An M/G/1 Queue with K Phases of Vacation and with Second Optional Service and with State Dependent Arrival Rate

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Abstract

We consider an M/G/1 queue with K-phase of vacation and with second optional service. The service policy is after completion of essential service, the customer chooses an optional service with probability p or leaves the system with probability (1-p). Both the essential service and optional service follows general distributions. In addition, after completion of essential service or second optional service, if there are no customers in the system, the server takes vacation consisting of K-phases. After completing the K^{th} phase of vacation, the server enters into the service station independent of the number of customers in the system. The vacation periods follows general distribution. For this model the supplementary variable technique has been applied to obtain the probability generating functions of number of customers in the queue at different server states. Some particular models are obtained, and a numerical study is also carried out.

Keywords: Vacation queue, Supplementary variable, Probability generating function, Performance measures.

1. Introduction

Many real-life queueing situations encountered in day-to-day as well as industrial scenario, the vacation models are the models which are the best fit. In queueing theory, the vacation period can be considered as the period during which the server is not available as he/she has left, when the system becomes empty. In the M/G/1 queueing system, the concept of vacation had been first studied by Keilson and Servi (1987), they introduced the concept of modified service time which has a main role in the system with general service and vacation times. In many real-life situations such as production system, bank services, computer and communication networks, we have the concept of vacation. Also, for overhauling or maintenance of a system the server (system) may go to vacation.

The classic M/G/1 queue with various vacation policies have been well studied by researchers (see Doshi (1986, 1990), Gross and Harris (1998), Ke (1986), Takagi (1991) collected the research results of the M/G/1 vacation queues. Chen et al. (2009) interruption where the vacation time follows a phase type distribution. Tian and Zhang (2001) treated the Geo/G/1 system with a variant vacation policy. In this system, they assumed that after serving all customers in the system the server take a random maximum number of vacations before returning to the service station. Tian and Zhang (2002, 2003) discussed the discrete time GI/Geo/1 queue with server vacations and the GI/M/1 queue with PH vacations or setup times, respectively. Ke and Chu (2006) analyzed the $M^{[X]}/G/1$ queue with modified vacation policy by stochastic decomposition property and Ke (2007) used supplementary variable technique to study an $M^{[X]}/G/1$ queue with balking under a variant vacation policy.

Ke (2003) made the contribution to the control policy of M/G/1 queue with server vacations, startup and breakdowns. He obtained the system characteristics of the model and obtained the total expected cost function per unit time to determine the optimal threshold of N policies at a minimum cost. Ke et al. (2010) studied the operating characteristics of an $M^{[X]}/G/1$ queueing system with N-policy and at most J vacations. In this model they assumed that the server takes at most J vacations repeatedly until at least N customers are waiting in the queue while returning from vacation.

In day to day life, we have numerous queueing situations in which all the arriving customers are moved over the essential service and merely some of them may need additional optional service. Such a model was first studied by Madan (1994). The other works to be noted here are Madan (2000), Medhi (2002), Al-Jararah and Madan (2003), Wang (2004), Kalyanaraman et al (2005) and Jau-chuan (2008). In this paper we consider an M/G/1 queue with K-phase of vacation and with second optional service. In this queue, after completion of essential service, the customer chooses an optional service with probability p or leaves the system with probability (1-p). After completion of essential service or second optional service, if there are no customers in the system, the server takes vacation consisting of K- phases. After completing the K^{th} phase of vacation, the server enters into the service station independent of the number of customers in the system. The service period and vacation period follows general distributions. The mathematical description and analysis of this model is given in section 2. In section 3, we derive some operating characteristics of the model analyzed in section 2. Section 4 deals with some particular models and section 5 presents some numerical results related to the model analyzed in this paper. The last section gives a conclusion.

2. The mathematical model and analysis

The arrival process is a Poisson process with positive arrival rate. The waiting room is of infinite capacity and service discipline is first in, first out (FIFO). The server provides two types of service called first essential service and second optional service and let B_1 and B_2 be the duration of the services provided by the server to a customer respectively. The service times are assumed to be a sequence of independent, identically distributed random variables with cumulative distribution function $B_j(t)$,$(j = 1, 2)$, Laplace-Stielties Transform (LST)

$B_j^*(s) = E(e^{-sB_j})$, and first and second moments b_{j1} and b_{j2} respectively. As soon as the first essential service is completed, with probability p, the customer may opt the second optional service or with probability 1-p, he may opt to leave the system. After completion of a service (first essential or second optional service), if there are no customers in the system, the server takes vacation consisting of K-phases with each phase respectively has time duration

$V_1, V_2, V_3, \dots, V_K$. The V_i□ s are independent random variables with distribution functions $V_i(x)$,$i = 1, 2, \dots, K$. After completing the K^{th} phase of vacation, the server enters into the service station independent of the number of customers in the system. That is if there are customers in the queue, the server starts service for the customer in the head of the queue otherwise, the server waits ideal for a new arrival. The arriving customers waiting in a queue of infinite capacity, if the service is not immediate due to server is busy or server is on vacation.

Now the modified vacation period is

$$V = V_1 + V_2 + V_3 + \cdots + V_K \quad (1)$$

and the LST of V is

$$V^*(s) = \prod_{i=1}^{K} V_i^*(s) \qquad (2)$$

Whose mean is $c = \sum_{j=1}^{K} c_j$, where $c = E(V), c_j = E(V_j), j = 1,2,\dots,K$, and second moment is $d = \sum_{j=1}^{K} d_j, d = E(V^2), d_j = E(V_j^2), j = 1,2,\dots,K$. Now, it is assumed that the arrival rates are state dependent, that is, the arrival rate □ is defined as,

$$□ = \begin{cases} □_0, & \text{if the arrival is during idle period} \\ □_1, & \text{if the arrival is during essential service period} \\ □_2, & \text{if the arrival is during second optional service period} \\ □_3, & \text{if the arrival is during vacation period} \end{cases}$$

The time required by a customer to complete a service cycle is $B_c = pB_1 + (1-p)B_2 + V$ where V is defined in equation (1). Now the LST of B_c is $B_c^*(s) = p(1-p)B_1^*(s)B_2^*(s)V^*(s)$, where $V^*(s)$ is given in equation (2) and $E(B_c) = pE(B_1) + (1-p)E(B_2) + E(V) = pb_{11} + (1-p)b_{21} + c$.

Assume that $B_j(0) = V_i(0) = 0, B_j(\infty) = V_i(\infty) = 1; i = 1,2,\dots,K, j = 1,2$. The elapsed first essential service time (second optional service time) of the customer in service at time t is denoted by $□_1(t)$ ($□_2(t)$) and the elapsed vacation time of phase i is denoted by $□_i(t)$.

Let $Y(t)$ be the state of the server at time t and is defined as

$$Y(t) = \begin{cases} 0, & \text{if the server is idle at time } t \\ i, & \text{if the server is at } i^{th} \text{ phase of vacation at time } t; i = 1,2,\dots,K \\ K+1 & \text{if the server is busy at time } t \end{cases}$$

Let the random variable $L(t)$ is defined as

$$L(t) = \begin{cases} 0, & \text{if } Y(t) = 0 \\ □_i(t), & \text{if } Y(t) = i, i = 1,2,\dots,K \\ □_1(t) & \text{if } Y(t) = K+1 \text{ with probability } p \\ □_2(t), & \text{if } Y(t) = K+1 \text{ with probability } (1-p) \end{cases}$$

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and let the random variable $N(t)$, is the number of customers in the queue. Now the following probabilities have been defined for the analysis:

$Q(t) = \Pr\{N(t) = 0, L(t) = 0\}$

$P_n(t, x)dx = \Pr\{N(t) = n, Y(t) = K + 1, x < \square_1(t) \leq x + dx\}, n \geq 0$

$Q_n(t, x)dx = \Pr\{N(t) = n, Y(t) = K + 1, x < \square_2(t) \leq x + dx\}, n \geq 0$

$R_{i,n}(t, x)dx = \Pr\{N(t) = n, Y(t) = i, x < \square_i(t) \leq x + dx\}, n \geq 0, i = 1, 2, \ldots, K$ where $\{N(t),$

$Y(t), t \geq 0\}$ is a bivariate Markov process with state space

$S = \{(0, 0)\} \cup \{(K + 1, j)\} \cup \{(i, j)\}, i = 1, 2, \ldots, K, j \geq 0$. In steady state, the corresponding probabilities are $Q = \lim_{n\to\infty} Q(t),\ P_n(x) = \lim_{n\to\infty} P_n(t,\ x)$,

$Q_n(x) = \lim_{n\to\infty} Q_n(t, x)$, and $R_{i,n}(x) = \lim_{n\to\infty} R_{i,n}(t, x)$. The hazard rate function of $B_1(B_2)$ is $\mu_1(x)dx = \frac{dB_1(x)}{1-B_1(x)}$ $(\mu_2(x)dx = \frac{dB_2(x)}{1-B_2(x)})$ is the conditional probability of completion of a first essential service (second optional service) during the time interval $(x, x + dx]$ given that the elapsed first essential service (second optional service) is x. The similar quantity for V_i is

$\square_i(x)dx = \frac{dV_i(x)}{1-V_i(x)}, i = 1, 2, \ldots, K$.

The model is governed by the following differential difference equations for $x > 0$,

$$\frac{d}{dx}P_0(x) + \left(\lambda_1 + \mu_1(x)\right)P_0(x) = 0 \quad (3)$$

$$\frac{d}{dx}P_n(x) + \left(\lambda_1 + \mu_1(x)\right)P_n(x) = \lambda_1 P_{n-1}(x), n \geq 1 \quad (4)$$

$$\frac{d}{dx}Q_0(x) + \left(\lambda_2 + \mu_2(x)\right)Q_0(x) = 0 \quad (5)$$

$$\frac{d}{dx}Q_n(x) + \left(\lambda_2 + \mu_2(x)\right)Q_n(x) = \lambda_2 Q_{n-1}(x), n \geq 1 \tag{6}$$

$$\frac{d}{dx}R_{i,0}(x) + \left(\lambda_3 + \mu_3(x)\right)R_{i,0}(x) = 0 \tag{7}$$

$$\frac{d}{dx}R_{i,n}(x) + \left(\lambda_3 + \mu_3(x)\right)R_{i,n}(x) = \lambda_3 R_{i,n-1}(x), i = 1, 2, \dots, K \tag{8}$$

$$\lambda_0 Q = \int_0^\infty \eta_K(x)R_{K,0}(x)dx \tag{9}$$

The boundary conditions at $x = 0$ are

$$P_0(0) = {}_0Q + \int_0^\infty {}_K(x)R_{K,1}(x)dx + (1-p)\int_0^\infty \mu_1(x)P_1(x)dx$$

$$+ \int_0^\infty \mu_2(x)Q_1(x)dx \tag{10}$$

$$P_n(0) = \int_0^\infty \eta_K(x)R_{K,n+1}(x)dx + (1-p)\int_0^\infty \mu_1(x)P_{n+1}(x)dx$$

$$+ \int_0^\infty \mu_2(x)Q_{n+1}(x)dx \tag{11}$$

$$Q_n(0) = p\int_0^\infty \mu_1(x)P_n(x)dx, \qquad n \geq 0 \tag{12}$$

$$R_{1,0}(0) = (1-p)\int_0^\infty \mu_1(x)P_0(x)dx + \int_0^\infty \mu_2(x)Q_0(x)dx, n \geq 0 \tag{13}$$

$$R_{1,n}(0) = 0, n \geq 1 \tag{14}$$

$$R_{i,n}(0) = \int_0^\infty \eta_{i-1}(x)R_{i-1.n}(x)dx, i = 2, 3, \dots, K; n \geq 0 \tag{15}$$

The normalizing condition is

$$Q + P(1) + Q(1) + \sum_{i=1}^{K} R_i(1) = 1$$

For the analysis, the following probability generating functions have been introduced $P(x,z) = \sum_{n=0}^{\infty} z^n P_n(x)$, $Q(x,z) = \sum_{n=0}^{\infty} z^n Q_n(x)$ and $R_i(x,z) = \sum_{n=0}^{\infty} z^n R_{i,n}(x), i = 1, 2, \dots, K$.

From equation (3), we have

$$P_0(x) = P_0(0)(1 - B_1(x))e^{-\square_1 x} \tag{16}$$

Multiplying equation (4) by z^n, summing from 1 to ∞ and adding equation (3), we get

$$P(x,z) = P(0,z)(1 - B_1(x))e^{-\square_1(1-z)x} \tag{17}$$

From equation (5), we have

$$Q_0(x) = Q_0(0)(1 - B_2(x))e^{-\square_2 x} \tag{18}$$

Multiplying equation (6) by z^n, summing from 1 to ∞ and adding equation (5), we get

$$Q(x,z) = Q(0,z)(1 - B_2(x))e^{-\square_2(1-z)x} \tag{19}$$

From equation (7), we have

$$R_{i,0}(x) = R_{i,0}(0)(1 - V_i(x))e^{-\square_3 x}, i = 1,2, \dots, K \tag{20}$$

Multiplying equation (8) by z^n, summing from 1 to ∞ and adding equation (7), we get

$$R_i(x,z) = R_i(0,z)(1 - V_i(x))e^{-\square_3(1-z)x}, i = 1, 2, \dots, K \tag{21}$$

Multiplying equation (11) by z^n, summing from 1 to ∞ and adding equation (10) and multiply by z, we get

$$zP(0,z) = z\lambda_0 Q + \int_0^{\infty} \eta_K(x) R_K(x,z)dx - \int_0^{\infty} \eta_K(x) R_{K,0}(x)dx$$

$$+(1-p)\left[\int_0^{\infty} \mu_1(x)P(x,z)dx - \int_0^{\infty} \mu_1(x)P_0(x)dx\right]$$

$$+\int_0^\infty \mu_2(x)Q(x,z)dx - \int_0^\infty \mu_2(x)Q_0(x)dx \qquad (22)$$

From equation (16), we have

$$\int_0^\infty \mu_1(x)P_0(x)dx = P_0(0)B_1^*(\lambda_1) \qquad (23)$$

Similarly from equations (18) and (20) we get

$$\int_0^\infty \mu_2(x)Q_0(x)dx = Q_0(0)B_2^*(\lambda_2) \qquad (24)$$

$$\int_0^\infty \eta_i(x)R_{i,0}(x)dx = R_{i,0}(0)V_i^*(\lambda_3), i = 1,2,\dots,K \qquad (25)$$

From equation (17), we have

$$\int_0^\infty \mu_1(x)P(x,z)dx = P(0,z)B_1^*(\lambda_1(1-z)) \qquad (26)$$

Similarly from equations (19) and (21) we get

$$\int_0^\infty \mu_2(x)Q(x,z)dx = Q(0,z)B_2^*(\lambda_2(1-z)) \qquad (27)$$

$$\int_0^\infty \eta_i(x)R_i(x,z)dx = R_i(0,z)V_i^*(\lambda_3(1-z)), i = 1,2,\dots,K \qquad (28)$$

Using equations (23), (24), (25), (26), (27) and (28) in (22), we get

$$zP(0,z) = z\lambda_0 Q + R_K(0,z)V_K{}^*(R_3) - R_{K,0}(0)V_K{}^*(\lambda_3) + (1-p)[P(0,z)B_1{}^*(R_1)$$

$$-P_0(0)B_1^*(\lambda_1)] + Q(0,z)B_2^*(R_2) - Q_0(0)B_2^*(\lambda_2) \qquad (29)$$

where $R_1 = \lambda_1(1-z)$, $R_2 = \lambda_2(1-z)$ and $R_3 = \lambda_3(1-z)$

Multiplying equation (12) by z^n, summing from 0 to ∞, we get

$$Q(0,z) = pP(0,z)B_1^*(R_1) \qquad (30)$$

Put $n = 0$ in equation (12), we get

$$Q_0(0) = pP_0(0)B_1{}^*(\square_1) \tag{31}$$

Multiplying equation (14) by z^n, summing from 1 to ∞ and adding with equation (13), we get

$$R_i(0, z) = (1 - p + pB_2{}^*(\square_2))P_0(0)B_1{}^*(\square_1) \tag{32}$$

Multiplying equation (15) by z^n, summing from 0 to ∞, we get

$$R_i(0,z) = \prod_{l=1}^{i-1} V_l^*\,(R_3)\big(1 - p + pB_2^*(\lambda_2)\big)P_0(0)B_1^*(\lambda_1), i = \ 2, \dots, K \tag{33}$$

Put $n = 0$ in equation (15), we get

$$R_{i,0}(0) = \prod_{l=1}^{i-1} V_l^*\,(\lambda_3)\big(1 - p + pB_2^*(\lambda_2)\big)P_0(0)B_1^*(\lambda_1), i = \ 2, \dots, K \tag{34}$$

From equation (29), we get

$$[z - \big(1 - p + pB_2^*(R_2)\big)B_1^*(R_1)]P(0,z) = z\lambda_0 Q + \big(1 - p + pB_2^*(\lambda_2)\big)P_0(0) \times B_1^*(\lambda_1)\left[\prod_{l=1}^{K} V_l^*\,(R_3) - 1 - \prod_{l=1}^{K} V_l^*\,(\lambda_3)\right] \tag{35}$$

From equation (9), we get

$$\lambda_0 Q = \prod_{l=1}^{K} V_l^*\,(\lambda_3)\big(1 - p + pB_2^*(\lambda_2)\big)P_0(0)B_1^*(\lambda_1) \tag{36}$$

Substituting equation (36) in (35), we get

$$P(0,z) = \frac{\big(1 - p + pB_2^*(\lambda_2)\big)P_0(0)B_1^*(\lambda_1)}{[z - \big(1 - p + pB_2^*(R_2)\big)B_1^*(R_1)]}\left\{(z - 1)\prod_{l=1}^{K} V_l^*\,(\lambda_3) - 1 + \prod_{l=1}^{K} V_l^*\,(R_3)\right\} \tag{37}$$

Now

$$\left.\begin{aligned} P(z) &= \int_0^\infty P(x,z)dx = P(0,z)\frac{\left(1-B_1^*(R_1)\right)}{R_1} \\ Q(z) &= \int_0^\infty Q(x,z)dx = Q(0,z)\frac{\left(1-B_2^*(R_2)\right)}{R_2} \\ and\ R_i(z) &= \int_0^\infty R_i(x,z)dx = R_i(0,z)\frac{\left(1-V_i^*(R_3)\right)}{R_3}, i = 1,2,\dots,K \end{aligned}\right\} \tag{38}$$

To find the unknown probability $P_0(0)$, we use the normalizing condition

$$Q + P(1) + Q(1) + \sum_{i=1}^{K} R_i\,(1) = 1$$
,

We get

$$P_0(0) = \frac{\lambda_0(1-\lambda_1 b_{11} - \lambda_2 p b_{21})}{\left(1-p+pB_2^*(\lambda_2)\right)B_1^*(\lambda_1)C_1} \tag{39}$$

where

$$C_1 = [1 + (\lambda_0 - \lambda_1)b_{11} + (\lambda_0 - \lambda_2)pb_{21}]\prod_{l=1}^{K} V_l^*\,(\lambda_3) + \lambda_0 c[1 + (\lambda_3 - \lambda_1)b_{11}$$

$$+(\square_3 - \square_2)pb_{21}]$$

and substituting equation (39) in (36), we get

$$Q = \frac{(1-\lambda_1 b_{11} - \lambda_2 p b_{21})}{C_1}\prod_{l=1}^{K} V_l^*\,(\lambda_3) \tag{40}$$

Equations in (38), together with (30), (32), (33), (36), (37), (39) and (40) gives the probability generating function of number of customers in the queue when server is busy the service is idle and the server is on the $i^{th}\ (i = 1,2,\dots,K)$ phase of vacation respectively.

3. Some operating characteristics

In this section we derive the operating characteristics mean and variance number of customers in the queue when the server is on the essential service, mean and variance number of customers in the queue when the server is on the second optional service and mean and variance number of customers in the queue when the server is on the i^{th} $(i = 1, 2, \ldots, K)$ phase of vacation.

Mean number of customers in the queue when the server is on the essential service is

$$L_b = \frac{\lambda_0 C_2}{2(1 - \lambda_1 b_{11} - \lambda_2 p b_{21}) C_1}$$

Variance of number of customers in the queue when the server is on the essential service is

$$V_b = \frac{\lambda_0 [2\, C_1\, C_3 - 3\lambda_0 C_2^2]}{12(1 - \lambda_1 b_{11} - \lambda_2 p b_{21})^2 C_1^2}$$

Mean number of customers in the queue when the server is on the second optional service is

$$L_s = \frac{p\lambda_0 C_4}{2(1 - \lambda_1 b_{11} - \lambda_2 p b_{21}) C_1}$$

Variance of number of customers in the queue when the server is on the second optional service is

$$V_s = \frac{p\lambda_0 [2\, C_1\, C_5 - 3p\lambda_0 C_4^2]}{12(1 - \lambda_1 b_{11} - \lambda_2 p b_{21})^2 C_1^2}$$

Mean number of customers in the queue when the server is on vacation is

$$L_v = \frac{\lambda_0 \lambda_3 d(1 - \lambda_1 b_{11} - \lambda_2 p b_{21})}{2C_1}$$

Variance of number of customers in the queue when the server is on vacation is

$$V_v = \frac{\lambda_0 \lambda_3 (1 - \lambda_1 b_{11} - \lambda_2 p b_{21})}{12 C_1^2} \{6dC_1 + 4\lambda_3 e C_1 - 3\lambda_0 \lambda_3 d^2 (1 - \lambda_1 b_{11} - \lambda_2 p b_{21})\}$$

where

$$d = \sum_{i=1}^{K} d_i + 2\sum_{i=1}^{K-1} c_i \sum_{j=i+1}^{K} c_j$$

$$e = \sum_{i=1}^{K} e_i + 3\left[\sum_{i=1}^{K-1} d_i \sum_{j=i+1}^{K} c_j + \sum_{i=1}^{K-1} c_i \sum_{j=i+1}^{K} d_j\right] + 6\sum_{i=1}^{K-2} c_i \sum_{j=i+1}^{K-1} c_j \sum_{m=j+1}^{K} c_m$$

$$C_2 = [\lambda_1 b_{12}(1 - p\lambda_2 b_{21}) + p\lambda_2 b_{11}(2\lambda_1 b_{11} b_{21} + \lambda_2 b_{22})]\left(\prod_{l=1}^{K} V_l^*(\lambda_3) + \lambda_3 c\right)$$

$$+\lambda_3^2 d b_{11}(1 - \lambda_1 b_{11} - \lambda_2 p b_{21})$$

$$C_3 = [3(1 - \lambda_1 b_{11} - p\lambda_2 b_{21} + \lambda_1^2 b_{12} + 2p\lambda_1\lambda_2 b_{11} b_{21} + p\lambda_2^2 b_{22})[\lambda_1 b_{12}$$

$$\times (1 - p\lambda_2 b_{21}) + p\lambda_2 b_{11}(2\lambda_1 b_{11} b_{21} + \lambda_2 b_{22})] + 2(1 - \lambda_1 b_{11} - \lambda_2 p b_{21})$$

$$\times (\lambda_1^2 b_{13}(1 - p\lambda_2 b_{21}) + 3p\lambda_1\lambda_2 b_{11}(\lambda_1 b_{12} b_{21} + \lambda_2 b_{11} b_{22}) + p\lambda_2^3 b_{11} b_{23})]$$

$$\times \left(\prod_{l=1}^{K} V_l^*(\lambda_3) + \lambda_3 c\right) + 3\lambda_3^2 d(1 - \lambda_1 b_{11} - \lambda_2 p b_{21})[\lambda_1 b_{12}(1 - p\lambda_2 b_{21})$$

$$+b_{11}(1 - \lambda_1 b_{11} - \lambda_2 p b_{21}) + p\lambda_2 b_{11}(2\lambda_1 b_{11} b_{21} + \lambda_2 b_{22})] + 2\lambda_3^3 e b_{11}$$

$$\times (1 - \lambda_1 b_{11} - \lambda_2 p b_{21})^2$$

$$C_4 = \left[(1 - \lambda_1 b_{11})(\lambda_2 b_{22} + 2\lambda_1 b_{11} b_{21}) + \lambda_1^2 b_{12} b_{21}\right]\left(\prod_{l=1}^{K} V_l^*(\lambda_3) + \lambda_3 c\right) + \lambda_3^2$$

$$\times d b_{21}(1 - \lambda_1 b_{11} - \lambda_2 p b_{21})$$

$$C_5 = [3((1 - \lambda_1 b_{11})(\lambda_2 b_{22} + 2\lambda_1 b_{11} b_{21}) + \lambda_1^2 b_{12} b_{21})(1 - \lambda_1 b_{11} - p\lambda_2 b_{21}$$

$$+\lambda_1^2 b_{12} + 2p\lambda_1\lambda_2 b_{11} b_{21} + p\lambda_2^2 b_{22}) + 2(1 - \lambda_1 b_{11} - \lambda_2 p b_{21})((1 - \lambda_1 b_{11})$$

$$\times \left(\lambda_2^2 b_{23} + 3\lambda_1(\lambda_1 b_{12} b_{21} + \lambda_2 b_{11} b_{22})\right) + \lambda_1^3 b_{13} b_{21})]\left(\prod_{l=1}^{K} V_l^*(\lambda_3) + \lambda_3 c\right)$$

$$+3\lambda_3^2(1-\lambda_1 b_{11}-\lambda_2 p b_{21})d[(1-\lambda_1 b_{11})(\lambda_2 b_{22}+2\lambda_1 b_{11} b_{21})+\lambda_1^2 b_{12} b_{21}$$

$$+b_{21}(1-\lambda_1 b_{11}-\lambda_2 p b_{21})]+2\lambda_3^3 e b_{21}(1-\lambda_1 b_{11}-\lambda_2 p b_{21})^2$$

4. Some particular cases

In this section, we present five particular cases by assuming particular form to the parameters and/or particular probability distribution to service time and/or vacation time.

Case 1: Now we take $\lambda_0=\lambda_1=\lambda_2=\lambda_3=\lambda$

$$Q=\frac{(1-\lambda b_{11}-\lambda p b_{21})}{D_1}\prod_{l=1}^{K} V_l^*(\lambda)$$

$$L_b=\frac{\lambda^2 D_2}{2(1-\lambda b_{11}-\lambda p b_{21})D_1}$$

$$V_b=\frac{\lambda^2[2D_1D_3-3\lambda^2 D_2^2]}{12(1-\lambda b_{11}-\lambda p b_{21})^2 D_1^2}$$

$$L_s=\frac{\lambda^2 p D_4}{2(1-\lambda b_{11}-\lambda p b_{21})D_1}$$

$$V_s=\frac{\lambda^2 p[2D_1D_5-3\lambda^2 p D_4^2]}{12(1-\lambda b_{11}-\lambda p b_{21})^2 D_1^2}$$

$$L_v=\frac{\lambda^2 d(1-\lambda b_{11}-\lambda p b_{21})}{2D_1}$$

$$V_v=\frac{\lambda^2(1-\lambda b_{11}-\lambda p b_{21})}{12D_1^2}\{4\lambda e D_1+6dD_1-3\lambda^2 d^2(1-\lambda b_{11}-\lambda p b_{21})\}$$

where

$$D_1=\prod_{l=1}^{K} V_l^*(\lambda)+\lambda c$$

$D_2 = [b_{12}(1 - p□b_{21}) + p□b_{11}(2b_{11}b_{21} + b_{22})]D_1 + □db_{11}(1 - □b_{11} - □pb_{21})$

$D_3 = [3(1 - □b_{11} - p□b_{21} + □2b_{12} + 2p□2b_{11}b_{21} + p□2b_{22})(b_{12}(1 - p□b_{21})$

$+p□b_{11}(2b_{11}b_{21} + b_{22})) + 2□(1 - □b_{11} - □pb_{21})(b_{13}(1 - p□b_{21}) + p□b_{11}$

$\times (3(b_{12}b_{21} + b_{11}b_{22}) + b_{23}))]D_1 + 3□d(1 - □b_{11} - □pb_{21})[p□2b_{11}(b_{22}$

$+2b_{11}b_{21}) + □b_{12}(1 - p□b_{21}) + b_{11}(1 - □b_{11} - □pb_{21})] + 2^{□}{}_2eb_{11}$

$\times (1 - □b_{11} - □pb_{21})^2$

$D_4 = [(1 - □b_{11})(b_{22} + 2b_{11}b_{21}) + □b_{12}b_{21}]D_1 + □db_{21}(1 - □b_{11} - □pb_{21})$

$D_5 = [3((1 - □b_{11})(b_{22} + 2b_{11}b_{21}) + □b_{12}b_{21})(1 - □b_{11} - p□b_{21} + □2(b_{12}$

$+2pb_{11}b_{21} + pb_{22})) - 2□(1 - □b_{11} - □pb_{21})((1 - □b_{11})(b_{23} + 3(b_{12}b_{21}$

$+b_{11}b_{22})) + □b_{13}b_{21})]D_1 + 3□(1 - □b_{11} - □pb_{21})d[□((1 - □b_{11})(b_{22}$

$+2b_{11}b_{21}) + □b_{12}b_{21}) + b_{21}(1 - □b_{11} - □pb_{21})] + (1 - □b_{11} - □pb_{21})^2$

$\times 2□^2eb_{21}$

<u>Case 2:</u> The service time and vacation time follows exponential distribution.

That is, $B_1(x) = 1 - e^{-\mu_1 x}$, $B_1^*(s) = \frac{\mu_1}{s+\mu_1}$, $B_1^{*'}(0) = \frac{-1}{\mu_1}$, $B_1^{*''}(0) = \frac{2}{\mu_1^2}$, $B_1^{*'''}(0) = \frac{-6}{\mu_1^3}$, $B_2(x) = 1 - e^{-\mu_2 x}$, $B_2^*(s) = \frac{\mu_2}{s+\mu_2}$, $B_2^{*'}(0) = \frac{-1}{\mu_2}$, $B_2^{*''}(0) = \frac{2}{\mu_2^2}$, $B_2^{*'''}(0) = \frac{-6}{\mu_2^3}$, $V_i(x) = 1 -$

$e^{-\eta_i x}$ * $= \frac{\eta_i}{}$ $^{*'}$ $= \frac{-1}{}$ $^{*''}$ $= \frac{2}{}$ $^{*'''}$ $= \frac{-6}{}$ $=$, $V_i(s)$ $s+□i$, $V_i(0)$ □i

, V_i (0) □$i2$, V_i (0) □$3i$, i 1, 2, ... , K.

$$Q = \frac{(\mu_1\mu_2 - \lambda_1\mu_2 - \lambda_2 p\mu_1)}{D_8}\prod_{l=1}^{K}\frac{\eta_l}{\lambda_3 + \mu_l}$$

$$L_b = \frac{\lambda_0 D_9}{(\mu_1\mu_2 - \lambda_1\mu_2 - \lambda_2 p\mu_1)D_8}$$

$$V_b = \frac{\lambda_0(D_8D_{10} - \lambda_0 D_9^2)}{(\mu_1\mu_2 - \lambda_1\mu_2 - \lambda_2 p\mu_1)^2 D_8^2}$$

$$L_s = \frac{\lambda_0 p D_{11}}{(\mu_1\mu_2 - \lambda_1\mu_2 - \lambda_2 p\mu_1)D_8}$$

$$V_s = \frac{\lambda_0 p(D_8D_{12} - \lambda_0 p D_{11}^2)}{(\mu_1\mu_2 - \lambda_1\mu_2 - \lambda_2 p\mu_1)^2 D_8^2}$$

$$L_v = \frac{\lambda_0\lambda_3(\mu_1\mu_2 - \lambda_1\mu_2 - \lambda_2 p\mu_1)D_6}{D_8}$$

$$V_v = \frac{\lambda_0\lambda_3(\mu_1\mu_2 - \lambda_1\mu_2 - \lambda_2 p\mu_1)}{D_8^2}\{2\lambda_3 D_7 D_8 + D_6 D_8 - \lambda_0\lambda_3(\mu_1\mu_2 - \lambda_1\mu_2$$

$$-\lambda_2 p\mu_1)D_6^2\}$$

where

$$D_6 = \sum_{i=1}^{K}\frac{1}{\nu_i} + \sum_{i=1}^{K-1}\frac{1}{\nu_i}\sum_{j=i+1}^{K}\frac{1}{\nu_j}$$

$$D_7 = \sum_{i=1}^{K}\frac{1}{\nu_i^2} + \sum_{i=1}^{K-1}\frac{1}{\nu_i^2}\sum_{j=i+1}^{K}\frac{1}{\nu_j} + \sum_{i=1}^{K-1}\frac{1}{\nu_i}\sum_{j=i+1}^{K}\frac{1}{\nu_j^2} + \sum_{i=1}^{K-2}\frac{1}{\nu_i}\sum_{j=i+1}^{K-1}\frac{1}{\nu_j}\sum_{m=j+1}^{K}\frac{1}{\nu_m}$$

$$D_8 = (\mu_1\mu_2 + (\lambda_0 - \lambda_1)\mu_2 + (\lambda_0 - \lambda_2)p\mu_1)\prod_{l=1}^{K}\frac{\eta_l}{\lambda_3 + \mu_l} + \lambda_0(\mu_1\mu_2 + \mu_2(\lambda_0$$

$$-\lambda_1) + (\lambda_0 - \lambda_2)p\mu_1)\sum_{i=1}^{K}\frac{1}{\nu_i}$$

$$D_9 = (\lambda_1\mu_2^2 + \lambda_2^2 p\mu_1)\left(\prod_{l=1}^{K}\frac{\eta_l}{\lambda_3 + \mu_l} + \lambda_3\sum_{i=1}^{K}\frac{1}{\nu_i}\right) + \lambda_3^2\mu_2 D_6(\mu_1\mu_2 - \lambda_1\mu_2$$

$$-\lambda_2 p\mu_1)$$

$$D_{10} = [\lambda_1 \mu_2^2(\mu_1\mu_2 + \lambda_1\mu_2 - \lambda_2 p\mu_1) + \lambda_2^2 p\mu_1\mu_2(3\lambda_1 + \mu_1) + \lambda_2^3 p\mu_1(2\mu_1 - p\mu_1$$

$$-2\lambda_1] \left(\prod_{l=1}^{K} \frac{\eta_l}{\lambda_3 + \mu_l} + \lambda_3 \sum_{i=1}^{K} \frac{1}{\nu_i}\right) + \lambda_3^2[2\lambda_2 p\mu_1^2\mu_2(\lambda_2 - \mu_2) + \mu_2^3(\mu_1^2 - \lambda_1^2)$$

$$+\lambda_2^3 p\mu_1\mu_2(p\mu_1 - 2\lambda_1) - 2\lambda_2^3 p^2\mu_1^2]D_6 + 2\lambda_3^3\mu_2 D_7(\mu_1\mu_2 - \lambda_1\mu_2 - \lambda_2 p\mu_1)^2$$

$$D_{11} = (\lambda_2\mu_1^2 + \lambda_1\mu_1\mu_2 - \lambda_1\lambda_2\mu_1)\left(\prod_{l=1}^{K} \frac{\eta_l}{\lambda_3 + \mu_l} + \lambda_3 \sum_{i=1}^{K} \frac{1}{\nu_i}\right) + \lambda_3^2\mu_1 D_6(\mu_1\mu_2$$

$$-\lambda_1\mu_2 - \lambda_2 p\mu_1)$$

$$D_{12} = [\lambda_2\mu_1^3(\mu_2^2 - \lambda_2 p) + \lambda_1\mu_1\mu_2(\mu_1\mu_2 + \lambda_1\mu_2 - \lambda_1\lambda_2) + \lambda_1\lambda_2 p\mu_1^2(3\lambda_2 - \mu_2)$$

$$+2\lambda_2^2\mu_1(\mu_1^2 - 2\lambda_1\mu_1 + \lambda_1^2)]\left(\prod_{l=1}^{K} \frac{\eta_l}{\lambda_3 + \mu_l} + \lambda_3 \sum_{i=1}^{K} \frac{1}{\nu_i}\right) + \lambda_3^2\mu_1[\mu_2^2(\mu_1^2 - \lambda_1^2)$$

$$+2\lambda_2(\lambda_1^2\mu_2 - \lambda_2 p\mu_1^2 + \lambda_1\lambda_2 p\mu_1) + 2\lambda_2\mu_1\mu_2(\mu_1 - p\mu_1 - 2\lambda_1) + \lambda_2^2 p^2\mu_1^2]D_6$$

$$+2\lambda_3^3\mu_1 D_7(\mu_1\mu_2 - \lambda_1\mu_2 - \lambda_2 p\mu_1)^2$$

Case 3: The service time follows exponential distribution.

That is, $B_1(x) = 1 - e^{-\mu_1 x}$, $B_1^*(s) = \frac{\mu_1}{s+\mu_1}$, $B_1^{*'}(0) = \frac{-1}{\mu_1}$, $B_1^{*''}(0) = \frac{2}{\mu_1^2}$, $B_1^{*'''}(0) = \frac{-6}{\mu_1^3}$,
$B_2(x) = 1 - e^{-\mu_2 x}$, $B_2^*(s) = \frac{\mu_2}{s+\mu_2}$, $B_2^{*'}(0) = \frac{-1}{\mu_2}$, $B_2^{*''}(0) = \frac{2}{\mu_2^2}$, $B_2^{*'''}(0) = \frac{-6}{\mu_2^3}$.

$$Q = \frac{(\mu_1\mu_2 - \lambda_1\mu_2 - \lambda_2 p\mu_1)}{D_{13}} \prod_{l=1}^{K} V_l^*(\lambda_3)$$

$$L_b = \frac{\lambda_0 D_{14}}{(\mu_1\mu_2 - \lambda_1\mu_2 - \lambda_2 p\mu_1)D_{13}}$$

$$V_b = \frac{\lambda_0(D_{13}D_{15} - \lambda_0 D_{14}^2)}{(\mu_1\mu_2 - \lambda_1\mu_2 - \lambda_2 p\mu_1)^2 D_{13}^2}$$

$$L_s = \frac{\lambda_0 p D_{16}}{(\mu_1\mu_2 - \lambda_1\mu_2 - \lambda_2 p\mu_1)D_{13}}$$

$$V_s = \frac{\lambda_0 p(D_{13}D_{17} - \lambda_0 p D_{16}^2)}{(\mu_1\mu_2 - \lambda_1\mu_2 - \lambda_2 p\mu_1)^2 D_{13}^2}$$

$$L_v = \frac{\lambda_0\lambda_3 d(\mu_1\mu_2 - \lambda_1\mu_2 - \lambda_2 p\mu_1)}{D_{13}}$$

$$V_v = \frac{\lambda_0\lambda_3(\mu_1\mu_2 - \lambda_1\mu_2 - \lambda_2 p\mu_1)}{D_{13}^2}\{4\lambda_3 e D_{13} + 6dD_{13} - 3\lambda_0\lambda_3 e^2(\mu_1\mu_2 - \lambda_1\mu_2$$

$$-\lambda_2 p\mu_1)\}$$ where

$$D_{13} = (\mu_1\mu_2 + (\lambda_0 - \lambda_1)\mu_2 + (\lambda_0 - \lambda_2)p\mu_1)\prod_{l=1}^{K} V_l^*(\lambda_3) + \lambda_0 c(\mu_1\mu_2 + \mu_2(\lambda_0$$

$$-\lambda_1) + (\lambda_0 - \lambda_2)p\mu_1)$$

$$D_{14} = 2(\lambda_1\mu_2^2 + \lambda_2^2 p\mu_1)\left(\prod_{l=1}^{K} V_l^*(\lambda_3) + \lambda_3 c\right) + \lambda_3^2\mu_2 d(\mu_1\mu_2 - \lambda_1\mu_2$$

$$-\lambda_2 p\mu_1)$$

$$D_{15} = 6[\lambda_1\mu_2^2(\mu_1\mu_2 + \lambda_1\mu_2 - \lambda_2 p\mu_1) + \lambda_2^2 p\mu_1\mu_2(3\lambda_1 + \mu_1) + \lambda_2^3 p\mu_1(2\mu_1 - p\mu_1$$

$$-2\lambda_1)]\left(\prod_{l=1}^{K} V_l^*(\lambda_3) + \lambda_3 c\right) + 3\lambda_3^2 d[2\lambda_2 p\mu_1^2\mu_2(\lambda_2 - \mu_2) + \mu_2^3(\mu_1^2 - \lambda_1^2)$$

$$+\lambda_2^3 p\mu_1\mu_2(p\mu_1 - 2\lambda_1) - 2\lambda_2^3 p^2\mu_1^2] + 2\lambda_3^3\mu_2 e(\mu_1\mu_2 - \lambda_1\mu_2 - \lambda_2 p\mu_1)^2$$

$$D_{16} = 2(\lambda_2\mu_1^2 + \lambda_1\mu_1\mu_2 - \lambda_1\lambda_2\mu_1)\left(\prod_{l=1}^{K} V_l^*(\lambda_3) + \lambda_3 c\right) + \lambda_3^2\mu_1 d(\mu_1\mu_2 - \lambda_1\mu_2$$

$$-\lambda_2 p\mu_1)$$

$$D_{17} = 6[\lambda_2\mu_1^3(\mu_2^2 - \lambda_2 p) + \lambda_1\mu_1\mu_2(\mu_1\mu_2 + \lambda_1\mu_2 - \lambda_1\lambda_2) + \lambda_1\lambda_2 p\mu_1^2(3\lambda_2$$

$$-\mu_2) + 2\lambda_2^2\mu_1(\mu_1^2 - 2\lambda_1\mu_1 + \lambda_1^2)]\left(\prod_{l=1}^{K} V_l^*(\lambda_3) + \lambda_3 c\right) + 3\lambda_3^2\mu_1 d[\lambda_2^2 p^2\mu_1^2$$

$$+2\lambda_2\left(\lambda_1^2\mu_2 - \lambda_2 p\mu_1^2 + \lambda_1\lambda_2 p\mu_1\right) + 2\lambda_2\mu_1\mu_2(\mu_1 - p\mu_1 - 2\lambda_1) + \mu_2^2(\mu_1^2$$

$$-\lambda_1^2)] + 2\lambda_3^3\mu_1 e(\mu_1\mu_2 - \lambda_1\mu_2 - \lambda_2 p\mu_1)^2$$

Case 4: The vacation time follows exponential distribution.

That is, $V_i(x) = 1 - e^{-\eta_i x}$, $V_i^*(s) = \frac{\eta_i}{s+\eta_i}$, $V_i^{*'}(0) = \frac{-1}{\nu_i}$, $V_i^{*''}(0) = \frac{2}{\nu_i^2}$, $V_i^{*'''}(0) = \frac{-6}{\nu_i^3}$, $i =$ 1, 2, ... , K.

$$Q = \frac{(1 - \lambda b_{11} - \lambda p b_{21})}{D_{18}} \prod_{l=1}^{K} \frac{\eta_l}{\lambda_3 + \mu_l}$$

$$L_b = \frac{\lambda_0 D_{19}}{2(1 - \lambda b_{11} - \lambda p b_{21}) D_{18}}$$

$$V_b = \frac{\lambda_0(2D_{18}D_{20} - 3\lambda_0 D_{19}^2)}{12(1 - \lambda b_{11} - \lambda p b_{21})^2 D_{18}^2}$$

$$L_s = \frac{\lambda_0 p D_{21}}{2(1 - \lambda b_{11} - \lambda p b_{21}) D_{18}}$$

$$V_s = \frac{\lambda_0 p(2D_{18}D_{22} - 3\lambda_0 p D_{21}^2)}{12(1 - \lambda b_{11} - \lambda p b_{21})^2 D_{18}^2}$$

$$L_v = \frac{\lambda_0\lambda_3(1 - \lambda b_{11} - \lambda p b_{21}) D_6}{D_{18}}$$

$$V_v = \frac{\lambda_0\lambda_3(1 - \lambda b_{11} - \lambda p b_{21})}{D_{18}^2}\{2\lambda_3 D_7 D_{18} + D_6 D_{18} - \lambda_0\lambda_3 D_6^2(1 - \lambda b_{11}$$

$$-\square p b_{21})\}$$

where

$$D_{18} = [1 + (\lambda_0 - \lambda_1)b_{11} + (\lambda_0 - \lambda_2)p b_{21}] \prod_{l=1}^{K} \frac{\eta_l}{\lambda_3 + \mu_l} + \lambda_0[1 + (\lambda_3 - \lambda_1)b_{11}$$

$$+(\lambda_3 - \lambda_2)p b_{21}] \sum_{i=1}^{K} \frac{1}{\nu_i}$$

$$D_{19} = \left(\prod_{l=1}^{K} \frac{\eta_l}{\lambda_3 + \mu_l} + \lambda_3 \sum_{i=1}^{K} \frac{1}{\nu_i}\right)[\lambda_1 b_{12}(1 - p\lambda_2 b_{21}) + p\lambda_2 b_{11}(2\lambda_1 b_{11} b_{21}$$

$$+\lambda_2 b_{22})] + 2\lambda_3^2 b_{11}(1 - \lambda_1 b_{11} - \lambda_2 p b_{21}) D_6$$

$$D_{20} = [3(1 - \lambda_1 b_{11} - p\lambda_2 b_{21} + \lambda_1^2 b_{12} + 2p\lambda_1\lambda_2 b_{11} b_{21} + p\lambda_2^2 b_{22})[\lambda_1 b_{12}$$

$$\times (1 - p\lambda_2 b_{21}) + p\lambda_2 b_{11}(2\lambda_1 b_{11} b_{21} + \lambda_2 b_{22})] + 2(1 - \lambda_1 b_{11} - \lambda_2 p b_{21})$$

$$\times (\lambda_1^2 b_{13}(1 - p\lambda_2 b_{21}) + 3p\lambda_1\lambda_2 b_{11}(\lambda_1 b_{12} b_{21} + \lambda_2 b_{11} b_{22}) + p\lambda_2^3 b_{11} b_{23})]$$

$$\times \left(\prod_{l=1}^{K} \frac{\eta_l}{\lambda_3 + \mu_l} + \lambda_3 \sum_{i=1}^{K} \frac{1}{\nu_i}\right) + 6\lambda_3^2 D_6(1 - \lambda_1 b_{11} - \lambda_2 p b_{21})[\lambda_1 b_{12}(1$$

$$-p\lambda_2 b_{21}) + b_{11}(1 - \lambda_1 b_{11} - \lambda_2 p b_{21}) + p\lambda_2 b_{11}(2\lambda_1 b_{11} b_{21} + \lambda_2 b_{22})]$$

$$+12\lambda_3^3 D_7 b_{11}(1 - \lambda_1 b_{11} - \lambda_2 p b_{21})^2$$

$$D_{21} = [(1 - \lambda_1 b_{11})(\lambda_2 b_{22} + 2\lambda_1 b_{11} b_{21}) + \lambda_1^2 b_{12} b_{21}]\left(\prod_{l=1}^{K} \frac{\eta_l}{\lambda_3 + \mu_l} + \lambda_3 \sum_{i=1}^{K} \frac{1}{\nu_i}\right)$$

$$+2\lambda_3^2 b_{21}(1 - \lambda_1 b_{11} - \lambda_2 p b_{21}) D_6$$

$$D_{22} = \left(\prod_{l=1}^{K} \frac{\eta_l}{\lambda_3 + \mu_l} + \lambda_3 \sum_{i=1}^{K} \frac{1}{\nu_i}\right)[3((1 - \lambda_1 b_{11})(\lambda_2 b_{22} + 2\lambda_1 b_{11} b_{21})$$

$$+\lambda_1^2 b_{12} b_{21})(1 - \lambda_1 b_{11} - p\lambda_2 b_{21} + \lambda_1^2 b_{12} + 2p\lambda_1\lambda_2 b_{11} b_{21} + p\lambda_2^2 b_{22})$$

$$+2(1 - \lambda_1 b_{11} - \lambda_2 p b_{21})((1 - \lambda_1 b_{11})\left(\lambda_2^2 b_{23} + 3\lambda_1(\lambda_1 b_{12} b_{21} + \lambda_2 b_{11} b_{22})\right)$$

$$+\lambda_1^3 b_{13} b_{21})] + 6\lambda_3^2(1 - \lambda_1 b_{11} - \lambda_2 p b_{21})[(1 - \lambda_1 b_{11})(\lambda_2 b_{22} + 2\lambda_1 b_{11} b_{21})$$

$$+\lambda_1^2 b_{12} b_{21} + b_{21}(1 - \lambda_1 b_{11} - \lambda_2 p b_{21})] D_6 + 12(1 - \lambda_1 b_{11} - \lambda_2 p b_{21})^2$$

$$\times \lambda_3^3 b_{21} D_7$$

<u>Case 5:</u> The probability $p = 0$

$$Q = \frac{(1-\lambda b_{11})}{D_{23}} \prod_{l=1}^{K} V_l^*(\lambda_3)$$

$$L^b \quad \frac{\lambda_0 D_{24}}{2(1-\lambda b_{11})D_{23}} =$$

$$V_b = \frac{\lambda_0[2D_{23}D_{25} - 3\lambda_0 D_{24}^2]}{12(1-\lambda b_{11})^2 D_{23}^2}$$

$$L_v = \frac{\lambda_0 \lambda_3 d(1-\lambda b_{11})}{2D_{23}}$$

$$V_v = \frac{\lambda_0 \lambda_3 (1-\lambda b_{11})}{12D_{23}^2}\{4\lambda_3 e D_{23} + 6dD_{23} - 3\lambda_0\lambda_3 d^2(1-\lambda b_{11})\}$$

where

$$D_{23} = [1 + (\lambda_0 - \lambda_1)b_{11}] \prod_{l=1}^{K} V_l^*(\lambda_3) + \lambda_0 c[1 + (\lambda_3 - \lambda_1)b_{11}]$$

$$D_{24} = \lambda_1 b_{12}\left(\prod_{l=1}^{K} V_l^*(\lambda_3) + \lambda_3 c\right) + \lambda_3^2 d b_{11}(1-\lambda_1 b_{11})$$

$$D_{25} = [3\lambda_1 b_{12}(1-\lambda_1 b_{11} + \lambda_1^2 b_{12}) + 2\lambda_1^2 b_{13}(1-\lambda_1 b_{11})]\left(\prod_{l=1}^{K} V_l^*(\lambda_3) + \lambda_3 c\right)$$
$$+3\lambda_3^2 d(1-\lambda_1 b_{11})(\lambda_1 b_{12} + b_{11}(1-\lambda_1 b_{11})) + 2\lambda_3^3 e b_{11}(1-\lambda_1 b_{11})^2$$

5. Numerical results

In this section, we present some numerical results in order to illustrate the effect of various parameters on the performance measures of the model (case 2) in section 4. The effect of the parameters arrival rate, service rate, vacation rate and the number of phases of vacation on the system performance measures (i) the mean number of customers in the queue when the server is on essential service (L_b), (ii) the mean number of customers in the queue when the server is on second optional service (L_s), (iii) the mean number of customers in the queue when the server is on vacation (L_v), (iv) the variance of number of customers in the queue when the server is on essential service (V_b), (v) the variance of number of customers in the queue when the server is on second optional service (V_s) and (vi) the variance of number of customers in the queue when the server is on vacation (V_v) have been numerically analyzed . Figure 1 represent the idle probability for different values of service rate. The idle probability increases for increasing value of service rate. Figure 2 represents the graph of mean number of customers when $K = 5$ by varying the service rate. From the figure, it is seen that the mean number of customers when the server is on essential and second optional service is decreases whereas the mean number of customers when the server is on vacation increases for increasing values of service rate. Table 1 shows the variance of number of customers. The variance value with respect to server is on essential and second optional service decreases as the service rate increases but in the case of variance with respect to vacation we encounter the contrary concept that is variance increases. For this analysis the values $\square_0 = 0.5, \square_1 = 0.4, \square_2 = 0.3, \square_3 = 0.2$ are fixed.

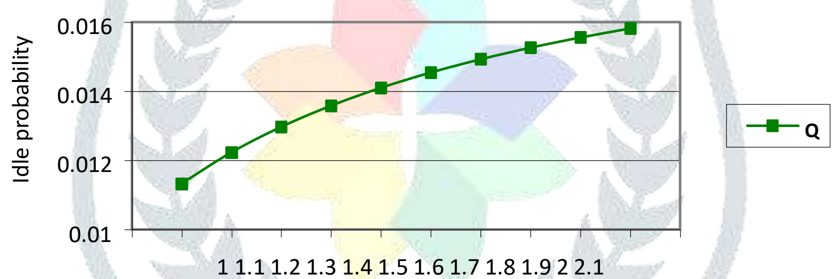

Figure 1: Idle probability

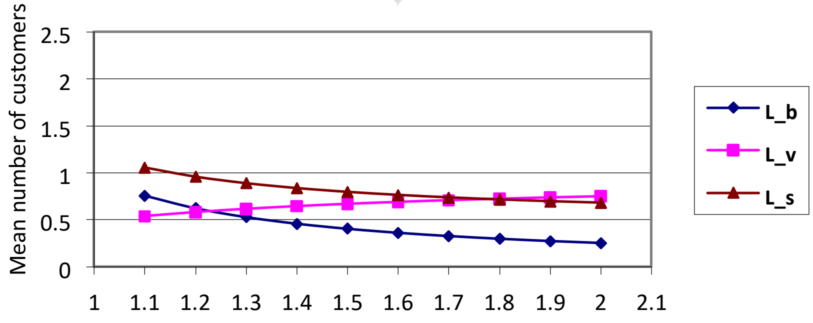

Figure 2: Mean number of customers for K=5

μ	V_b	V_s	V_v
1.1	3.7779	4.7543	1.3778
1.2	2.9671	4.0301	1.4629
1.3	2.4389	3.5580	1.5296
1.4	2.0693	3.2282	1.5832
1.5	1.7968	2.9859	1.6271
1.6	1.5879	2.8008	1.6638
1.7	1.4228	2.6552	1.6948
1.8	1.2889	2.5378	1.7214
1.9	1.1783	2.4411	1.7444
2.0	1.0853	2.3603	1.7645

Table 1: Variance of number of customers for K=5

6. Conclusion

In the forgoing analysis an M/G/1 queue with K-phase of optional vacation and with second optional service has been considered. For this model the queue length distribution and the mean queue length are obtained. An extensive numerical work has been carried out to observe the nature of the operating characteristics.

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