A CHARACTERIZATION OF OPEN DISTANCE PATTERN UNIFORM CHORDAL GRAPHS AND DISTANCE HEREDITARY GRAPHS

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Abstract

Given an arbitrary nonempty subset A of vertices in a (p, q)-graph $G = (V, E)$, each vertex u in G is associated with the set $f^o_A(u) = \{d(u, v) : v \in A, u \neq v\}$, where $d(x, y)$ denotes the usual distance between the vertices x and y in G, called its *open A-distance pattern*. G is called an *open distance-pattern uniform* (or, in short, *odpu*)*-graph* if there exists a nonempty subset $A \subseteq V(G)$ such that $f^o_A(u)$ is independent of the choice of $u \in V(G)$, where the *set-valued function* (or, *set-valuation*) f^o_A is called the *open distance pattern uniform* (or, an *odpu*-) *labeling* of G and A is called an *odpu-set* of G. The minimum cardinality of an odpu-set in G, if it exists, is the odpu-number $\varsigma(G)$ of G. Given any property P, we establish characterization of odpu-graph with property P. In this paper, we characterize odpu-chordal graphs and thereby characterize interval graphs, split graphs, strongly chordal graphs and ptolemaic graphs that are odpu-graphs. Wealso characterize odpu-distance hereditary graphs.

Key Words and Phrases: Open distance-pattern uniform graphs, Open distance- pattern uniform (odpu-) set, Odpu-number, odpu-chordal graphs, odpu-interval graphs, odpu-split graphs, odpu-strongly chordal graphs, odpu-ptolemaic graphs,odpu-distance hereditary graphs.

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1 Introduction

All graphs considered in this paper are finite, simple, connected and non-trivial, as treated in F. Harary [5]. New terms and notations will be introduced as and when required.

Given an arbitrary nonempty subset A of vertices in a (p, q)-graph $G = (V, E)$, the *A-distance pattern* of a vertex u in G has been defined to be the set $f_A(u) = \{d(u, v) : v \in A\}$, where $d(x, y)$ denotes the usual distance between the vertices x and y in G; clearly, $0 \in f_A(u) \Leftrightarrow d(u, v) = 0$ for some $v \in A \Leftrightarrow u = v$ and $u \in A$. The last observation motivated associating with each vertex u of G its *open A- distance pattern* (or, A-'odp' in short) $f^o(u) = \{d(u, v) : v \in A, u \neq v\}$. We intend to study graphs G which possess a nonempty $A \subseteq V(G)$ such that $f^o(u)$ is independent of the choice of u (see [2]); we call such graphs *odp-uniform graphs* (or, simply, 'odpu-graphs'), where the *set-valued function* (or, *set-valuation*) f^o is called the *open distance pattern uniform* (or, a *odpu-*) *labeling* and A is called an *odpu-set* of G. The minimum cardinality of an odpu-set in G, if it exists, is the *odpu-number* $\varsigma(G)$ of G.

It is proved that a graph G with radius $r(G)$ is an odpu graph if and only if the open distance pattern of every vertex in G is $\{1, 2, \cdots, r(G)\}$ and proved that a graph is an odpu-graph if and only if its centre $C(G)$ is an odpu-set, thereby characterizing odpu-graphs, which in fact invokes a method to check the existence of an odpu-set for any given graph (see [2]).

In this paper, we characterize odpu-chordal graphs and thereby characterize interval graphs, split graphs, strongly chordal graphs and ptolemaic graphs that are odpu-graphs. We also characterize odpu-distance hereditary graphs.

We shall need the following definitions and previous results.

Proposition 1. [2] For any graph G, $\varsigma(G) = 2$ if and only if there exist at least two vertices $x, y \in V(G)$ such that $d(x) = d(y) = |V(G)| - 1$.

Proposition 2. [2] There is no graph having odpu-number three.

Proposition 3. [2] A graph G is an odpu graph if and only if its centre $Z(G)$ is an odpu set and hence $|Z(G)| \geq 2$.

Proposition 4. [2] All self-centered graphs are odpu graphs.

Theorem 1.1. [2] The shadow graph of any complete graph K_n, $n \geq 3$ is an odpu-graph with odpu-number $n + 2$ (The shadow graph $S(G)$ of a graph G is obtained from G by adding for each vertex v of G a new vertex v', called the shadow vertex of v, and joining v' to all the neighbors of v in G).

Theorem 1.2. [2] Every odpu-graph G satisfies, $r(G) \leq d(G) \leq r(G)+1$ where $r(G)$ and $d(G)$ denote the radius and diameter of G respectively.

A graph G is said to be chordal if every cycle of length at least 4 has a chord, that is, an edge joining non-consecutive vertices of the cycle. A subset $S \subseteq V(G)$ is called a vertex separator of G for non-adjacent vertices a and b (or $a - b$ separator of G) if, in $G - S$, (the graph obtained from G by the removal of the vertices of S and incident edges) the vertices a and b belong to distinct connected components. Let $S(a, b, G)$ be the set of all $a - b$ separators of G. If no proper subset of $S \in S(a, b, G)$ belongs to $S(a, b, G)$, then S is called a minimal $a - b$ separator of G. Let $S_0(a, b, G)$ be the set of all minimal $a - b$ separators of G. The following propositions (cf. [8]) needed in proving our main results.

Proposition 5. [8] If G is a chordal graph and x, y are distinct nonadjacent vertices in $\langle Z(G)\rangle$, the central graph of G, and $S_0 \in S_0(x, y, G)$, then the following conditions hold.

(1)$S_0 \subseteq Z(G)$.

(2)There are at least two distinct vertices $z_1, z_2 \in S_0$ such that for every $i = 1, 2$either $d_G(x, z_i) = 1$ or $d_G(y, z_i) = 1$. In particular $|S_0| \geq 2$.

Proposition 6. [8] If G is a chordal graph and x, y are distinct nonadjacent vertices in $\langle Z(G)\rangle$, the central subgraph of G, then, $S_0(x, y, G) = S_0(x, y, \langle Z(G)\rangle)$.

Proposition 7. [8] If G is a chordal graph, then $\langle Z(G)\rangle$ is connected.

Proposition 8. [8] If G is a chordal graph, then $d(\langle Z(G)\rangle) \leq 3$.

Proposition 9. [8] If G is a chordal graph with $(r, d) = (2, 3)$ then $d(\langle Z(G)\rangle) \leq$
2.

Proposition 10. [8] There are no self-centered chordal graph G with $(r, d) = (3, 3)$. Consequently for a self-centered chordal graph G, $r(G) \leq 2$.

A distance-hereditary graph (cf. [6]) is a connected graph, which preserves the distance function for induced subgraphs. That is, the distance between any two non-adjacent vertices of any connected induced subgraph of such a graph is same as the distance between these two vertices in the original graph.

Proposition 11. [6] Let G be a distance-hereditary graph, and let u be a vertex of G. Then, for each positive integer $k \leq$ max $d(u, v) : v \in V$, and vertices v, w of the same connected component of $N_k(u)$ we have, $N(v) \cup N_{k-1}(u) = N(w) \cup N_{k-1}(u)$, where the i^{th} neighborhood of the vertex u, $N_i(u) = \{v : d(u, v) = i\}$.

A graph G is an interval graph if and only if there is a one-to-one correspon- dence between its vertices and a set of intervals on the real line, such that two vertices are adjacent if and only if the corresponding intervals have an intersec- tion. It is also well-known that a graph G is an interval graph if and only if G is chordal and asteroidal triple free, where asteroidal triple is a set of three distinct vertices (v_1, v_2, v_3) such that there exists a path connecting v_i and v_j that

contains no neighbor of v_k; $(I, k\ j)$, for every combination of $1 \leq i, j, k \leq 3$

(cf. [4]).

A graph is a split graph if and only if its vertices can be partitioned into an independent set and vertices which induces a clique. For simplicity, given a split graph G, we call a vertex partition of $V(G) = (V_I, V_C)$, such that V_I is an independent set and vertices in V_C induce a clique, as an I-C-decomposition of

G. It is known that, a chordal graph whose complement is also a chordal graph is equivalent to a split graph (cf. [4]).

A graph G is strongly chordal if and only if G is chordal and every even cycle of length six or more contains a chord splitting the cycle into two odd length paths (cf. [4]).

Ptolemaic graphs are exactly those graphs that are both chordal and distance- hereditary (cf. [7]).

2 ODPU-Chordal Graphs

Following theorem establish the characterization for odpu-chordal graphs.

Theorem 2.1. *A chordal graph G is an odpu-graph if and only if $r(G) = r(\langle Z(G)\rangle)$.*

Proof. Let G be a chordal graph with $r(G) = r(\langle Z(G)\rangle)$. Since, $r(\langle Z(G)\rangle) \leq 2$,there are two possibilities.

(i) $r(G) = r(\langle Z(G)\rangle) = 1$. Then, there are two universal vertices in G and hence by Proposition 1, G is an odpu-graph.

(ii) $r(G) = r(\langle Z(G)\rangle) = 2$. By proposition 6, $d_G(u, v) = d_{\langle Z(G)\rangle}(u, v)\ \forall u, v \in Z(G)$. Let $u \in Z(G)$. Since $r(\langle Z(G)\rangle) = 2$, there exists a vertex $v \in Z(G)$ such that $d(u, v) = 2$. Hence $2 \in f^o_M(u)\ \forall u \in Z(G)$. Now the connectedness of $\langle Z(G)\rangle$ ensures the existence of a vertex $w \in Z(G)$ such that $d(u, w) = 1$. Therefore $f^o_M(u) = \{1, 2\}\ \forall u \in Z(G)$. Now let $u \in V(G) - Z(G)$.

Claim:i $2 \in f^o_M(u)$.

If there is no vertex $v \in Z(G)$ such that $d(u, v) = 2$, then u is adjacent to all the vertices of $Z(G)$. Since $r(\langle Z(G)\rangle) = 2$, there exist two vertices x and y in $Z(G)$ such that $d(x, y) = 2$. Now consider $S_0(x, y)$. Since $u \notin Z(G)$, $u \notin S_0(x, y)$. But $G - S_0(x, y)$ contains an $x-y$ (xuy)-path, which is a contradiction to the fact that G is chordal and hence $S_0 \subseteq Z(G)$. Hence, there exists a vertex $v \in Z(G)$ such that $d(u, v) = 2$, implies $2 \in f^o_M(u)$.

Claim:ii $1 \in f^o_M(u)$.

Let u is not adjacent to any of the vertices of $Z(G)$. Let $x, y \in Z(G)$ such that $d(x, y) = 2$. Since $r(G) = r(\langle Z(G)\rangle) = 2$, $d(x, u) = d(y, u) = 2$. Thus, there exist vertices w and z in $V(G) - Z(G)$ such that (xwu) and (yzu) are paths of length 2 in G. If $w = z$, then the path $(xwy) \in V(G) - S_0(x, y)$, leads the contradiction. Otherwise the path $(xwuzy) \in V(G) - S_0(x, y)$, a contradiction that $S_0(x, y) \subseteq Z(G)$. Hence, there exists a vertex $v \in Z(G)$ such that $d(u, v) = 1$ and hence $1 \in f^o_M(u)\ \forall u \in V(G) - Z(G)$. Therefore $f^o_M(u) = \{1, 2\}\ \forall u \in V(G) - Z(G)$.

Hence G is an odpu-graph.

Conversely, assume G is a chordal-odpu-graph. Since $r(G) \leq 2$ and $\langle Z(G)\rangle$ is connected, $r(\langle Z(G)\rangle) \leq 2$.

If $r(\langle Z(G)\rangle) = 1$, then $f^o\ (u) = \{1\}_M$ $\forall u \in V\ (G)$. Thus there exists at least two universal vertices in G. Hence $r(G) = 1$.

If $r(\langle Z(G)\rangle) = 2$, $f^o\ (u) = \{1,_M 2\}$ $\forall u \in V\ (G)$. Since $Z(G)$ is an odpu-set, the distance from any vertex of G to any vertex of $Z(G)$ is less than or equal to 2. Thus $r(G) = 2$ and hence, $r(G) = r(\langle Z(G)\rangle)$.

Corollary 2.2. *A chordal graph G is an odpu-graph then $\langle Z(G)\rangle$ is self- centered.*

Proof. Let G be a chordal, odpu-graphs. If $\langle Z(G)\rangle$ is not self-centered, then $r(\langle Z(G)\rangle)$ $e_{\langle Z(G)\rangle}(u)$ $d(\langle Z(G)\rangle)$. Hence there exist vertices $u, v \in Z(G)$ such that
$e_{\langle Z(G)\rangle}(v)$. Let $e_{\langle Z(G)\rangle}(u) = d(\langle Z(G)\rangle)$, $e_{\langle Z(G)\rangle}(v) = r(\langle Z(G)\rangle)$ and
$M = Z(G)$. Since $d_{\langle Z(G)\rangle}(u, v) = d_G(u, v)$ $\forall u, v \in Z(G)$, $f^o\ (u)$ has an$_M$ element
$d(\langle Z(G)\rangle)$, a contradiction.

Remark 2.3. Converse of Corollary 2.2 need not be true. For example, P_4 ischordal with $\langle Z(P_4)\rangle = P_2$, which is self-centered but P_4 is not an odpu-graph.

Since, interval graphs, split graphs, block graphs, Ptolemaic graphs, strongly chordal graphs and maximal outerplanar graphs are subclasses of chordal graphs, (cf. [4]), the following Corollary is immediate from Theorem 2.1 and Corollary2.2.

Corollary 2.4. *The following classes of graphs G: interval graphs, split graphs, block graphs, Ptolemaic graphs, strongly chordal graphs or maximal outerplanar graphs are odpu-graphs if and only if $r(G) = r(\langle Z(G)\rangle)$ and hence $\langle Z(G)\rangle$ is self-centered.*

3 ODPU-Distance Hereditary Graphs

Following Theorem establishes the characterization of odpu-distance hereditary graphs. Further we show that the central subgraph $\langle Z(G)\rangle$ of distance hereditary- odpu-graph G is either self-centered or disconnected.

Theorem 3.1. *Let G be a distance hereditary graph with connected $\langle Z(G)\rangle$. Then G is an odpu-graph if and only if $r(G) = r(\langle Z(G)\rangle)$.*

Proof. Let G be a distance hereditary graph with $\langle Z(G)\rangle$ is connected. Let G is an odpu-graph. If $r(G) \neq$ $r(\langle Z(G)\rangle)$, then $r(\langle Z(G)\rangle) < r(G)$. Since $\langle Z(G)\rangle$ is connected, $d_{\langle Z(G)\rangle}(u, v) = d_G(u, v)$ $\forall u, v \in Z(G)$. But since $r(\langle Z(G)\rangle) < r(G)$, $r(G) \notin f^o\ (x)$ $\forall x \in Z(G)$, which is not possible. Hence, $r(G) = r(\langle Z(G)\rangle)_M$

Conversely, let $r(G) = r(\langle Z(G)\rangle) = r$. Let $u \in Z(G)$. Then there exists a vertex $v \in Z(G)$

such that $d(u, v) = r$. Since $\langle Z(G)\rangle$ is connected, $f^o_M(u) = \{1, 2, \ldots, r\}$. Now let $u \in V(G) - Z(G)$. If $1 \notin f^o_M(u)$, then $N_1(u) \cap Z(G) = \varphi$, where $N_t(u) = \{v \in V(G) : d(u, v) = t\}$. Let k be the largest integer such that $N_k(u) \cap Z(G) \neq \varphi$ and let $w \in N_k(u) \cap Z(G)$. Then, since, $d_G(u, w) = k$, $e_G(w) \geq k$. But since $u, N_1(u) \not\subseteq Z(G)$, by Proposition 11, $e_{Z(G)}(w) \leq k - 2$. Thus $r(G) \neq r(\langle Z(G)\rangle)$, a contradiction. Therefore $1 \in f^o_M(u)\ \forall u \in V(G) - Z(G)$. Now if $r \notin f^o_M(u)$, then $N_r(u) \cap Z(G) = \varphi$. Since $u \notin Z(G)$, by Proposition 11, $e_{\langle Z(G)\rangle}(w) \leq r - 1$ for every $w \in Z(G)$. Thus $r(\langle Z(G)\rangle) < r(G)$, a contradiction. Hence $r \in f^o_M(u)$.

Since $1, r \in f^o_M(u)$, $f^o_M(u) = \{1, 2, \ldots, r\}\ \forall v \in V(G) - Z(G)$. Hence the theorem.

Corollary 3.2. *For a distance hereditary odpu-graph G, either, $\langle Z(G)\rangle$ is disconnected or it is self-centered.*

Proof. Let G be a distance hereditary odpu-graph. If $\langle Z(G)\rangle$ is connected, then we prove $\langle Z(G)\rangle$ is self-centered. If not, let $r(\langle Z(G)\rangle) \neq d(\langle Z(G)\rangle)$. Let $u, v \in Z(G)$ such that $e_{\langle Z(G)\rangle}(u) = r(\langle Z(G)\rangle)$ and $e_{\langle Z(G)\rangle}(v) = d(\langle Z(G)\rangle)$. Since G is distance hereditary, $d_{(\langle Z(G)\rangle)}(x, y) = d_G(x, y)\ \forall\ x, y \in Z(G)$. Thus $d(\langle Z(G)\rangle) \in f^o_M(v)$ and $d(\langle Z(G)\rangle) \in f^o_M(u)$. Which is a contradiction. Hence the theorem.

Remark 3.3. Converse of Corollary 3.2 need not be true. For, P_4 is distance hereditary and $\langle Z(P_4)\rangle = P_2$, which is self-centered, but P_4 is not an odpu-graph.

Theorem 3.4. *For every integer $n \geq 2$ there is an odpu-graph G of order $n(n+2)$ such that its central subgraph $\langle Z(G)\rangle$ is disconnected.*

Proof. Consider two disjoint complete graphs $G_1 = K_n$ and $G_2 = K_n$ of order n. Now add all edges between these two complete graphs and subdivide each of the new edges of the bipartite subgraph between G_1 and G_2 by one (a degree 2 vertex) to get a graph G of order $n(n + 2)$. Let the vertices of G_1 are $\{v_1, v_2, \ldots, v_n\}$ and the vertices of G_2 are $\{u_1, u_2, \ldots, u_n\}$ and $w_{i,j}$ be the vertex which subdivides the earlier edge v_iu_j of the bipartite graph. Then all vertices $w_{i,j}$'s have eccentricity 3 in G and the new graph has radius 2 and diameter 3. Also the central subgraph $\langle Z(G)\rangle$ is the disjoint union of the complete graphs G_1 and G_2. Hence $\langle Z(G)\rangle$ is disconnected.

Now, we prove that G is an odpu-graph. Let $M = Z(G) = V(G_1) \cup V(G_2)$. For each $v_i \in V(G_1)$, there exist a $v_j \in V(G_1)$ such that $d(v_i, v_j) = 1$ and hence $1 \in f^o_M(v_i)$. Now $d(v_i, u_j) = 2;\ \forall u_j \in V(G_2)$ and hence $2 \in f^o_M(v_i)$. Hence $f^o_M(v_i) = \{1, 2\}\ \forall\ v_i \in V(G_1)$. Similarly, $f^o_M(u_j) = \{1, 2\}\ \forall\ u_j \in V(G_2)$. Now, each vertex $w_{i,j}$ is adjacent to exactly v_i and u_j and hence, $d(w_{i,j}, v_i) = 1$ and $d(w_{i,j}, u_j) = 1$. Hence $1 \in f^o_M(w_{i,j})\ \forall\ i, j$. Since, $d(w_{i,j}, v_k) = 2;\ \forall\ i \neq k$ and $d(w_{i,j}, u_k) = 2;\ \forall\ j \neq k$, $2 \in f^o_M(w_{i,j})\ \forall\ i, j$. Hence $f^o_M(w_{i,j}) = \{1, 2\}\ \forall\ i, j$ and hence G is an odpu-graph.

Conclusion

The characterization of odpu-graphs leads to an interesting condition $r(G) = r(\langle Z(G)\rangle)$, for many important classes of graphs such as chordal graphs, interval graphs, split graphs, strongly chordal graphs, ptolemaic graphs, distance heredi- tary graphs. However, this characterization is not in general, a characterization for all odpu-graphs. For example, by Theorem 3.4 there are classes of odpu- graphs with radius 2 and disconnected centre. ie., $r(\langle Z(G)\rangle) = \infty$. Thus there are more classes of odpu-graphs which do not come under this characterization. We leave it for further scope of investigations.

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