The Dual Geometry of Boolean Rings

Dr. Abhay Kumar Lalan
Department of Mathematics,
B.S.S Inter College Begusarai.

Abstract

In this paper we discuss the properties of Boolean rings and semi rings. We discuss the geometrical and topological structure of Boolean rings in respect of lattice structure. We derive theorems on dualities and its isomorphic structure and compare it with recursive structure. We discuss the different applications of Boolean rings in Automata theory and physical sciences. We examine here that the variety of Boolean semi rings, which is generated by the three elements semi ring is dual to the category of partially stone spaces. We discuss this duality in the context of natural dualities.

Key words: geometrical topological, automata, duality, recursive, isomorphic

Introduction

It has been discussed by S. Eilenberg and W. Kuich to include semi ring connected to Boolean rings in automata theory. Guzman Studied varitey of Boolean semi ring generated by 2-element semi ring. We reformulate here the relationship between variety generated by 3-element semi ring and the category of partially stone –space. Clark and Davey studied natural dualities between algebraic and topological quasi- variety. We analyse the concept of duality in richer context.

We introduce here a topological structure S and obtain an optimal natural duality between the quasi variety IPS (S) and the category IS_c P + (S). We further constructed an optimal and its small structure S_{os} that yields a strong duality. The geometry of some of the partially stone spaces as they are n-dimensional cubes with unique incomparable covers for each element of the cube.

The main thrust is on the quasi variety A=ISP (S) generated by S. on the other hand, we obtain the category of structured topological spaces X= IS_c P+ (S) , generated by some appropriate structure S= < S, G, H, R, T > having S= { o, h, 1 } as underlying set and T the discrete topology We proceed here with three binary relations on S: r_1, r_2, and r_3 and introduce the concept of natural duality

Definition

A partially complimented distributive lattice is a type <0,0,0,2,2,>algebra A = <A, 0,h,1,V,^> such that <A,o,1,v,^> is a bounded distributive lattice and <[h,1],h, 1,v,> is a complimented distributive lattice i.e. a Boolean algebra where [h,1] = { a € A/ h ≤ α ≤ 1}

Lemma

Given a partially complimented distributive lattice <A, O,h, 1, V, ^>, the bar operation satisfies

(1) $X \vee \bar{X} = 1$

(2) $X \wedge \bar{X} = X \wedge \bar{1}$

These two properties characterize partially complimented distributive lattices. From (I) and (II) , we derive that BSR is dually equivalent to the category of partially stone spaces, P S S denote this dual equivalence as BSR ⇆PSS

We introduce here the notion of natural dualities similar to A. Davey. The main aim is to impose on the carrier S of the semi ring ring S, the discrete topology together with operations, partial operations and relations to form a dual topological structure $\underline{S}$ as the generator of the dual category X. In particular, X= IS_c P + ($\underline{S}$) is the category of isomorphic copies, topologically closed structures of non-empty products of copies of S. On the basis of above construction we obtain a dual adjuction <D. E, e, € > between the categories A and X with the many desirable properties. If S yields a full duality on a and it is injective in X, $\underline{S}$ is said to yield a strong duality on A.

We construct here three dualities each one coming from a different topological structure. In all three of them the algebra side of the duality is A = ISP (S) . The first topological struct are $\underline{S}$ yields an optimal duality X= IS_c P + (s) ⇆ A. The Second one $\underline{S}$. yields an optimal strong duality S= IS_c P + ($\underline{S}$)⇆ A. The third one $\underline{S}$,, yields an optimal strong duality X $_{os}$ = ISc P+ ($\underline{S}$,,) ⇆ A. in the first duality , labeling the oppropriate confravariant functions D and E, A is isomorphic to ED (A) = X [A (A, S), S]. in particular, when L= X (S" , S), we identify the prime filters of X ($\underline{S}''$, $\underline{S}$) with its join irreducible elements.

** **Lemma** *

(I) If a morphism preserves a binary relation r then it also preserves r^{-1}, and

(II) If a morphism preserves two k – ary relations r and s then it also preserves their intersection r ∩ s.

Lemma

If $\underline{S}$ ' is a structure that yields duality on A , then the finitary them functions on S must be exactly the morphism from finite powers of $\underline{S}$ ' into $\underline{S}$ '. Therefore, for any n€ N

x' (S'' , S) = x (S'', S).

Theorem

The structure $\underline{S}$ = ≤ S , (r_1, r_2, r_3), T > yields an optimal natural duality on A.

Proof.

We show here that if we eliminate r_2 or r_3 from $\underline{S}$ to from $\underline{S}'$, duality is not preserved. The map (1,h,1) preserves r_1 and r_2 but not r_3. The map (0,0,h) preserves r, and r_3 but not r_2.

Let $\underline{S}'$ = < S, (r_2,r_3), T>, x_1= I S_cP + ($\underline{S}'$) and suppose that $\underline{S}$ yields a duality on A. By the duality and and entailment theorem (r_2 ,r_3) entail r_1. By entailment lemma, for any x ≤ s^2 and α€ X^1 (x, $\underline{S}'$), α preserves r_1. Let x={(h,o), (o,1)} let us define α:X→ $\underline{S}$ $by\ \alpha\ (h,o) = 1\ and\ \ \alpha$ (0,1) =0. then α preservs r_2 and r_3 butnot r_1. Therefore, duality without r_1 is not retained.

Reference:

1. Foster A. L., The idempotent elements of a commutative ring form a Boolean algebra, ring duality and transformation theory, Duke Math J. 12 (1945),
2. R. Gilmer, multiplicative ideal theory , Marcel Dukker, Inc. New York, 1972.
3. Harary F., Atomic Boolean-like rings with finite radical, Duke Math J. 17 (1950), 273-276
4. J. Von Neumann, continuous geometry , Princeton University Press, Princeton 1960.

5. Swaminathan V., Arithmetical Boolean like rings, proceedings of the International sysmposium on Algebra and its Applications, Marcel Dekker Inc., New York, 1984, 379-384.
6. R.A. Melter, Contributions to Boolean Geometry of P-rings, Pacific Journal of Math 1964, 995-1017