$

Therefore $\tilde{g}(\tilde{\nabla}_X L, Y) = -D(X,Y)$

Therefore $(L_X\tilde{g})(Y,Z) = -2D_2(X,Y), \quad \forall X,Y \in \Gamma(TM)$

Thus we have

$$D_2(X,Y) = \delta g(X,Y), \quad \forall X,Y \in \Gamma(TM)$$

Putting $X = \xi$ and $Y = \xi$, we get

$$\delta = 0$$

Therefore L is killing vector field. □

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