BOOLEAN NETWORKS OF BOUNDED DEPTH

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Abstract : A directed path from an input to an output of the network S is called a *chain.* The number of vertices in a chain which differ from the input is called the *length* of the chain. The length of the longest chain in S is called the depth of the network S and is denoted by $l(S)$. In the present paper we present some results about the size and length of Boolean networks in the basis {&, ω} that compute specific monotone Boolean functions.

Keywords : Boolean network, monotone Boolean functions, *chain, length.*

In papers on Boolean functions the term 'depth' of a Boolean network is customarily used for two different notions. Sometimes by the depth of a Boolean network one means the following parameter.

Let S be an arbitrary Boolean network. A directed path from an input to an output of the network S is called a *chain.* The number of vertices in a chain which differ from the input is called the *length* of the chain. The length of the longest chain in S is called the depth of the network S and is denoted by $l(S)$.

To distinguish this notion from the depth of Boolean networks used in another sense we call $l(S)$ the *length* of the network S (this name is motivated by the well-known notion of 'width' of a Boolean network).

Another parameter is often called the depth of a Boolean network. Let S be an arbitrary Boolean network in the basis ε and let the inputs of S be the variables and their negations. To each directed chain C directed from an input to an output of S, one assigns the chain C' obtained from C by deleting the input vertex. The chain C' is partitioned into subchains C'_1 , C'_2…. C'_d such that

1) every vertex of C' belongs to only one subchain;
2) one and the same function in E is realized at any vertex of any given sub-chain;
3) the function realized at the vertices of a subchain $C'_i, 1 \le i \le d-1$, differs from the function realized at the vertices of the subchain C'_{i+1}.

A chain C' with maximally many subchains of this kind is said to be *maximal.* The number of subchains in a maximal chain C' is called the *depth* of the network S and is denoted by $D(S)$.

If a Boolean network S consists only of two-input OR gates and AND gates, then its depth is sometimes defined as follows. In the network S every non-extendable chain consisting solely of OR gates (AND gates) is replaced by a single OR gate (AND gate) with the necessary number of inputs. As a result, we obtain a Boolean network S' realizing the same function as the network S does, and the height of the network S' is equal to the depth of the network S.

Let

$$l_\varepsilon (f) = \min l_\varepsilon(S),$$

where the minimum is taken over all unramified Boolean networks S in the basis ε that compute the Boolean function f , and

$$l_\varepsilon (n) = \max l_\varepsilon (f).$$

where the maximum is taken over all monotone Boolean functions of n variables.

The following theorem holds.

Theorem 1. *For any* $n \ge 2$

$$n-\frac{1}{2}\log_2 n-\log_2\log_2 n+c_1 \le l_{\{\&,\vee\}}(n)$$
$$\le n-\frac{1}{2}\log_2 n-\log_2\log_2 n+c_2,$$

where c_1 *and* c_2 *are constants*

It can be easily verified that if a Boolean network with a single output in the basis {&, ω} consists of N elements, then $l_{\{\&,\omega\}}(S) > \log_2 N$. For a proof of the upper estimate for $l_{\{\&,\omega\}}(n)$, see [1].

Let H be an arbitrary closed class of monotone Boolean functions, let $H(n)$ be the set of functions in H depending on n variables, let ε be an arbitrary finite basis for the functions in H, and let

$$l_\varepsilon (H(n)) = \max l_\varepsilon (f),$$

where the maximum is taken over all functions f in $H(n)$.

Ugol'nikov [2] obtained a complete description (up to order) of the behaviour of the functions $l_\varepsilon (H(n))$ for each closed class H of monotone Boolean functions and any finite basis ε for the functions in H.

We present some results about the size and length of Boolean networks in the basis {&, ω} that compute specific monotone Boolean functions.

Yao [3] proved that every Boolean network in the basis {&, ω} recognizing connectivity for undirected n-vertex graphs has length of order $\Omega((\log n)^{3/2}/ \log\log n)$.

The following assertions were proved in [4] and [5].

Theorem 2. 1. *If a Boolean network S in the basis* {&, ω} *is of size* n^d *and rccognizes connectivity for undirected n-vertex graphs and if* $d = o((\log n)^{1/2})$, *then the length of S is of order* $\Omega((\log n)^2/ \log d)$.

2. *If an unramified Boolean network S in the basis* {&, ω} *recognizes connectivity for undirected n-vertex graphs, then the length of the network S is of order* $\Omega(\log n)^2/ \log\log n)$.

Similar Boolean networks were also studied in [6] and [7].

In [8] and [9], Boolean networks in the basis {&, ω} were considered which recognize the existence of matchings with given number of edges in undirected graphs. The following theorem was proved in these papers.

Theorem 3. *Let m be an arbitrary positive integer and let n = 3m. Then the length of any Boolean network in the basis* {&, ω} *which recognizes the existence of matchings with m edges in undirected n-vertex graphs is of order* $\Theta(n)$.

Let us proceed to results about the size of Boolean networks of fixed depth that compute individual monotone Boolean functions.

In 1970 Lupanov [10] showed that maximal (with respect to the number of elements) unramified Boolean networks having distinct depth and computing the same monotone Boolean function can have substantially different sizes. For functions of this kind he considered a sequence of monotone Boolean functions

$$f_n(x_1,\dots,x_n : y_1,\dots,y_{1n}) = \bigvee_{i=1}^{n} x_i y_i y_{n+1} \dots y_{n.} \qquad (1)$$

$n = 1, 2, \dots$ and proved the following assertion.

Let $L^d_{\&}(f_n)(L^d_{\vee}(f_n))$ be the number of elements in a minimal unramified Boolean network in the basis {&, ω} which has depth d and computes the function f_n in (1) and let the output element of the network be an AND gate (an OR gate).

Theorem 4. *For any fixed* $d \geq 2$

$$L^d_{\&}(f_n) \sim L^d_{\vee}(f_n) \sim \frac{n(d-1)}{d}((d-1)!n)^{1/(d-1)}$$

as $n \to \infty$.

The size of unramified Boolean networks of given depth that compute the functions f_n in (1) was later studied in [11-13], where the following results were obtained.

In [13] it was shown (for a sketch of the proof, see [12]) that the function f_n can be computed by an unramified Boolean network S in the basis {&, ⋁} for which the product of the number of elements in S and the depth of S is equal to $\Omega(n \log^2 n)$. Moreover, it is proved that among these networks one can find a network of depth $\Theta(\log n)$.

In [13] it was proved that the function f_n in (1) can be computed by an unramified Boolean network S in the basis {&, ⋁} such that the product of the number of elements in S and the depth of S is of the order of $\dfrac{n \log n \log\log\log n}{\log\log\log\log n}$.

A symmetric monotone Boolean function $f(x_1,\dots,x_n)$ is called the *majority* function if for any n-tuple $(\sigma_1,\dots,\sigma_n) \in E^n$ we have

$$f(\sigma_1,\dots,\sigma_n) = \begin{cases} 0 & \text{if } \sigma_1+\dots+\sigma_n \lceil n/2 \rceil, \\ 1 & \text{otherwise} \end{cases}$$

The majority function of n variables is often denoted by MAJ(n).

We denote by $L^d_{\varepsilon}(\mathrm{MAJ}(n))$ the number of elements in a minimal unramified Boolean network of depth d in the basis ε that computes the function MAJ(n).

Boppana studied the behaviour of the function $L^d_{\{\&,\vee\}}(\mathrm{MAJ}(n))$. In [14 & 15] he proved the following assertion.

Theorem 5. *For any fixed d and an arbitrary n*

$$L^d_{\{\&,\vee\}}(\mathrm{MAJ}(n)) = \exp\{\Omega(n^{1/(d-1)})\}.$$

A similar result was obtained by Hastad (see [16], [17]).

At the same time; Valiant [18] proved that for any even n there is an unramified Boolean network in the basis {&, ⋁} which computes the function MAJ(n) and has size of order at most, $n^{5.3}$. For the proof of this assertion, see also [19].

Razborov studied the behaviour of the function $L^d_{\{\&,\vee\}}(\mathrm{MAJ}(n))$ for the cases in which ε={&, ⊕} and ε={&, ω, ⊕}. In [20] he proved the following theorem.

Theorem 6. *For any fixed d and for an arbitrary n*

$$L^d_{\{\&,\oplus\}}(\mathrm{MAJ}(n)) = \exp\{\Omega(n^{1/(d+1)})\}$$

$$L^d_{\{\&,\vee,\oplus\}}(\mathrm{MAJ}(n)) = \exp\{\Omega(n^{1/(2d+1)})\}. \qquad (2)$$

The relations (2) and a sketch of the proof can be found in [21]. Another proof is given in [19].

A generalization of the assertion of Theorem 2.19 is given in [22].

An unramified Boolean network of depth 3 in the basis {&, ω} in which the input poles are marked by variables and their negations is called a ∑∏∑*-network* if the output element of this network is an OR gate. If the output element of such a network is an AND gate, then the network is called a ∏∑∏*-network*.

Suppose that a ∑∏∑-network (a ∏∑∏-network) has $2\binom{n}{2}$ inputs marked by the variables $x_{1,2}, x_{1,3},\dots,x_{1,n}, x_{2,3},\dots x_{ij},\dots,x_{n-1,n}$ and their negations and computes the function CLIQUE$_S$ $(x_{1,2}, x_{1,3},\dots,x_{1,n}, x_{2,3},\dots x_{ij},\dots,x_{n-1,n})$ defined in 2.8.

Valiant [23] proved the following assertion.

Theorem 7. *Let m be an arbitrary positive integer, let* $n = 7m^4$, *and let* $s = \sqrt{n/7}$. *Then* any ∑∏∑-network (∏∑∏-network) *computing the function*

CLIQUE$_S$ $(x_{1,2}, x_{1,3},\dots,x_{1,n}, x_{2,3},\dots x_{ij},\dots,x_{n-1,n})$

contains at least $c^{n^{1/4}}$ *elements, where c is a constant,* $0 < c < 2$.

The size of ∑∏∑-networks and ∏∑∏-networks computing the symmetric monotone functions $f_r(x_1,\dots,x_n)$ (recall that the function $f_r(x_1,\dots,x_n)$ is equal to 1 at any n-tuple of values of the variables $x_1,\dots,x_n$ containing at least r ones and is equal to zero on the other n-tuples of these variables) was studied in [24] and [25]. These functions are often called *threshold* functions with threshold r. The following assertions were proved in these papers.

Theorem 8 [24]. *As* $n \to \infty$, *a minimal* ∑∏∑*-network computing the threshold function* $f_r(x_1,\dots,x_n)$ *contains* $\exp\{\Theta((r/\log r)^{1/2})\}\, n \log n$ *elements for* $2 \leq r \leq n/2$.

Theorem 9 [25]. *For* $4 \leq r \leq \log n$, *a minimal* $\prod\sum\prod$*-network computing the threshold function* $f_r(x_1,...,x_n)$ *contains at least* $\exp\{\Theta(r)\}n^2$ *and at most* $e^r n^2 \log_2 n$ *elements. For* $r = 2$ *and* $r = 3$ *the number of elements in these* $\prod\sum\prod$ *-networks is of the order of* $n^{3/2}$ *and* n^2 *respectively.*

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