GENERALIZED DIFFERENTIAL TRANSFORM METHOD FOR SOLUTIONS OF NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

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Abstract: ***In the present paper, Generalized Differential Transform Method (GDTM) is used for obtaining the approximate analytic solutions of non-linear partial differential equations of fractional order. The fractional derivatives are described in the Caputo sense.***

Keywords: Fractional differential equations; Caputo fractional derivative; Generalized Differential transform method; Analytic solution.

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1. Introduction:

Differential equations with fractional order are generalizations of classical differential equations of integer order and have recently been proved to be valuable tools in the modeling of many physical phenomena in various fields of science and engineering. By using fractional derivatives a lot of works have been done for a better description of considered material properties. Based on enhanced rheological models Mathematical modeling naturally leads to differential equations of fractional order and to the necessity of the formulation of the initial conditions to such equations. Recently, various analytical and numerical methods have been employed to solve linear and nonlinear fractional differential equations. The differential transform method (DTM) was proposed by Zhou [1] to solve linear and nonlinear initial value problems in electric circuit analysis. This method has been used for solving various types of equations by many authors [2-15]. DTM constructs an analytical solution in the form of a polynomial and different from the traditional higher order Taylor series method. For solving two-dimensional linear and nonlinear partial differential equations of fractional order DTM is further developed as Generalized Differential Transform Method (GDTM) by Momani, Odibat, and Erturk in their papers [16-18].Recently, Vedat Suat Ertiirka and Shaher Momanib applied generalized differential transform method to solve fractional integro-differential equations [19]. The GDTM is implemented to derive the solution of space-time fractional telegraph equation by Mridula Garg,Pratibha Manohar and Shyam L.Kalla [20]. Manish Kumar Bansal,Rashmi Jain applied generalized differential transform method to solve fractional order Riccati differential equation [21]. Aysegul Cetinkaya, Onur Kiymaz and Jale Camli applied generalized differential transform method to solve non linear PDE's of fractional order [22].

2 Mathematical Preliminaries on Fractional Calculus:

Many definitions of fractional calculus have been developed to solve the problems of fractional differential equations. The most frequently encountered definitions include **Riemann-Liouville**, **Caputo**, **Wely**, **Rize** fractional operator. Introducing the following definitions [23, 24] in the present analysis:

2.1 Definition: Let $\alpha \in \boldsymbol{R}^{+}$. The integral operator I^{α} defined on the usual Lebesgue space L (a, b) by

$$I^{\alpha} f(x) = \frac{d^{-\alpha} f(x)}{dx^{-\alpha}} = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt$$

$$I^0 f(x) = f(x),$$

for $x \in [a,b]$ is called **Riemann-Liouville** fractional **integral operator** of order α (≥0).

It has the following properties:

(i) $I^{\alpha} f(x)$exists for any $x\epsilon[a,b]$

$(ii) I^{\alpha} I^{\beta} f(x) = I^{\alpha+\beta} f(x)$

(iii) $I^{\alpha} I^{\beta} f(x) = I^{\beta} I^{\alpha} f(x)$

(iv) $I^{\alpha} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$,

where $f(x)\epsilon L[a,b], \alpha, \beta \geq 0, \gamma > -1$

2.2 Definition: The Riemann-Liouville definition of fractional order derivative is

$${}_{0}^{RL}D_x^{\alpha} f(x) = \frac{d^n}{dx^n}\, {}_{0}I_x^{n-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} f(t)\, dt\,,$$

where n is an integer that satisfies $\alpha \in (n-1, n)$

2.3 Definition: A modified fractional differential operator ${}_0^cD_x^\alpha$ proposed by **Caputo** is given by

$$ {}_0^cD_x^\alpha f(x) = {}_0I_x^{n-\alpha}\frac{d^n}{dx^n}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1}f^{(n)}(t)\,dt, $$

where $\alpha \in \boldsymbol{R}^+$ is the order of operation and n is an integer that satisfies $\alpha \in (n-1, n)$.
It has the following two basic properties[25]:
(i) If $f \in L_\infty(a,b)$ or $f \in C[a,b]$ and $\alpha > 0$ then
${}_0^cD_x^\alpha\ {}_0I_x^\alpha f(x) = f(x)$
(ii) If $f \in C^n[a,b]$ and if $\alpha > 0$,then
${}_0I_x^\alpha\,{}_0^cD_x^\alpha f(x) = f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(0^+)}{k!}x^k$; $\alpha \in (n-1,n)$

2.4 Definition: For m being the smallest integer that exceeds α, the **Caputo time-fractional** derivative operator of order $\alpha > 0$, is defined as in [26]

$$ D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \dfrac{\partial^m u(x,\xi)}{\partial \xi^m}\ ; & \alpha = m\epsilon\mathbb{N} \\ \dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t (t-\xi)^{m-\alpha-1}\frac{\partial^m u(x,\xi)}{\partial \xi^m}d\xi\ ; & \mathrm{m}-1 \le \alpha < m \end{cases} $$

Relation between Caputo derivative and Riemann-Liouville derivative:
${}_0^cD_t^\alpha f(x) = {}_0^{RL}D_t^\alpha f(x) - \sum_{k=0}^{m-1}\frac{f^{(k)}(0^+)}{\Gamma(k-\alpha+1)}x^{k-\alpha}$; $\alpha \in (m-1,m)$.
Integrating by parts, we get the following formulae as given in [27]:
(i) $\int_a^b g(x)\,{}_a^cD_x^\alpha f(x)dx = \int_a^b f(x)\,{}_x^{RL}D_b^\alpha g(x)dx + \sum_{j=0}^{n-1}\left[{}_x^{RL}D_b^{\alpha+j-n}g(x)\,{}_x^{RL}D_b^{n-1-j}f(x)\right]_a^b$
(ii) For $n=1$, $\int_a^b g(x)\,{}_a^cD_x^\alpha f(x)dx = \int_a^b f(x)\,{}_x^{RL}D_b^\alpha g(x)dx + \left[{}_x\mathrm{I}_b^{1-\alpha}\mathrm{g(x).f(x)}\right]_a^b$

3 Generalized two dimensional differential transform method:-

Consider a function of two variables $u(x,y)$ be a product of two single-variable functions, i.e. $u(x,y) = f(x)g(y)$, which is analytic and differentiated continuously with respect to *x* and *y* in the domain of interest. Then the generalized two-dimensional differential transform of the function $u(x,y)$ is given by [16-18]

$$ U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[\left(D_{x_0}^\alpha\right)^k\left(D_{y_0}^\beta\right)^h u(x,y)\right]_{(x_0,y_0)} \quad \textbf{(3.1)} $$

where $0 < \alpha, \beta \le 1$; $U_{\alpha,\beta}(k,h) = F_\alpha(k)G_\beta(h)$ is called the spectrum of $u(x,y)$ and $\left(D_{x_0}^\alpha\right)^k = D_{x_0}^\alpha, D_{x_0}^\alpha, \dots\dots, D_{x_0}^\alpha$ (k-times).
The inverse generalized differential transform of $U_{\alpha,\beta}(k,h)$ is given by

$$ u(x,y) = \sum_{k=0}^\infty\sum_{h=0}^\infty U_{\alpha,\beta}(k,h)(x-x_0)^{k\alpha}(y-y_0)^{h\beta} \quad \textbf{(3.2)} $$

It has the following properties:
(i) if $u(x,y) = v(x,y) \pm w(x,y)$ then $U_{\alpha,\beta}(k,h) = V_{\alpha,\beta}(k,h) \pm W_{\alpha,\beta}(k,h)$
(ii) if $(x,y) = av(x,y)$,$a \in \mathbb{R}$ then $U_{\alpha,\beta}(k,h) = aV_{\alpha,\beta}(k,h)$
(iii) if $u(x,y) = v(x,y)w(x,y)$ then $U_{\alpha,\beta}(k,h) = \sum_{r=0}^k\sum_{s=0}^h V_{\alpha,\beta}(r,h-s)\,W_{\alpha,\beta}(k-r,s)$
(iv) if $u(x,y) = (x-x_0)^{n\alpha}(y-y_0)^{m\beta}$ then $U_{\alpha,\beta}(k,h) = \delta(\mathrm{k-n})\delta(\mathrm{h-m})$
(v) if $u(x,y) = D_{x_0}^\alpha v(x,y), 0 < \alpha \le 1$ then $U_{\alpha,\beta}(k,h) = \frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}V_{\alpha,\beta}(k+1,h)$
(vi) if $u(x,y) = D_{x_0}^\gamma v(x,y), 0 < \gamma \le 1$ then $U_{\alpha,\beta}(k,h) = \frac{\Gamma(\alpha k+\gamma+1)}{\Gamma(\alpha k+1)}V_{\alpha,\beta}\left(k+\frac{\gamma}{\alpha},h\right)$
(vii) if $u(x,y) = D_{y_0}^\gamma v(x,y), 0 < \gamma \le 1$ then $U_{\alpha,\beta}(k,h) = \frac{\Gamma(\beta h+\gamma+1)}{\Gamma(\beta h+1)}V_{\alpha,\beta}\left(k,h+\frac{\gamma}{\beta}\right)$
(viii) if $u(x,y) = f(x)g(y)$ and the function $f(x) = x^\lambda h(x)$ where $\lambda > -1$, $h(x)$ has the generalized Taylor series expansion
$h(x) = \sum_{n=0}^\infty a_n\left(x-x_0\right)^{\alpha k}$ and

(a) $\beta < \lambda+1$ and α is arbitrary or

(b) $\beta \ge \lambda+1$, α is arbitrary and $a_n = 0$ for $n = 0,1,2,\dots\dots m-1$, where $m-1 < \beta \le m$.

Then (3.1) becomes

$$ U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[D_{x_0}^{\alpha k}\left(D_{y_0}^\beta\right)^h u(x,y)\right]_{(x_0,y_0)} $$

(ix) if $v(x,y) = f(x)g(y)$, the function $f(x)$ satisfies the conditions given in (viii) and $u(x,y) = D_{x_0}^\gamma v(x,y)$,then

$$U_{\alpha,\beta}(k,h)=\frac{\Gamma(\alpha(k+1)+\gamma)}{\Gamma(\alpha k+1)}V_{\alpha,\beta}\left(k+\frac{\gamma}{\alpha},h\right)$$

where $U_{\alpha,\beta}(k,h), V_{\alpha,\beta}(k,h)$ and $W_{\alpha,\beta}(k,h)$ are the differential transformations of the functions $u(x,y), v(x,y)$ and $w(x,y)$ respectively and

$$\delta(k-n)=\begin{cases}1 & ; \ k=n\\ 0 & ; \ k\neq n\end{cases}$$

4 Test Problems

In this section, we present three examples to illustrate the applicability of Generalized Differential Transform Method (GDTM) to solve non linear partial differential equations of fractional order.

4.1 Example: Consider the non-linear inhomogeneous time-fractional partial differential equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}+u(x,t)\frac{\partial u(x,t)}{\partial x}+\frac{\partial^2 u(x,t)}{\partial x^2}=2t^\alpha+2x^2+2 \ ; \ t>0$$

subject to initial condition $u(x,0)=x^2$

(4.1.1)

where $\frac{\partial^\alpha}{\partial t^\alpha}$ is the fractional differential operator(Caputo derivative) of order $0<\alpha\le 1$

Applying generalized two-dimensional differential transform (3.1) with $(x_0,t_0)=(0,0)$ on (4.1.1) we obtain

$$U_{1,\alpha}(k,h)=\frac{\Gamma(\alpha(h-1)+1)}{\Gamma(\alpha h+1)}\Bigg[-\sum_{r=0}^{k}\sum_{s=0}^{h-1}U_{1,\alpha}(r,h-s-1)(k-r+1)U_{1,\alpha}(k-r+1,s)-(k+2)(k+1)U_{1,\alpha}(k+2,h-1)$$
$$+2\delta(k)\delta(h-2)+2\delta(k-2)\delta(h-1)+2\delta(k)\delta(h-1)\Bigg] \quad \textbf{(4.1.2)}$$

and $$U_{1,\alpha}(k,0)=\delta(k-2)=\begin{cases}1 & ;k=2\\ 0 & ;k\neq 2\end{cases} \quad \textbf{(4.1.3)}$$

Now utilizing the recurrence relation (4.1.2) and the initial condition (4.1.3), we obtain after a little simplification the following values of $U_{1,\alpha}(k,h)$ for $k=0,1,2,\ldots$ and $h=0,1,2,3\ldots$

$$U_{1,\alpha}(2,0)=1;\ U_{1,\alpha}(k,0)=0\ \ \forall k\in W-\{2\};$$

$$U_{1,\alpha}(2,1)=\frac{2}{\Gamma(\alpha+1)};\ U_{1,\alpha}(0,3)=0;\ U_{1,\alpha}(2,2)=0;$$

$$U_{1,\alpha}(0,1)=0;\ U_{1,\alpha}(0,2)=\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\left[2-\frac{4}{\Gamma(\alpha+1)}\right];$$

$$U_{1,\alpha}(1,1)=0;\ U_{1,\alpha}(4,1)=0;\ U_{1,\alpha}(0,4)=\frac{312}{\Gamma(4\alpha+1)};$$

$$U_{1,\alpha}(2,3)=-\frac{156}{\Gamma(3\alpha+1)};U_{1,\alpha}(1,2)=\frac{12}{\Gamma(2\alpha+1)};$$

$$U_{1,\alpha}(3,1)=-\frac{2}{\Gamma(\alpha+1)};\ U_{1,\alpha}(4,2)=\frac{10}{\Gamma(2\alpha+1)};$$

$$U_{1,\alpha}(6,1)=0;\ U_{1,\alpha}(1,3)=\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\left[\frac{2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\left\{2-\frac{4}{\Gamma(\alpha+1)}\right\}+\frac{48}{\Gamma(2\alpha+1)}\right];$$

$$U_{1,\alpha}(3,2)=-\frac{8}{\Gamma(2\alpha+1)};\ U_{1,\alpha}(5,1)=0;\ U_{1,\alpha}(1,4)=\frac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}\left[\frac{8}{\Gamma(2\alpha+1)}-\frac{16}{\Gamma(2\alpha+1)\Gamma(\alpha+1)}+\frac{48\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)(\Gamma(\alpha+1))^2}\right];$$

$$U_{1,\alpha}(3,3)=-\frac{8\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)(\Gamma(\alpha+1))^2};\ U_{1,\alpha}(5,2)=0;\ U_{1,\alpha}(7,1)=0;$$

$$U_{1,\alpha}(2,4)=\frac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}\left[\frac{12}{\Gamma(2\alpha+1)}-\frac{600}{\Gamma(3\alpha+1)}-\frac{12\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}-\frac{96}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}-\frac{240\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)(\Gamma(\alpha+1))^2}\right];$$

$$U_{1,\alpha}(4,3)=\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\left[\frac{40}{\Gamma(2\alpha+1)}+\frac{20}{(\Gamma(\alpha+1))^2}\right];U_{1,\alpha}(6,2)=0;\ U_{1,\alpha}(8,1)=0$$

and so on

Now, from (3.2), we have

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U_{1,\alpha}(k,h)x^k t^{\alpha h} \tag{4.1.4}$$

Using the above values of $U_{1,\alpha}(k,h)$ in (4.1.4), the solution of (4.1.1) is obtained as

$$u(x,t)=\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\left[2-\frac{4}{\Gamma(\alpha+1)}\right]t^{2\alpha}+\frac{312}{\Gamma(4\alpha+1)}t^{4\alpha}+\frac{12}{\Gamma(2\alpha+1)}xt^{2\alpha}+\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\left[\frac{4\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}-\frac{40}{\Gamma(2\alpha+1)}\right]xt^{3\alpha}$$
$$+\frac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}\left[\frac{8}{\Gamma(2\alpha+1)}-\frac{16}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}+\frac{48\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)(\Gamma(\alpha+1))^2}\right]xt^{4\alpha}+x^2+\frac{2}{\Gamma(\alpha+1)}x^2t^{\alpha}$$
$$-\frac{156}{\Gamma(3\alpha+1)}x^2t^{3\alpha}+\frac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}\left[\frac{12}{\Gamma(2\alpha+1)}-\frac{600}{\Gamma(3\alpha+1)}-\frac{12\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}-\frac{96}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right.$$
$$\left.-\frac{240\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)(\Gamma(\alpha+1))^2}\right]x^2t^{4\alpha}+\ldots\ldots \tag{4.1.5}$$

4.2 Example: Consider the non-linear space-fractional telegraph partial differential equation

$$\frac{\partial^{3\alpha}u(x,t)}{\partial x^{3\alpha}}=\frac{\partial^2 u(x,t)}{\partial t^2}+\frac{\partial u(x,t)}{\partial t}+u(x,t)+cu^2(x,t);\ x>0$$

subject to initial condition $u(0,t)=e^{-t}$; $\frac{\partial u(0,t)}{\partial x}=e^{-t}$ and $\frac{\partial^2 u(0,t)}{\partial x^2}=e^{-t}$

(4.2.1)

where $\frac{\partial^\alpha}{\partial x^\alpha}$ is the fractional differential operator(Caputo derivative) of order $0<\alpha\le 1$

and c= constant

Applying generalized two-dimensional differential transform (3.1) with $(x_0,t_0)=(0,0)$ on (4.2.1) we obtain

$$U_{\alpha,1}(k,h)=\frac{\Gamma(\alpha(k-3)+1)}{\Gamma(\alpha k+1)}\left[\frac{\Gamma(h+3)}{\Gamma(h+1)}U_{\alpha,1}(k-3,h+2)+\frac{\Gamma(h+2)}{\Gamma(h+1)}U_{\alpha,1}(k-3,h+1)+U_{\alpha,1}(k-3,h)\right.$$
$$\left.+c\sum_{r=0}^{k-3}\sum_{s=0}^{h}U_{\alpha,1}(r,h-s)U_{\alpha,1}(k-r-3,s)\right] \tag{4.2.2}$$

and $U_{\alpha,1}(0,h)=\frac{(-1)^h}{h!}$; $U_{\alpha,1}\left(1/\alpha,h\right)=\frac{(-1)^h}{h!}$; $U_{\alpha,1}\left(2/\alpha,h\right)=\frac{(-1)^h}{2h!}$ **(4.2.3)**

Taking $\alpha = \frac{1}{2}$, then (4.2.2) and (4.2.3) becomes

$$U_{\frac{1}{2},1}(k,h) = \frac{\Gamma\left(\frac{1}{2}(k-3)+1\right)}{\Gamma\left(\frac{1}{2}k+1\right)}\left[\frac{\Gamma(h+3)}{\Gamma(h+1)}U_{\frac{1}{2},1}(k-3,h+2)+\frac{\Gamma(h+2)}{\Gamma(h+1)}U_{\frac{1}{2},1}(k-3,h+1)+U_{\frac{1}{2},1}(k-3,h)\right.$$
$$\left.+c\sum_{r=0}^{k-3}\sum_{s=0}^{h}U_{\frac{1}{2},1}(r,h-s)U_{\frac{1}{2},1}(k-r-3,s)\right] \quad \textbf{(4.2.4)}$$

and $U_{\frac{1}{2},1}(0,h) = \frac{(-1)^h}{h!}$; $U_{\frac{1}{2},1}(2,h) = \frac{(-1)^h}{h!}$ **(4.2.5)**

Utilizing the recurrence relation (4.2.4) and the initial condition (4.2.5), we obtain after a little simplification the following values of $U_{1/2,1}(k,h)$ for $k = 0,1,2,\ldots$ and $h = 0,1,2,3\ldots$

$U_{\frac{1}{2},1}(0,0)=1$; $U_{\frac{1}{2},1}(0,1)=-1$; $U_{\frac{1}{2},1}(0,2)=\frac{1}{2!}$; $U_{\frac{1}{2},1}(0,3)=-\frac{1}{3!}$; $U_{\frac{1}{2},1}(0,4)=\frac{1}{4!}$; $U_{\frac{1}{2},1}(0,5)=-\frac{1}{5!}$; $U_{\frac{1}{2},1}(0,6)=\frac{1}{6!}$;

$U_{\frac{1}{2},1}(0,7)=-\frac{1}{7!}$; $U_{\frac{1}{2},1}(1,h)=0 \ \ \forall h\in W$; $U_{\frac{1}{2},1}(2,0)=1$; $U_{\frac{1}{2},1}(2,1)=-1$; $U_{\frac{1}{2},1}(2,2)=\frac{1}{2!}$; $U_{\frac{1}{2},1}(2,3)=-\frac{1}{3!}$;

$U_{\frac{1}{2},1}(2,4)=\frac{1}{4!}$; $U_{\frac{1}{2},1}(2,5)=-\frac{1}{5!}$;

$U_{\frac{1}{2},1}(2,6)=\frac{1}{6!}$; $U_{\frac{1}{2},1}(2,7)=-\frac{1}{7!}$; $U_{\frac{1}{2},1}(3,0)=\frac{1}{\Gamma\left(\frac{5}{2}\right)}(1+c)$; $U_{\frac{1}{2},1}(3,1)=-\frac{1}{\Gamma\left(\frac{5}{2}\right)}(1+2c)$;

$U_{\frac{1}{2},1}(3,2)=\frac{1}{\Gamma\left(\frac{5}{2}\right)}\left(\frac{1}{2!}+\frac{4c}{2!}\right)$; $U_{\frac{1}{2},1}(3,3)=-\frac{1}{\Gamma\left(\frac{5}{2}\right)}\left(\frac{1}{3!}+\frac{8c}{3!}\right)$; $U_{\frac{1}{2},1}(3,4)=\frac{1}{\Gamma\left(\frac{5}{2}\right)}\left(\frac{1}{4!}+\frac{16c}{4!}\right)$;

$U_{\frac{1}{2},1}(3,5)=-\frac{1}{\Gamma\left(\frac{5}{2}\right)}\left(\frac{1}{5!}+\frac{32c}{5!}\right)$; $U_{\frac{1}{2},1}(4,h)=0 \ \ \forall h\in W$; $U_{\frac{1}{2},1}(5,0)=\frac{1}{\Gamma\left(\frac{7}{2}\right)}(1+2c)$; $U_{\frac{1}{2},1}(5,1)=-\frac{1}{\Gamma\left(\frac{7}{2}\right)}(1+4c)$;

$U_{\frac{1}{2},1}(5,2)=\frac{1}{\Gamma\left(\frac{7}{2}\right)}\left(\frac{1}{2!}+\frac{8c}{2!}\right)$; $U_{\frac{1}{2},1}(5,3)=-\frac{1}{\Gamma\left(\frac{7}{2}\right)}\left(\frac{1}{3!}+\frac{16c}{3!}\right)$; $U_{\frac{1}{2},1}(5,4)=\frac{1}{\Gamma\left(\frac{7}{2}\right)}\left(\frac{1}{4!}+\frac{32c}{4!}\right)$;

$U_{\frac{1}{2},1}(5,5)=-\frac{1}{\Gamma\left(\frac{7}{2}\right)}\left(\frac{1}{5!}+\frac{64c}{5!}\right)$

and so on

Now, from (3.2) and using the above values of $U_{1/2,1}(k,h)$, the solution of (4.2.1) is obtained as

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U_{\frac{1}{2},1}(k,h)x^{\frac{k}{2}}t^h$$
$$=1-t+\frac{1}{2!}t^2-\frac{1}{3!}t^3+\frac{1}{4!}t^4-\frac{1}{5!}t^5+\frac{1}{6!}t^6-\frac{1}{7!}t^7+\left(1-t+\frac{1}{2!}t^2-\frac{1}{3!}t^3+\frac{1}{4!}t^4-\frac{1}{5!}t^5+\frac{1}{6!}t^6\right)x$$
$$+\left(\frac{1}{\Gamma\left(\frac{5}{2}\right)}(1+c)-\frac{1}{\Gamma\left(\frac{5}{2}\right)}(1+2c)t+\frac{1}{\Gamma\left(\frac{5}{2}\right)}\left(\frac{1}{2!}+\frac{4c}{2!}\right)t^2-\frac{1}{\Gamma\left(\frac{5}{2}\right)}\left(\frac{1}{3!}+\frac{8c}{3!}\right)t^3+\frac{1}{\Gamma\left(\frac{5}{2}\right)}\left(\frac{1}{4!}+\frac{16c}{4!}\right)t^4\right.$$
$$\left.-\frac{1}{\Gamma\left(\frac{5}{2}\right)}\left(\frac{1}{5!}+\frac{32c}{5!}\right)t^5\right)x^{\frac{3}{2}}+\left(\frac{1}{\Gamma\left(\frac{7}{2}\right)}(1+2c)-\frac{1}{\Gamma\left(\frac{7}{2}\right)}(1+4c)t+\frac{1}{\Gamma\left(\frac{7}{2}\right)}\left(\frac{1}{2!}+\frac{8c}{2!}\right)t^2-\frac{1}{\Gamma\left(\frac{7}{2}\right)}\left(\frac{1}{3!}+\frac{16c}{3!}\right)t^3\right.$$

$$+\frac{1}{\Gamma\left(\frac{7}{2}\right)}\left(\frac{1}{4!}+\frac{32c}{4!}\right)t^4-\frac{1}{\Gamma\left(\frac{7}{2}\right)}\left(\frac{1}{5!}+\frac{64c}{5!}\right)t^5\Bigg)x^{5/2}+......$$

(4.2.6)

Taking $\alpha = 1/4$, then (4.2.2) and (4.2.3) becomes

$$U_{1/4,1}(k,h)=\frac{\Gamma\left(\frac{1}{4}(k-3)+1\right)}{\Gamma\left(\frac{1}{4}k+1\right)}\left[\frac{\Gamma(h+3)}{\Gamma(h+1)}U_{1/4,1}(k-3,h+2)+\frac{\Gamma(h+2)}{\Gamma(h+1)}U_{1/4,1}(k-3,h+1)+U_{1/4,1}(k-3,h)\right.$$
$$\left.+c\sum_{r=0}^{k-3}\sum_{s=0}^{h}U_{1/4,1}(r,h-s)U_{1/4,1}(k-r-3,s)\right]$$

(4.2.7)

and $U_{1/4,1}(0,h)=\frac{(-1)^h}{h!}$; $U_{1/4,1}(4,h)=\frac{(-1)^h}{h!}$ **(4.2.8)**

Utilizing the recurrence relation (4.2.7) and the initial condition (4.2.8),we obtain after a little simplification the following values of $U_{1/4,1}(k,h)$ for $k = 0,1,2,...$ and $h = 0,1,2,3 ...$

$U_{1/4,1}(0,0)=1;U_{1/4,1}(0,1)=-1;\ U_{1/4,1}(0,2)=\frac{1}{2!};\ U_{1/4,1}(0,3)=-\frac{1}{3!};U_{1/4,1}(0,4)=\frac{1}{4!};\ U_{1/4,1}(0,5)=-\frac{1}{5!};\ U_{1/4,1}(0,6)=\frac{1}{6!};$
$U_{1/4,1}(0,7)=-\frac{1}{7!};\ U_{1/4,1}(1,h)=0\ \ \forall h\in W;$
$U_{1/4,1}(2,h)=0\ \ \forall h\in W;U_{1/4,1}(3,h)=0\ \ \forall h\in W;U_{1/4,1}(4,0)=1;\ U_{1/4,1}(4,1)=-1;$
$U_{1/4,1}(4,2)=\frac{1}{2!};U_{1/4,1}(4,3)=-\frac{1}{3!};\ U_{1/4,1}(4,4)=\frac{1}{4!};\ U_{1/4,1}(4,5)=-\frac{1}{5!};U_{1/4,1}(4,6)=\frac{1}{6!};\ U_{1/4,1}(4,7)=-\frac{1}{7!};\ U_{1/4,1}(5,h)=0$
$\forall h\in W;U_{1/4,1}(6,h)=0\ \ \forall h\in W;U_{1/4,1}(7,0)=\frac{1}{\Gamma\left(\frac{11}{4}\right)}(1+2c);\ U_{1/4,1}(7,1)=-\frac{1}{\Gamma\left(\frac{11}{4}\right)}(1+4c);$
$U_{1/4,1}(7,2)=\frac{1}{\Gamma\left(\frac{11}{4}\right)}\left(\frac{1}{2!}+\frac{8c}{2!}\right);\ U_{1/4,1}(7,3)=-\frac{1}{\Gamma\left(\frac{11}{4}\right)}\left(\frac{1}{3!}+\frac{16c}{3!}\right);U_{1/4,1}(7,4)=\frac{1}{\Gamma\left(\frac{11}{4}\right)}\left(\frac{1}{4!}+\frac{32c}{4!}\right);$
$U_{1/4,1}(7,5)=-\frac{1}{\Gamma\left(\frac{11}{4}\right)}\left(\frac{1}{5!}+\frac{64c}{5!}\right);U_{1/4,1}(8,h)=0\ \ \forall h\in W;U_{1/4,1}(9,h)=0\ \ \forall h\in W$

and so on

Now, from (3.2) and using the above values of $U_{1/4,1}(k,h)$, the solution of (4.2.1) is obtained as

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U_{1/4,1}(k,h)x^{k/4}t^h$$
$$=1-t+\frac{1}{2!}t^2-\frac{1}{3!}t^3+\frac{1}{4!}t^4-\frac{1}{5!}t^5+\frac{1}{6!}t^6-\frac{1}{7!}t^7+\left(1-t+\frac{1}{2!}t^2-\frac{1}{3!}t^3+\frac{1}{4!}t^4-\frac{1}{5!}t^5+\frac{1}{6!}t^6-\frac{1}{7!}t^7\right)x$$
$$+\Bigg(\frac{1}{\Gamma\left(\frac{11}{4}\right)}(1+2c)-\frac{1}{\Gamma\left(\frac{11}{4}\right)}(1+4c)t+\frac{1}{\Gamma\left(\frac{11}{4}\right)}\left(\frac{1}{2!}+\frac{8c}{2!}\right)t^2-\frac{1}{\Gamma\left(\frac{11}{4}\right)}\left(\frac{1}{3!}+\frac{16c}{3!}\right)t^3$$
$$+\frac{1}{\Gamma\left(\frac{11}{4}\right)}\left(\frac{1}{4!}+\frac{32c}{4!}\right)t^4-\frac{1}{\Gamma\left(\frac{11}{4}\right)}\left(\frac{1}{5!}+\frac{64c}{5!}\right)t^5\Bigg)x^{7/4}+......$$

(4.2.9)

4.3 Example: Consider the non-linear space time fractional wave partial differential equation

$$\frac{\partial^{3\beta}u(x,t)}{\partial t^{3\beta}}=\frac{1}{2}x^2\frac{\partial^{5\alpha}u(x,t)}{\partial x^{5\alpha}}+cu^2(x,t);\ x>0,\ t>0$$

subject to initial condition $u(x,0)=\sum_{n=0}^{\infty}a_nx^n$; $u_t(x,0)=\sum_{n=0}^{\infty}b_nx^n$ and $u_t^2(x,0)=\sum_{n=0}^{\infty}c_nx^n$ **(4.3.1)**

where $\frac{\partial^{\alpha}}{\partial x^{\alpha}}$ and $\frac{\partial^{\beta}}{\partial t^{\beta}}$ are fractional differential operators (Caputo derivative) of order $0<\alpha,\beta\le 1$ Applying generalized two-dimensional differential transform (3.1) with $(x_0,t_0)=(0,0)$ on (4.3.1) we obtain

$$U_{\alpha,\beta}(k,h)=\frac{\Gamma(\beta(h-3)+1)}{\Gamma(\beta(h-3)+4)}\left[\frac{1}{2}\sum_{r=0}^{k}\sum_{s=0}^{h-3}\delta\left(r-\frac{2}{\alpha}\right)\delta(h-s-3)\frac{\Gamma(\alpha(k-r)+6)}{\Gamma(\alpha(k-r)+1)}U_{\alpha,\beta}(k-r+5,s)\right.$$
$$\left.+c\sum_{r=0}^{k}\sum_{s=0}^{h-3}U_{\alpha,\beta}(r,h-s-3)U_{\alpha,\beta}(k-r,s)\right]$$ **(4.3.2)**

and $U_{\alpha,\beta}(k,0)=a_k;U_{\alpha,\beta}\left(k,\frac{1}{\beta}\right)=b_k$; $U_{\alpha,\beta}\left(k,\frac{2}{\beta}\right)=\frac{c_k}{2!}$ **(4.3.3)** Taking $\alpha=1/4$ and $\beta=1/2$, then (4.3.2) and (4.3.3) becomes

$$U_{1/4,1/2}(k,h)=\frac{\Gamma\left(\frac{1}{2}(h-3)+1\right)}{\Gamma\left(\frac{1}{2}(h-3)+4\right)}\left[\frac{1}{2}\sum_{r=0}^{k}\sum_{s=0}^{h-3}\delta(r-8)\delta(h-s-3)\frac{\Gamma\left(\frac{1}{4}(k-r)+6\right)}{\Gamma\left(\frac{1}{4}(k-r)+1\right)}U_{1/4,1/2}(k-r+5,s)\right.$$
$$\left.+c\sum_{r=0}^{k}\sum_{s=0}^{h-3}U_{1/4,1/2}(r,h-s-3)U_{1/4,1/2}(k-r,s)\right]$$ **(4.3.4)**

and $U_{1/4,1/2}(k,0)=a_k;U_{1/4,1/2}(k,2)=b_k$; $U_{1/4,1/2}(k,1)=0$ **(4.3.5)** Now utilizing the recurrence relation (4.3.4) and the initial conditions (4.3.5),we obtain after a little simplification the following values of $U_{1/4,1/2}(k,h)$ for $k=0,1,2,\ldots$ and $h=0,1,2,3\ldots$

$U_{1/4,1/2}(0,0)=a_0$; $U_{1/4,1/2}(0,1)=0$; $U_{1/4,1/2}(0,2)=b_0$; $U_{1/4,1/2}(0,3)=\frac{1}{\Gamma(4)}ca_0^2$; $U_{1/4,1/2}(0,4)=0$; $U_{1/4,1/2}(0,5)=\frac{1}{\Gamma(5)}2ca_0b_0$;

$U_{1/4,1/2}(0,6)=\frac{\Gamma(5/2)}{\Gamma(11/2)\Gamma(4)}2ca_0^3$; $U_{1/4,1/2}(0,7)=\frac{\Gamma(3)}{\Gamma(6)}cb_0^2$; $U_{1/4,1/2}(0,8)=\frac{\Gamma(7/2)}{\Gamma(13/2)\Gamma(4)}3c^2a_0^2b_0$;

$U_{1/4,1/2}(0,9)=\frac{1}{\Gamma(7)}c^2a_0^4\left[\frac{4\Gamma(5/2)}{\Gamma(11/2)}+\frac{1}{\Gamma(4)}c\right]$; $U_{1/4,1/2}(0,10)=\frac{\Gamma(9/2)}{\Gamma(15/2)\Gamma(6)}24c^2a_0b_0^2$; $U_{1/4,1/2}(1,0)=a_1$; $U_{1/4,1/2}(1,1)=0$;

$U_{1/4,1/2}(1,2)=b_1$;

$U_{1/4,1/2}(1,3)=\frac{1}{\Gamma(4)}2ca_0a_1$; $U_{1/4,1/2}(1,4)=0$; $U_{1/4,1/2}(1,5)=\frac{1}{\Gamma(5)}2c(a_0b_1+b_0a_1)$;

$U_{1/4,1/2}(1,6)=\frac{\Gamma(5/2)}{\Gamma(11/2)\Gamma(4)}6c^2a_0^2a_1$; $U_{1/4,1/2}(1,7)=\frac{\Gamma(3)}{\Gamma(6)}2cb_0b_1$;

$U_{1/4,1/2}(1,8)=\frac{\Gamma(7/2)}{\Gamma(13/2)\Gamma(4)}2c^2a_0\left(a_1b_0+\frac{3}{2}a_0b_1+2a_1b_0\right)$;

$$U_{\frac14,\frac12}(1,9)=\frac{1}{\Gamma(7)}4c^2a_0^3a_1\left[\frac{\Gamma(5/2)}{\Gamma(11/2)}+\frac{1}{\Gamma(4)}c+\frac{\Gamma(5/2)}{\Gamma(11/2)}3c\right];$$

$$U_{\frac14,\frac12}(1,10)=\frac{\Gamma(9/2)}{\Gamma(15/2)\Gamma(6)}24c^2b_0(a_1b_0+2a_0b_1);U_{\frac14,\frac12}(2,0)=a_2;\ U_{\frac14,\frac12}(2,1)=0;\ U_{\frac14,\frac12}(2,2)=b_2;$$

$$U_{\frac14,\frac12}(2,3)=\frac{1}{\Gamma(4)}c(2a_0a_2+a_1^2);\ U_{\frac14,\frac12}(2,4)=0;$$

$$U_{\frac14,\frac12}(3,0)=a_3;\ U_{\frac14,\frac12}(3,1)=0;\ U_{\frac14,\frac12}(3,2)=b_3;U_{\frac14,\frac12}(3,3)=\frac{1}{\Gamma(4)}2c(a_0a_3+a_1a_2);\ U_{\frac14,\frac12}(3,4)=0;U_{\frac14,\frac12}(4,0)=a_4$$

$$;\ U_{\frac14,\frac12}(4,1)=0;\ U_{\frac14,\frac12}(4,2)=b_4;U_{\frac14,\frac12}(4,3)=\frac{1}{\Gamma(4)}c\left(2(a_0a_4+a_1a_3)+a_2^2\right);\ U_{\frac14,\frac12}(4,4)=0;U_{\frac14,\frac12}(5,0)=a_5;$$

$$U_{\frac14,\frac12}(5,1)=0;\ U_{\frac14,\frac12}(5,2)=b_5;U_{\frac14,\frac12}(5,3)=\frac{1}{\Gamma(4)}2c(a_0a_5+a_1a_4+a_2a_3);\ U_{\frac14,\frac12}(5,4)=0;$$

$$U_{\frac14,\frac12}(6,0)=a_6;\ U_{\frac14,\frac12}(6,1)=0;\ U_{\frac14,\frac12}(6,2)=b_6;$$

$$U_{\frac14,\frac12}(6,3)=\frac{1}{\Gamma(4)}c\left(2(a_0a_6+a_1a_5+a_2a_4)+a_3^2\right);\ U_{\frac14,\frac12}(6,4)=0;U_{\frac14,\frac12}(7,0)=a_7;\ U_{\frac14,\frac12}(7,1)=0;\ U_{\frac14,\frac12}(7,2)=b_7;$$

$$U_{\frac14,\frac12}(7,3)=\frac{1}{\Gamma(4)}2c(a_0a_7+a_1a_6+a_2a_5+a_3a_4);\ U_{\frac14,\frac12}(7,4)=0;U_{\frac14,\frac12}(8,0)=a_8;\ U_{\frac14,\frac12}(8,1)=0;\ U_{\frac14,\frac12}(8,2)=b_8;$$

$$U_{\frac14,\frac12}(8,3)=\frac{1}{\Gamma(4)}\left(\frac12\Gamma(6)a_5+c\left(2(a_0a_8+a_1a_7+a_2a_6+a_3a_5)+a_4^2\right)\right);U_{\frac14,\frac12}(8,4)=0;$$

and so on

Now, from (3.2), we have

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U_{\frac14,\frac12}(k,h)x^{k/4}t^{h/2} \qquad \textbf{(4.3.6)}$$

Using the above values of $U_{1/4,1/2}(k,h)$ in (4.3.6**)**, the solution of (4.3.1) is obtained as

$$u(x,t)=a_0+b_0t+\frac{1}{\Gamma(4)}ca_0^2t^{3/2}+\frac{1}{\Gamma(5)}2ca_0b_0t^{5/2}+\frac{\Gamma(5/2)}{\Gamma(11/2)\Gamma(4)}2ca_0^3t^3+\frac{1}{\Gamma(6)}2cb_0^2t^{7/2}+\frac{\Gamma(7/2)}{\Gamma(13/2)\Gamma(4)}3c^2a_0^2b_0t^4$$

$$+\frac{1}{\Gamma(7)}c^2a_0^4\left[\frac{4\Gamma(5/2)}{\Gamma(11/2)}+\frac{1}{\Gamma(4)}c\right]t^{9/2}+\frac{\Gamma(9/2)}{\Gamma(15/2)\Gamma(6)}24c^2a_0b_0^2t^5+\left\{a_1+b_1t+\frac{1}{\Gamma(4)}2ca_0a_1t^{3/2}+\frac{1}{\Gamma(5)}2c(a_0b_1+b_0a_1)t^{5/2}\right.$$

$$+\frac{\Gamma(5/2)}{\Gamma(11/2)\Gamma(4)}6c^2a_0^2a_1t^3+\frac{1}{\Gamma(6)}4cb_0b_1t^{7/2}+\frac{\Gamma(7/2)}{\Gamma(13/2)\Gamma(4)}2c^2a_0\left(a_1b_0+\frac32a_0b_1+2a_1b_0\right)t^4$$

$$\left.+\frac{1}{\Gamma(7)}4c^2a_0^3a_1\left[\frac{\Gamma(5/2)}{\Gamma(11/2)}+\frac{1}{\Gamma(4)}c+\frac{\Gamma(5/2)}{\Gamma(11/2)}3c\right]t^{9/2}+\frac{\Gamma(9/2)}{\Gamma(15/2)\Gamma(6)}24c^2b_0(a_1b_0+2a_0b_1)t^5\right\}x^{1/4}$$

$$+\left\{a_2+b_2t+\frac{1}{\Gamma(4)}c(2a_0a_2+a_1^2)t^{3/2}\right\}x^{1/2}+\left\{a_3+b_3t+\frac{1}{\Gamma(4)}2c(a_0a_3+a_1a_2)t^{3/2}\right\}x^{3/4}$$

$$+\left\{a_4+b_4t+\frac{1}{\Gamma(4)}c\left(2(a_0a_4+a_1a_3)+a_2^2\right)t^{3/2}\right\}x+\left\{a_5+b_5t+\frac{1}{\Gamma(4)}2c(a_0a_5+a_1a_4+a_2a_3)t^{3/2}\right\}x^{5/4}$$

$$+\left\{a_6+b_6t+\frac{1}{\Gamma(4)}c\left(2\left(a_0a_6+a_1a_5+a_2a_4\right)+a_3^2\right)t^{3/2}\right\}x^{3/2}+\left\{a_7+b_7t+\frac{1}{\Gamma(4)}2c\left(a_0a_7+a_1a_6+a_2a_5+a_3a_4\right)t^{3/2}\right\}x^{7/4}$$

$$+\left\{a_8+b_8t+\frac{1}{\Gamma(4)}\left(\frac{1}{2}\Gamma(6)a_5+c\left(2\left(a_0a_8+a_1a_7+a_2a_6+a_3a_5\right)+a_4^2\right)t^{3/2}\right)\right\}x^2+..... \qquad \textbf{(4.3.7)}$$

5 Conclusions

In the present study, we considered three examples to exhibits the applicability of the generalized differential transform method (GDTM). It may be concluded that GDTM is a reliable technique to handle linear and nonlinear fractional differential equations. Compared with other approximate methods this technique provides more realistic series solutions.

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