Some Important Applications Of Fixed Point Theorems To Explain Invariant Approximation

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Abstract:- In this paper, we prove some fixed point theorems. In this paper, we prove some fixed point results for nonexpansive and generalized nonexpansive mappings. Invariant approximation results are also obtained for these types of mappings as applications. we obtain Brosowski – Meinardus type theorems on invariant approximations on a class of nonconvex sets in locally bounded topological vector spaces.

Keywords:- Fixed point theorems, Invariant approximations, Vector spaces and nonconvexsets etc.

(1)-Introduction:- Fixed point theory is one of the famous and traditional theory in mathematics and has a lot of applications. In fixed point theory the importance of various contractive inequalities can not be over emphasized. Fixed point theorems for different types of mappings have been investigated extensively by various researchers. Brosowski initiated the study of invariant approximations using fixed point theory and subsequently

various generalizations of Brosowski's results have appeared in the literature . In this paper, we extend some important fixed point theorems due to Dotson, Anderson, Nelson and Singh, Khan and Sessa to a locally bounded topological vector space and as applications ,we obtain several Brosowski – Meinardus type theorems for nonexpansive maps defined on a class of nonconvex sets containing the subclass of starshaped sets in a locally bounded topological vector space which is not necessarily locally convex. Some recent results of Habinaik, Khan and Khan , Khan and Sessa follow as a consequence of our results.

(2)-Preliminaries:- Here first, we recall some important definitions, well known fact andnotations.

Let (X, d) be a metric space, T be a self-map of X, F(T) , the set of all fixed points of T. The map T is nonexpansive on a subset S of X if d (Tx, Ty) ≤ d (x, y) for all x, y ϵ S . Thus the contractive mappings are nonexpansive and any nonexpansive map is continuous. A subset S of a linear space E is called starshaped if there exists at least one point z ϵ S such that tz + (1-t)x ϵ S for all x ϵ S , t ϵ (0, 1); z is called a star center of S. Let 0 < p ≤ 1. A real valued map

II . IIp on a liner space E is called a p- norm if (i) II x IIp ≥ 0 and II x IIp = 0 iff x = 0 ,
(ii) II λx IIp = I λ I ^{p} II x IIp , and (iii) II x + y IIp ≤ II x IIp + II y IIp for all x, y ϵ E and λ ϵ C. The formula dp (x, y) = II x – y IIp defines a translation invariant metric on E . It is well known topology is generated by a p- norm. Let S ⊆ E and F = { fα } for each α ϵ S a family of
function from [0, 1] into S such that fα(1) = α for each α ϵ S. The family F is said to be
contractive it there exists a function ϕ: (0, 1) → (0, 1) such that for all α, β ϵ S and
all t ϵ [0, 1] , we have II fα(t) – $f_β$(t) IIp ≤ [ϕ(t)]^{p} II α – β IIp. The family F is said to be jointly continuous if t → t_0 in [0, 1] and α → $α_0$ in S , then $f_α$(t) → f$α_0$(t_0) in S . Here α → $α_0$ denotes the weak convergence in S. If for a subset S of E , there exists a contractive jointly continuous family of functions F = { $f_α$}$_{α ϵ S}$, then we say that S has the property of contractiveness and joint continuity. We observe that if S ⊆ E is starshaped with z as star center and f_z(t) = (1- t) z + tx , (x ϵ S , t ϵ [0, 1]), then F = { f_z}$_{z ϵ S}$ is a contractive jointly continuous family with ϕ(t) = t . Thus the class of subsets of E with the property of contractiveness and joint continuity contains

the class of starshaped sets which in turn contains the class of convex sets. For a subset S of E, a mapping T: S $\to E$ is said to be

(i) demicompact if every bounded sequence (x_n) in S such that ($Tx_n \to x_n$) is strongly convergent in E has a strongly convergent subsequence.

(ii) completely continuous if whenever (x_n) converges weakly to x , then (Tx_n) converges strongly to Tx . Let S be a subset of a metric space (X, d) . For an element x ϵ X ,

we set d (x, S) = inf { d (x, y) : y ϵ S } . and $P_s(x)$ = { y ϵ S, d (x, y) = d(x, S) } , $P_s(x)$ is called the set of all best approximations of x from S . The map $P_s : Z \to 2^s$ is called metric projection onto S . It is well known that $P_s(x)$ is always bounded , it is closed if S is closed .

(3) Results:- First we shall need the following result of Khan and Sessa which is an analogue of a fixed point theorem of Subrahmanyam.

Theorem (3.1) – Let S be a closed subset of a metric space (X, d) and T: S → S a continuous map with T(S) compact. Suppose that d (Tx, T^2x) ≤ k d (x, Tx) for all x ϵ S , 0 < k < 1. Then T has a fixed point in S.

We now establish some common fixed point theorems for two maps.

Theorem (3.2)- Let I , T be two self maps of E, u ϵ F(T) ∩ F(I) and S compact T invariant subset of E. Suppose that I & T are commuting on D = Ps(u), I is continuous on D, T is I – nonexpansive on D U {u} and I(D) = D. Suppose that D has a contractive jointly continuous family F = { fx(α) } $x \in D$ such that I(fx(α)) = $f_{I(x)}$ (α) for all x ϵ D and all α ϵ [0, 1]. Then I, T have a common fixed point in D.

Proof:- We note that D is non empty , T – invariant and compact . Define T_n: D →D as in the proof of below theorem . Since I and T commute on D, it follows from the property of F that

$T_n (I (x)) = f_{T(I(x))} (\lambda_n) = f_{I(T(x))} (\lambda_n) = I (f_{T(x)} (\lambda_n)) = I (T_n (x)), \quad x \in D.$

Thus for each n, T_n commutes with I and T_n (D) ⊆ D = I (D). Since F is contractive and T = I ($f_{T(x)} (\lambda_n)$) = I (T_n (x)) , x ϵ D.

Thus for each n, T_n commutes with I and T_n (D) ⊆ D = I (D). Since F is contractive and T is I – nonexpansive , we get

$\| T_n(x) - T_n(y) \|_p \leq [\phi(\lambda_n)]^p \| T(x) - T(y) \|_p$

$\leq [\phi(\lambda_n)^p] \| I(x) - I(y) \|_p$ L $\| I(x) - I(y) \|_p$, I(x) ≠ I(y).

So we get x_n ϵ D such that x_n ϵ F (T_n) ∩ F(I) for each n, in particular , I (x_n) = x_n. By compactness of D , (x_n) has a subsequence (x_{nj}) which converges to z ϵ D and hence T (x_{nj}) → T (z) . The joint continuity of F and the uniqueness of the limit give

$X_{nj} = T_{nj} (x_{nj}) = f_{T(x_{nj})} (\lambda_{nj}) \to f_{T(z)} (I) = T(z) = z$.

And hence by the continuity of I ,

I(z) = I (lim j tends to ∞ x_{nj}) = lim j tends to ∞ I (x_{nj}) = lim j tends to ∞ x_{nj} = z . This completes the proof.

Theorem (3.3):- Suppose that E is complete and S is a weakly compact subset of E , T, I are commuting self maps of S with I being continuous in the weak and strong topologies on S , T functions such that I { $f_x(\alpha)$ } = $f_{I(x)}$ (α) for all x ϵ S and all α ϵ [0,1]. Then each of the following cases I , T have a common fixed point in S .

(i) E^* separates points of E, T is weakly continuous and family F = { $f_x(\alpha)$ } $_{x \in s}$ is jointly weak continuous .

(ii) T, is completely continuous and F is jointly continuous .

(iii) I, is demicompact and F is jointly continuous .Nonexpansive imply

$\| T_n(x) - T_n(y) \| \leq [\phi(\lambda_n)]^p \| T(x) - T(y) \|_p \leq [\phi(\lambda_n)]^p \| I(x) - I(y) \|_p$ for all $x, y \in S$

So there exist a unique $x_n \in S$ such that $x_n = T_n x_n = I x_n$ for each n. Now S is weakly compact implies that there is a subsequence (x_j) of (x_n) converging weakly to some $a \in S$ and I being weakly continuous gives $Ia = a$.

(i) T is weakly continuous so $Tx_j \to T(a)$ and hence $x_j = f_{T(x)} (\lambda_j) \to f_{T(a)} (I) = Ta$. Also $x_j \to a$. As the weak topology is Hausdorff , we get $Ta = a$.

(ii) As $x_j \to a$, so $T(x_j) \to Ta$. Also , $x_j = f_{Txj} (\lambda_j) \to f_{T(a)} (I) = Ta$. thus $T(x_j) \to T^2a$ and consequently $T^2a = Ta$ implies $Tw = w$, where $w = Ta$. Also $ITa = Tia = Ta = w$.

(iii) Suppose that (x_n) is a bounded sequence and $(Tx_n - x_n)$ converges strongly to 0 . BY demicompactness of I , (x_n) has a subsequence (x_k) converges strongly to x in S and hence $x_k = I x_k \to Ix$ implies that $x = I x$. Also $Tx_k \to Tx$. Further , $x_k = f_{Txk}(\lambda_k) \to f_{Tx} (I) = Tx$. Since the strong topology is Hausdorff , we get $Tx = x$.

Theorem(3.4):- Let E be complete , T, I, selfmaps of E and $u \in F(T) \cap F(I)$, S a subset of E such that $T(aS) \subseteq S$, where aS is the boundry of S in E , I continuous in the weak and the strong topologies on $D = Ps(u)$, $ID = D$, I, T commute on D and

$$\| Tx - Ty \| \leq \| I^m x - I^n y \|_p \quad \text{for all } x, y \in D \cup \{u\}.........................(*)$$

And for some $m = m (x, y)$, $n = n (x, y)$ in $N_0 = \{ 0, 1, 2, 3,...............\}$. If D is nonempty weakly compact and has a contractive family of functions $F = \{ f_x(\alpha) \}_{x\in D}$ such that $I (f_x(\alpha)) = f_{I (x)} (\alpha)$

For all $x \in D$ and all $\alpha \in [0, 1]$, then I, T have a common fixed point in D under the condition

(i) – (iii) of theorem (3.3).

Proof:- Let $y \in D$. Then $I^n y \in D$ for $n \in N_0$ because $I (D) \subseteq D$. Moreover, by definition of D , $y \in aS$. Since $T(aS) \subseteq S$, therefore, $Ty \in S$. By equation (*) we have

$\| Ty - u \|_p = \| Ty - Tu \|_p \leq \| I^n y - I^m u \|_p$ for some $n , m \in N_0$. Also $I^m u = u$, so

$\| Ty - u \|_p \leq \| I^n y - u \|_p$. Since $Ty \in S$ and $I^n y \in D$, so by definition of D , $Ty \in D$ and hence $T, I : D \to D$ satisfy by the hypotheses of theorem (3.3) and the result follows.

Let X be a Hausdorff locally convex space whose topology is defined by a family Q of continuous seminorms. A subset M of X is approximatively compact in X iff for each $y \in E$ and a net (x_n) in M such that $\lim_{x \to \infty} p (x_n - y) = \inf p (y - m)$ for each $m \in M$, $p \in Q$ implies that there exists a subnet (x_{nj}) converging to an element of M . A compact set is approximatively compact but the converce is not true , in general , for instance a closed convex set in a Hilbert space is approximatively compact but fails to be compact.

Theorem(3.5):- Let M be a nonempty approximatively compact subset of a locally convex space X and $P_M : E \to 2^M$ be the metric projection . Then

(i) $P_M (x) \neq \varphi$ for each $x \in M$.

(ii) P_M maps a compact subset of X onto a compact subset of M.

The above remarks lead to another Brosowski – Meinardus type theorem on invariant approximations.

Theorem(3.6):- Let (X, τ) be a Hausdorff locally convex space , T a nonexpansive selfmap of E, $a \in F(T)$ and S a nonempty approximatively compact T – invariant subset of X. If $P_s(u)$ has the property of contractiveness and joint continuity , then T has a fixed point in $P_s(u)$.

Proof:- The topology τ is determined by a family Q of seminorms and hence by theorem B, $P_s(u)$ is a nonempty compact subset of S . If y ϵ $P_s(u)$, then by hypothesis T(y) ϵ S . As before we can show that p(u) – T(y) = d_p (u, S) for all pϵ Q and hence $P_s(u)$ is T – invariant . Further

T: $P_s(u) \rightarrow P_s(u)$ is nonexpansive , therefore T has a fixed point in $P_s(u)$ as required .

Theorem(3.7):- Let T be a nonexpansive selfmap of E, u ϵ F(T) and S a compact T – invariant subset of E such that T(S) is compact . If $P_S(u)$ has the property of contractiveness and joint continuity , then T has a fixed point in $P_s(u)$.

Proof:- The $P_s(u)$ is nonempty . As S is closed so is $P_s(u)$ and hence $P_s(u)$ is compact . To show that $P_s(u)$ is T – invariant , Let y ϵ $P_s(u)$ and set d_p(u, S) = r . By hypothesis

r ≤ d_p(u, T(y)) = d_p(T(u) , T(y)) ≤ d_p(u, y) = r . That is r = d_p(u, T(y)) and hence

T(y) ϵ $P_s(u)$ and consequently $P_s(u)$ is T – invariant . By theorem we prove that T has a fixedpoint in $P_s(u)$.

Theorem(3.8):- Let E be complete with a separating dual E , T be a nonexpansive selfmap of S , u ϵ F(T) and S a T – invariant subset of E . Suppose that $P_s(u)$ is nonempty weakly compact and has the property of contractiveness and joint continuity
. Then T has a fixed point in $P_s(u)$.

Proof:- As in the proof of theorem (3.7) , we can show that $P_s(u)$ is T – invariant and hence by the above theorems , T has a fixed point in $P_s(u)$.

Conclusion:- In this paper we define some important applications of fixed point theorems to explain the invariant approximations.

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