

Estimation of Error in Runge-Kutta Fourth Order Method

Veena Singh

Mathematics Department

M. L. K. P. G. College, Balrampur, U. P., India

Email: veenasingh1907@gmail.com

Abstract: Ordinary differential equations with boundary conditions have been solved by using Runge-Kutta fourth order method with the help of a computer program written in C++. At the points 0.01, 0.02, 0.3, ..., 1.0 of the interval [0, 1], the values of y have been calculated and compared with the exact values of y at these points. Calculations for minimum error, maximum error, minimum percentage error and maximum percentage error have been done for each differential equation. The Runge-Kutta fourth order method has been found accurate up to seven digits after the decimal point in some cases. In the worst case, it is accurate up to two digits after the decimal point.

Key words: Differential equation, Error, Exact value, Euler's method, Runge-Kutta fourth order method.

Introduction

Heun's Method is used to generate a numerical solution to an initial value problem of the form $y' = f(x, y)$ where $y(x_0) = y_0$. Heun's Method was an improvement over the rather simple Euler Method, and that though it uses Euler's method as a basis, it goes beyond it, attempting to compensate for the Euler Method's failure to take the curvature of the solution curve into account. Heun's Method is one of the simplest of a class of methods called predictor-corrector algorithms. One of the most powerful predictor-corrector algorithms of all—one which is so accurate, that most computer packages designed to find numerical solutions for differential equations will use it by default—the fourth order Runge-Kutta Method.^[1-8] The Runge-Kutta algorithm is a little hard to follow even when one only considers it from a geometric point of view. In reality the formula was not originally derived in this fashion, but with a purely analytical approach.

Material and Method

One member of the family of Runge-Kutta methods is so commonly used, that it is often referred to as "RK4" or simply as "the Runge-Kutta method".^[9-12]

Let an initial value problem be specified as follows.

$$y' = f(t, y), \quad y(t_0) = y_0$$

Then, the RK4 method for this problem is given by the following equation:

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$k_1 = f(t_n, y_n)$$

$$k_2 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right)$$

$$k_3 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right)$$

$$k_4 = f(t_n + h, y_n + hk_3)$$

Thus, the next value (y_{n+1}) is determined by the present value (y_n) plus the product of the size of the interval (h) and an estimated slope. The slope is a weighted average of slopes:

- k_1 is the slope at the beginning of the interval;

- k_2 is the slope at the midpoint of the interval, using slope k_1 to determine the value of y at the point $t_n + h/2$ using Euler's method;
- k_3 is again the slope at the midpoint, but now using the slope k_2 to determine the y -value;
- k_4 is the slope at the end of the interval, with its y -value determined using k_3 .

When the four slopes are averaged, more weight is given to the slopes at the midpoint:

$$\text{slope} = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$

The RK4 method is a fourth-order method, meaning that the error per step is on the order of h^5 , while the total accumulated error has order h^4 .

We have solved the differential equations given in Table-1 at the points

$$0=x_0, x_1, x_2, \dots, x_n=1, n=100, x_i - x_{i-1} = 0.01, i=1, 2, \dots, 100$$

with the help of Runge-Kutta fourth order method by using the computer program developed in C++.^[1]

With the help of above computer program, we have calculated the value of y , exact value of y , difference between calculated and exact values of y and percentage error in the value of y . We have calculated the values of y in the interval $[0, 1]$ by assuming the step size of 0.01. Percentage error in the value of y is defined as

$$PE_y = \frac{\text{Difference between calculated and exact values of } y}{\text{Exact value of } y} \times 100$$

Table-1: Differential equations with boundary conditions along with exact solutions

S. No.	Differential Equation	Boundary Condition	Exact Solution
1	$dy/dx = -y$	$y=1$ when $x=0$	$y=e^{-x}$
2	$dy/dx = 3x^2 + y$	$y=1$ when $x=0$	$y=-3x^2-6x-6+2e^x$
3	$dy/dx = x$	$y=0$ when $x=0$	$y=x^2/2$
4	$dy/dx = x + xy$	$y=1$ when $x=0$	$y=2e^{x^2/2} - 1$
5	$dy/dx = 1 + x^2$	$y=0$ when $x=0$	$y=x + x^3/3$

Result and Discussion

Solution of Differential equation $dy/dx = -y, y=1$ when $x=0$, by Runge-Kutta fourth order method

This differential equation has been solved by Runge-Kutta fourth order method with the help of computer program at different values of x in the interval $[0,1]$. Exact values of y at these points have also been calculated from the solution of differential equations. Exact value of y , calculated value of y by Runge-Kutta fourth order method and difference between them (error in the calculated value of y) at different points is included in Table-2. A close look to this Table indicates the following-

Maximum Error	= 0.0024948261156
Minimum Error	= 0.0000000000969
Maximum Percentage Error	= 0.6781640495255
Minimum Percentage Error	= 0.0000000100855

It is clear that the minimum error in Runge-Kutta fourth order method occurs after eight places of the decimal point and the maximum error occurs after two places of the decimal point. Graph between errors in the value of y calculated by Runge-Kutta fourth order method at different points of the interval $[0,1]$ is shown in Graph-1. Graph between percentage errors in the value of y calculated by Runge-Kutta fourth order method at different points of the interval $[0,1]$ is shown in Graph-2.

Table-2: Exact value of y for $dy/dx = -y, y=1$ when $x=0$, calculated value of y for this differential equation by Runge-Kutta fourth order method and difference between these values at different points of interval $[0,1]$

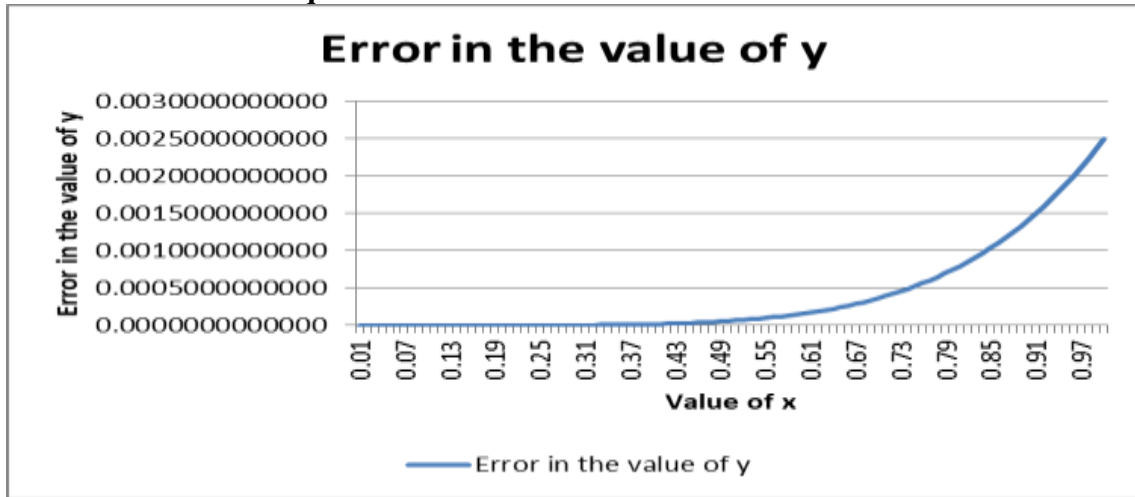
Value of x	Exact value of y for $dy/dx = -y, y=1$ when $x=0$	Value of y calculated by Runge-Kutta fourth order method	Error in the value of y calculated by Runge-Kutta fourth order method	Percentage error in the value of y calculated by Runge-Kutta fourth order method
0.00	1.0000000000000000	1.0000000000000000	0.0000000000000000	0.0000000000000000
0.01	0.9900498337491681	0.9900498337491681	0.0000000000000000	0.0000000000000000
0.02	0.9801986532163121	0.9801986532163121	0.0000000000000000	0.0000000000000000
0.03	0.9704561945261800	0.9704561945261800	0.0000000000000000	0.0000000000000000
0.04	0.9608204576691960	0.9608204576691960	0.0000000000000000	0.0000000000000000
0.05	0.9512894426416100	0.9512894426416100	0.0000000000000000	0.0000000000000000
0.06	0.9418612503608970	0.9418612503608970	0.0000000000000000	0.0000000000000000
0.07	0.9325340817369560	0.9325340817369560	0.0000000000000000	0.0000000000000000
0.08	0.9233061367697870	0.9233061367697870	0.0000000000000000	0.0000000000000000
0.09	0.9141757155503910	0.9141757155503910	0.0000000000000000	0.0000000000000000
0.10	0.9051412180896720	0.9051412180896720	0.0000000000000000	0.0000000000000000

Value of x	Exact value of y for $dy/dx = -y$, $y=1$ when $x=0$	Value of y calculated by Runge-Kutta fourth order method	Error in the value of y calculated by Runge-Kutta fourth order method	Percentage error in the value of y calculated by Runge-Kutta fourth order method
0.01	0.9900498339705	0.9900498341822	0.0000000002117	0.0000000213828
0.02	0.9801986737449	0.9801986734868	0.0000000002582	0.0000000263396
0.03	0.9704455341992	0.9704455346608	0.0000000004616	0.0000000475606
0.04	0.9607894400113	0.9607894401082	0.0000000000969	0.0000000100855
0.05	0.9512294273356	0.9512294259442	0.0000000013914	0.0000001462781
0.06	0.9417645348473	0.9417645328453	0.0000000020020	0.0000002125744
0.07	0.9323938196281	0.9323938156894	0.0000000039387	0.0000004224277
0.08	0.9231163480373	0.9231163464341	0.0000000016032	0.0000001736683
0.09	0.9139311888121	0.9139311810429	0.0000000077692	0.0000008500837
0.10	0.9048374234292	0.9048374102442	0.0000000131850	0.0000014571712
0.11	0.8958341358305	0.8958341160731	0.0000000197574	0.0000022054730
0.12	0.8869204390961	0.8869204027466	0.0000000363495	0.0000040983947
0.13	0.8780954351077	0.8780953817868	0.0000000533208	0.0000060723252
0.14	0.8693582348806	0.8693581552370	0.0000000796436	0.0000091611981
0.15	0.8607079841204	0.8607078746506	0.0000001094698	0.0000127185761
0.16	0.8521437920137	0.8521436406013	0.0000001514125	0.0000177684155
0.17	0.8436648150878	0.8436645973140	0.0000002177738	0.0000258128366
0.18	0.8352702178835	0.8352699232177	0.0000002946658	0.0000352778950
0.19	0.8269591359150	0.8269587547517	0.0000003811633	0.0000460921614
0.20	0.8187307628380	0.8187302692679	0.0000004935701	0.0000602847899
0.21	0.8105842512848	0.8105836159530	0.0000006353318	0.0000783794922
0.22	0.8025187989192	0.8025179835868	0.0000008153324	0.0001015966693
0.23	0.7945336110278	0.7945325781589	0.0000010328688	0.0001299968681
0.24	0.7866278652864	0.7866265923054	0.0000012729810	0.0001618276006
0.25	0.7788007830714	0.7787992025451	0.0000015805263	0.0002029435890
0.26	0.7710515931569	0.7710496502269	0.0000019429299	0.0002519844271
0.27	0.7633795088972	0.7633771580570	0.0000023508402	0.0003079517046
0.28	0.7557837405548	0.7557808891527	0.0000028514021	0.0003772775090
0.29	0.7482635738226	0.7482601597499	0.0000034140727	0.0004562660578
0.30	0.7408182339286	0.7408141585579	0.0000040753707	0.0005501174922
0.31	0.7334469544756	0.7334421023639	0.0000048521117	0.0006615490937
0.32	0.7261490422675	0.7261433174387	0.0000057248288	0.0007883820644
0.33	0.7189237454303	0.7189170204321	0.0000067249982	0.0009354257976
0.34	0.7117703202171	0.7117624477387	0.0000078724784	0.0011060419613
0.35	0.7046880939190	0.7046789264557	0.0000091674633	0.0013009249580
0.36	0.6976763368831	0.6976657008175	0.0000106360656	0.0015244985400
0.37	0.6907343273437	0.6907220550332	0.0000122723105	0.0017767048812

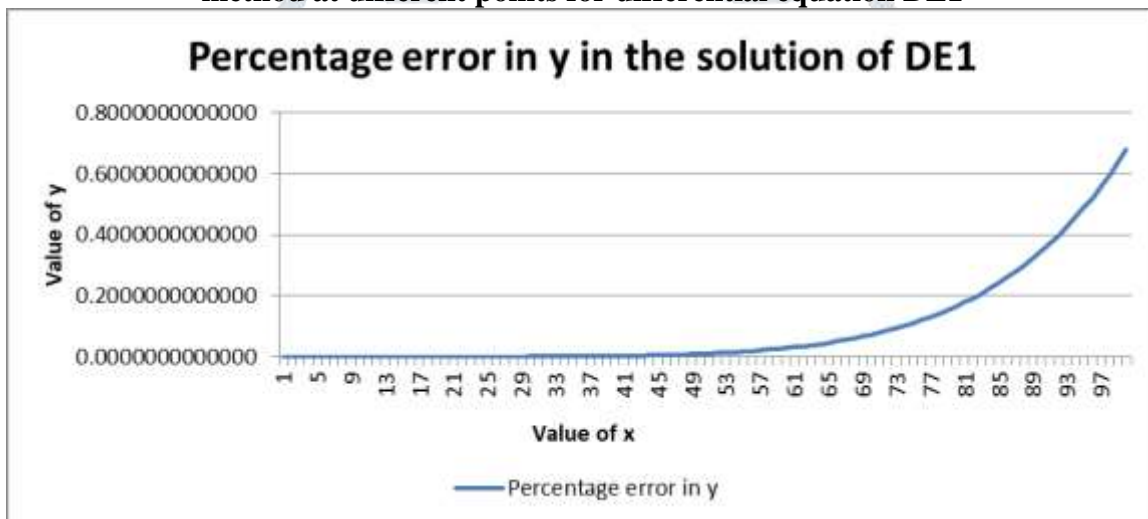
Value of x	Exact value of y for $dy/dx = -y$, $y=1$ when $x=0$	Value of y calculated by Runge-Kutta fourth order method	Error in the value of y calculated by Runge-Kutta fourth order method	Percentage error in the value of y calculated by Runge-Kutta fourth order method
0.38	0.6838614124733	0.6838472809316	0.0000141315417	0.0020664335525
0.39	0.6770568841835	0.6770406734525	0.0000162107310	0.0023942938044
0.40	0.6703200620173	0.6703015443003	0.0000185177170	0.0027625186936
0.41	0.6636502525097	0.6636291747931	0.0000210777166	0.0031760278174
0.42	0.6570468284309	0.6570228859971	0.0000239424338	0.0036439463344
0.43	0.6505091094572	0.6504819953506	0.0000271141066	0.0041681363483
0.44	0.6440364226186	0.6440058146885	0.0000306079302	0.0047525153990
0.45	0.6376281592229	0.6375936646719	0.0000344945510	0.0054098224037
0.46	0.6312836590528	0.6312449102470	0.0000387488058	0.0061380973948
0.47	0.6250022690278	0.6249588449622	0.0000434240656	0.0069478252707
0.48	0.6187833984450	0.6187348500243	0.0000485484207	0.0078457859118
0.49	0.6126264065996	0.6125722275840	0.0000541790157	0.0088437284235
0.50	0.6065306597126	0.6064703581365	0.0000603015762	0.0099420491305
0.51	0.6004955845390	0.6004285870777	0.0000669974613	0.0111570281373
0.52	0.5945205593098	0.5944462769736	0.0000742823362	0.0124944941006
0.53	0.5886049865185	0.5885227701280	0.0000822163905	0.0139680078088
0.54	0.5827482746041	0.5826574552812	0.0000908193229	0.0155846575405
0.55	0.5769498035027	0.5768496544752	0.0001001490275	0.0173583606220
0.56	0.5712090624870	0.5710988177557	0.0001102447313	0.0193002419815
0.57	0.5655254427445	0.5654042962554	0.0001211464891	0.0214219343576
0.58	0.5598983759097	0.5597654488163	0.0001329270934	0.0237412893392
0.59	0.5543272992723	0.5541816376818	0.0001456615906	0.0262771815029
0.60	0.5488116557211	0.5486523184555	0.0001593372656	0.0290331416797
0.61	0.5433508613018	0.5431768023785	0.0001740589233	0.0320343512317
0.62	0.5379444350296	0.5377545790656	0.0001898559640	0.0352928576981
0.63	0.5325918035465	0.5323849751311	0.0002068284154	0.0388343219015
0.64	0.5272924315860	0.5270674028824	0.0002250287036	0.0426762627586
0.65	0.5220457892076	0.5218012678940	0.0002445213136	0.0468390548559
0.66	0.5168513517435	0.5165859917553	0.0002653599882	0.0513416453889
0.67	0.5117085997467	0.5114209684770	0.0002876312697	0.0562099737648
0.68	0.5066169887420	0.5063055414830	0.0003114472590	0.0614758813682
0.69	0.5015760702619	0.5012392598518	0.0003368104101	0.0671504144714
0.70	0.4965853097112	0.4962214500191	0.0003638596920	0.0732723431250
0.71	0.4916442080105	0.4912515339665	0.0003926740440	0.0798695555896
0.72	0.4867522710466	0.4863289773261	0.0004232937205	0.0869628650278
0.73	0.4819090096225	0.4814531000240	0.0004559095985	0.0946049128376
0.74	0.4771139109709	0.4766234180819	0.0004904928890	0.1028041475466

Value of x	Exact value of y for $dy/dx = -y$, $y=1$ when $x=0$	Value of y calculated by Runge-Kutta fourth order method	Error in the value of y calculated by Runge-Kutta fourth order method	Percentage error in the value of y calculated by Runge-Kutta fourth order method
0.75	0.4723665527410	0.4718393134429	0.0005272392981	0.1116165602900
0.76	0.4676664314699	0.4671002498626	0.0005661816073	0.1210652655805
0.77	0.4630130771425	0.4624055955828	0.0006074815597	0.1312018147477
0.78	0.4584060244203	0.4577548344102	0.0006511900101	0.1420552905982
0.79	0.4538448125952	0.4531473497492	0.0006974628460	0.1536787083699
0.80	0.4493289855429	0.4485825779973	0.0007464075456	0.1661160462836
0.81	0.4448580651623	0.4440599564964	0.0007981086659	0.1794074848635
0.82	0.4404316576562	0.4395789391363	0.0008527185199	0.1936097247069
0.83	0.4360492935989	0.4351389537182	0.0009103398807	0.2087699473485
0.84	0.4317105347511	0.4307393802450	0.0009711545062	0.2249550168448
0.85	0.4274149472343	0.4263797266587	0.0010352205756	0.2422050474132
0.86	0.4231621014868	0.4220593568163	0.0011027446705	0.2605962742518
0.87	0.4189515472499	0.4177777265219	0.0011738207281	0.2801805449282
0.88	0.4147829136594	0.4135343109858	0.0012486026736	0.3010255804812
0.89	0.4106557586268	0.4093284788899	0.0013272797369	0.3232098196670
0.90	0.4065696694340	0.4051597023365	0.0014099670975	0.3467959376982
0.91	0.4025242374693	0.4010274574553	0.0014967800141	0.3718484192332
0.92	0.3985190581871	0.3969310648797	0.0015879933074	0.3984736174410
0.93	0.3945537075495	0.3928700132285	0.0016836943211	0.4267338739577
0.94	0.3906278362899	0.3888437776959	0.0017840585940	0.4567156838885
0.95	0.3867410280648	0.3848517176396	0.0018893104253	0.4885208157779
0.96	0.3828928941911	0.3808933397889	0.0019995544022	0.5222229068508
0.97	0.3790830498529	0.3769679756744	0.0021150741784	0.5579448037154
0.98	0.3753111140632	0.3730751345206	0.0022359795426	0.5957669407503
0.99	0.3715767096261	0.3692142123730	0.0023624972531	0.6358033729959
1.00	0.3678794411714	0.3653846150558	0.0024948261156	0.6781640495255

Graph-1: Graph between errors in the value of y calculated by Runge-Kutta fourth order method at different points for differential equation DE1



Graph-2: Graph between percentage errors in the value of y calculated by Runge-Kutta fourth order method at different points for differential equation DE1



In the above manner, the remaining differential equations have been solved with the help of developed computer program.

Conclusion

Maximum error in the value of y and maximum percentage error in it for each differential equation have been calculated and included in Table-2. Average of the maximum error is 0.0014620589518 which indicates that the Runge-Kutta fourth order method is accurate at least up to two places after the decimal point in worst cases. In some cases, the accuracy of Runge-Kutta fourth order method has been found to be accurate up to eight places after the decimal point. Value of average of minimum percentage error has been found to be 0.0332014263613.

Table-2: Maximum Error and maximum percentage error in the value of y calculated by Runge-Kutta fourth order method

Differential Equation	Maximum Error in the value of y calculated by Runge-Kutta fourth order method	Maximum Percentage Error in the value of y calculated by Runge-Kutta fourth order method
DE1	0.0024948261156	0.0000000000969
DE2	0.0023318937057	0.0578924889010
DE3	0.0000000310817	0.0000104471599
DE4	0.0024834813812	0.1080976492986
DE5	0.0000000624748	0.0000065463499
Average	0.0014620589518	0.0332014263613

Minimum error in the value of y and minimum percentage error in it for each differential equation have been calculated and included in Table-3. The values of average of minimum error and minimum percentage error have been found to be 0.0000000000287 and 0.000000141442 respectively.

Table-3: Minimum Error and minimum percentage error in the value of y calculated by Runge-Kutta fourth order method

Differential Equation	Minimum Error in the value of y calculated by Runge-Kutta fourth order method	Minimum Percentage Error in the value of y calculated by Runge-Kutta fourth order method
DE1	0.0000000000969	0.0000000100855
DE2	0.0000000000388	0.000000038424
DE3	0.0000000000007	0.0000000338588
DE4	0.0000000000051	0.000000005069
DE5	0.0000000000022	0.0000000224272
Average	0.0000000000287	0.000000141442

References

1. Srivastava, R.B.; Pandey V.K.; J. Chem. Bio. Phy. Sci. Sec. C, **2012**, Vol. 2, No. 2, 901-999.
2. Blanes, S.; Casas, F.; Ros, J.; *Special issue: Themes in geometric integration. Appl. Numer. Math.*, 39, 3-4, **2001**, 245-259.
3. Murugesan, K.; Dhayabaran, D. Paul; Amirtharaj, E. C. Henry; Evans, David J., *Int. J. Comput. Math.*, 79, 2, **2002**, 247-269.
4. Papageorgiou, G.; Famelis, I.; *Comput. Math. Appl.*, 42, 8-9, **2001**, 1165-1176.
5. Ling, Xiang; Tu, Shan-Tung; Gong, Jian-Ming; *International Journal of Pressure Vessels and Piping*, 77, 5, **2000**, 243-248.
6. Chou, Lin-Yi; Sharp, P. W.; *J. Appl. Math. Decis. Sci.*, 4, 2, **2000**, 143-150.
7. Tsitouras, Ch.; *Celestial Mech. Dynam. Astronom.*, 74, 4, **1999**, 223-230.
8. Burnton, Christopher; Scherer, Rudolf; *BIT*, 38, 1, **1998**, 12-21.
9. Day, J. D.; Murthy, D. N. P.; *Mathematics of Computation*, 39, 160, **1982**, 491-509.
10. Micula, Maria; *Studia Univ. Babeedla s-Bolyai Math.*, 26, 2, **1981**, 56-61.
11. Coroian, Iulian; *Stud. Cerc. Mat.*, 26, **1974**, 505-511.
12. Micula, Maria; *Stud. Cerc. Mat.*, 25, **1973**, 33-40.