THE VISCOUS INCOMPRESSIBLE FLOW BETWEEN TWO PARALLEL POROUS FLAT PLATES

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ABSTRACT : In this paper we study the exact solution of Navier-Stokes equation for unsteady flow of a viscous incompressible flow through a channel bounded by two parallel porous flat plates, one in uniform motion and the other at rest with uniform suction at the lower plate and equal injection at the upper plate under the influence of pressure gradient expressed as a linear function of time.

Keywords: unsteady laminar flow, viscous incompressible flow, parallel porous flat plates, Navier-Stokes equation.

1. Equation of Motion

Let us consider the dimensional laminar flow of a viscous incompressible fluid between two parallel porous flat plate kept at a distance 2*h*. Let a rectangular Cartesian system of co-ordinates axis be selected with the axis of *X* along the midway between the plates and *Y* axis perpendicular to it. In the present problem, the laminar flow of a viscous incompressible fluid confined between two parallel porous flat plates, one of which is at rest and other moving in its own plane with uniform velocity *U*. Let *u*, *v* be the velocity components along *X* and *Y* axis respectively. The Navier-Stokes equations for unsteady laminar flow of viscous incompressible fluid in absence of body forces are given by

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\upsilon\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right] \tag{1}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\upsilon\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right] \tag{2}$$

and equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{3}$$

when υ and ρ denotes the kinetic viscosity and density of the fluid.
Boundary conditions are

$$\left.\begin{array}{ll} t>0, \quad \text{for } y=0 & \\ u(0,t)=0 & \\ v(0,t)=v_0<0 & \text{(suction)} \\ \text{for } y=-h & \\ u(h,t)=U & \\ v(h,t)=v_0<0 & \text{(injection)} \end{array}\right\} \tag{4}$$

where v_0 being constant and *h* is the distance between the plate.
It assumed that the longitudinal velocity is independent of *x*, so that

$$\frac{\partial u}{\partial x}=0 \tag{5}$$

As the suction is uniform

$$\left.\begin{array}{l} \dfrac{\partial v}{\partial t}=0 \\ \dfrac{\partial v}{\partial x}=0 \\ \text{and hence} \quad \dfrac{\partial^2 v}{\partial x^2}=0 \end{array}\right\} \tag{6}$$

Thus using (5) equation (3) becomes

$$\left.\begin{array}{l} \dfrac{\partial v}{\partial y}=0 \\ \dfrac{\partial^2 v}{\partial y^2}=0 \end{array}\right\} \tag{7}$$

The pressure *p* is independent of *y*. Thus equation (1) becomes

$$\frac{\partial u}{\partial t}+v_0\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\upsilon\frac{\partial^2 u}{\partial y^2} \tag{8}$$

Now

$$\left.\begin{aligned}\frac{\partial u}{\partial t}&=\frac{U\upsilon}{h^2}\frac{\partial \bar{u}}{\partial \bar{t}}\\ \frac{\partial u}{\partial y}&=\frac{U}{h}\frac{\partial \bar{u}}{\partial \eta}\\ \frac{\partial^2 u}{\partial y^2}&=\frac{U}{h^2}\frac{\partial^2 \bar{u}}{\partial \eta^2}\\ \frac{\partial p}{\partial x}&=\frac{\mu U}{h^2}\frac{\partial \bar{p}}{\partial \bar{x}}\end{aligned}\right\} \tag{9}$$

The equation (8) with the help of (9) becomes

$$\frac{\partial \bar{u}}{\partial \bar{t}}+\frac{\partial \bar{u}}{\partial \eta}=-\frac{\partial \bar{p}}{\partial \bar{x}}+\frac{\partial^2 \bar{u}}{\partial \eta^2} \tag{10}$$

Where suction parameter $\sigma=\frac{v_0 h}{\upsilon}$.

Boundary condition (4) becomes

$$\left.\begin{aligned}&\bar{u}(0,\bar{t})=0, \qquad \eta=0\\ &\bar{u}(1,\bar{t})=1, \qquad \eta=1\end{aligned}\right\} \tag{11}$$

2. Method of Solution

The equation (10) can be written as

$$\frac{\partial \bar{u}}{\partial \bar{t}}+\sigma\frac{\partial \bar{u}}{\partial \eta}-\frac{\partial^2 \bar{u}}{\partial \eta^2}=-\frac{\partial \bar{p}}{\partial \bar{x}} \tag{12}$$

Let

$$-\frac{\partial \bar{p}}{\partial \bar{x}}=f(\bar{t}) \tag{13}$$

and

$$\bar{u}(\eta,\bar{t})=\bar{u}_0(\eta)f(\bar{t})=\bar{u}_1(\eta,\bar{t})=\bar{u}_1(\eta)f(\bar{t}) \tag{14}$$

Now,

$$\left.\begin{aligned}\frac{\partial u}{\partial t}&=\bar{u}_0(\eta)f'(\bar{t})-\bar{u}_1(\eta)f''(\bar{t})\\ \frac{\partial u}{\partial \eta}&=\bar{u}_0(\eta)f(\bar{t})-\bar{u}_1'(\eta)f'(\bar{t})\\ \frac{\partial^2 u}{\partial \eta^2}&=\bar{u}_0''(\eta)f(\bar{t})-\bar{u}_1''(\eta)f'(\bar{t})\end{aligned}\right\} \tag{15}$$

From equation (13) and (15) in equation (10) we get

$$\bar{u}_1 f''+f'\left(\sigma\bar{u}_1-\bar{u}_0-\bar{u}_1''\right)+f(1-\sigma\bar{u}_0+\bar{u}_0'')=0 \tag{16}$$

Thus we have

$$f''(\bar{t})=0 \qquad \text{(for all } \bar{t}\text{)} \tag{17}$$

$$1-\sigma\bar{u}_0'+\bar{u}_1''=0 \tag{18}$$

and

$$\sigma\bar{u}_1'-\bar{u}_0-\bar{u}_1''=0 \tag{19}$$

$$f''(\bar{t})=0$$

From (17) which gives

$$f(\bar{t})=A+B\bar{t} \tag{20}$$

Where *A* and *B* are dimensionless constant. Thus from (13) and (20)

$$\frac{\partial \bar{p}}{\partial x}=A+B\bar{t} \tag{21}$$

Again, (18)

$$\bar{u}_0''(\eta)-\sigma\bar{u}_0'(\eta)=-1 \tag{22}$$

Boundary conditions are

$$\left.\begin{aligned}&\bar{u}_0(0)=0 \qquad \text{for } \eta=1\\ &\bar{u}_0(1)=1 \qquad \text{for } \eta=1\end{aligned}\right\} \tag{23}$$

From (22)

$$\frac{d^2\bar{u}_0}{d\eta^2}-\sigma\frac{d\bar{u}_0}{d\eta}=-1$$

Put $\frac{d\bar{u}_0}{d\eta}=z$

The above differential equation assume the form

$$\frac{dz}{d\eta}=\sigma z=-1 \tag{24}$$

Solution of (24) is given by

$$\frac{d\bar{u}_0}{d\eta}e^{-\sigma\eta} = K - \frac{e^{-\sigma\eta}}{\sigma} \tag{25}$$

The solution of differential equation (25) is given by

$$\bar{u}_0 = \frac{\eta}{\sigma} + \frac{K}{\sigma}e^{\sigma\eta} + L \tag{26}$$

where K and L are constant of integration using the boundary conditions, we have

$$\left.\begin{aligned} K &= -\frac{1}{e^{\sigma}-1} \\ L &= -\frac{1}{\sigma\left(e^{\sigma}-1\right)} \end{aligned}\right\} \tag{27}$$

Thus the solution of differential equation (22) is given by

$$\bar{u}_0 = \frac{\eta}{\sigma} - \frac{1}{\sigma\left(e^{\sigma}-1\right)}e^{\sigma\eta} + \frac{1}{\sigma\left(e^{\sigma}-1\right)} \tag{28}$$

and it also becomes

$$\frac{d^2\bar{u}_1}{d\eta^2} - \sigma\frac{d\bar{u}_1}{d\eta} = -\frac{\eta}{\sigma} + \frac{1}{\sigma\left(e^{\sigma}-1\right)}e^{\sigma\eta} - \frac{1}{\sigma\left(e^{\sigma}-1\right)} \tag{29}$$

Boundary conditions are

$$\left.\begin{aligned} \bar{u}_0(0) &= 0, & \eta &= 0 \\ \bar{u}_0(1) &= -\frac{1}{B}, & \eta &= 0 \end{aligned}\right\} \tag{30}$$

The solution of (29) is given by

$$\bar{u}_1(\eta) = \frac{\eta^2}{2\sigma^2} + \frac{\eta}{\sigma^3} + \frac{\eta}{\sigma^2\left(e^{\sigma}-1\right)} + \frac{M}{\sigma}e^{\sigma\eta} + \frac{1}{\sigma\left(e^{\sigma}-1\right)}\left[\frac{\eta}{\sigma}e^{\sigma\eta} - \frac{1}{\sigma^2}e^{\sigma\eta}\right] + N \tag{31}$$

where

$$\begin{aligned} M &= \frac{\sigma}{B\left(e^{\sigma}-1\right)} - \frac{3e^{\sigma}+1}{2\sigma^2\left(e^{\sigma}-1\right)^2} \\ N &= \frac{1}{\sigma^3\left(e^{\sigma}-1\right)} + \frac{\sigma}{B\left(e^{\sigma}-1\right)} + \frac{3e^{\sigma}+1}{2\sigma^2\left(e^{\sigma}-1\right)^2} \end{aligned} \tag{32}$$

From (31) and (32), we get

$$\bar{u}_1(\eta) = \frac{\eta^2}{2\sigma^2} + \frac{\eta}{\sigma^3} + \frac{\eta\left(e^{\sigma\eta}+1\right)}{\sigma^2\left(e^{\sigma}-1\right)} + \frac{e^{\sigma\eta}+1}{\sigma^3\left(e^{\sigma}-1\right)} + \frac{1}{B}\frac{e^{\sigma\eta}-1}{e^{\sigma}-1} - \frac{3e^{\sigma}+1}{2\sigma^2\left(e^{\sigma}-1\right)^2}\left(e^{\sigma\eta}-1\right) \tag{33}$$

Now substituting the value of $\bar{u}_0(\eta)$ and $\bar{u}_1(\eta)$ from (28) and (33) in (14) and using (20), we get

$$\begin{aligned} \bar{u}_1(\eta,\bar{t}) &= \frac{\left(e^{\sigma}-1\right)\eta - e^{\sigma\eta}+1}{\sigma^2\left(e^{\sigma}-1\right)}(A+B\bar{t}) + \frac{e^{\sigma\eta}-1}{e^{\sigma}-1} \\ &-B\left[\frac{\eta^2}{2\sigma^2} + \frac{\eta\left(e^{\sigma\eta}+1\right)}{\sigma^2\left(e^{\sigma}-1\right)} - \frac{\left(3e^{\sigma}+1\right)\left(e^{\sigma\eta}-1\right)}{2\sigma^2\left(e^{\sigma}-1\right)^2} + \frac{\eta\left(e^{\sigma}-1\right)-e^{\sigma\eta}+1}{\sigma^3\left(e^{\sigma}-1\right)}\right] \end{aligned} \tag{34}$$

Thus (34) represents the complete solution for the longitudinal velocity in the present case when $\sigma \neq 0$, i.e. when the wall are porous.

References

[1] Bhatnagar, P.L. (1960) : Boundary layer and flate plate with suction. Proc. Ind. Acad. Sci. A, 51, 1-13.

[2] Sinha, K.D. and Choudhary, R.C. (1965) : Flow of viscous in compressible fluid between two parallel plates, one in uniform motion and other at rest with suction at the stationary plate. Proc. Ind. Acad. Sci. Soci. 61, 308-18.

[3] Mishra, B.N. and Choudhary, R.C. (1971): Plane couette flow with pressure gradient & suction. Rau. Univ. Math. Jour. Vol. 2.

[4] Nanda, R.S. and Jain, M.K. (1960) : Viscous flow near a stagnation point with suction. J. Sci. Eng. Res. 387-394.