Derive the NTU-Effectiveness Formula for the Double Pipe Heat Exchanger in Parallel Flow Arrangement

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Abstract: In Heat Exchangers text books, the derivation of the effectiveness(ε)and the number of transfer units(NTU), (ε-NTU) relations, are based on figures which are difficult to understand and the physical meaning of the steps in the derivation is unclear. The NTU-Effectiveness heat exchanger design formula for the double pipe heat exchanger in parallel flow arrangement is derived from the first principles. Energy equations of parallel flow arrangement for both the hot and the cold fluids are determined. The actual rate of heat transfer for parallel flow arrangement along with the heat exchanger effectiveness are stated. The relationship between the effectiveness(ε)and the number of transfer units(NTU) for the double pipe heat exchanger in parallel flow arrangement is plotted.

Keywords: exchanger, effectiveness, parallel flow, transfer units, double pipe.

I. Introduction

Heat exchangers are devices that provide the transfer of thermal energy between two or more fluids at different temperatures. Heat exchangers are used in a wide variety of applications, such as power production, process, chemical and food industries, electronics, environmental engineering, waste heat recovery, manufacturing industry, air-conditioning, refrigeration, space applications [1]. There are two ways in which heat is transferred in the heat exchanger: conduction through the wall that separates the two fluids and convection in each fluid. The double pipe heat exchanger is consider the simplest type of heat exchanger which consists from two pipes that have the same center with different diameter, as shown in figure (1-1). One fluid flows through the smaller pipe, however; the other fluid flows in the circular area between the two pipes. There are two flow arrangement types in a double pipe heat exchanger: parallel flow and counter-flow [2]. In the parallel flow heat exchangers, the two fluids enter the exchanger at the same end and flow parallel to one another in the same direction to another side. For all types of heat exchangers, there are two methods of analysis: The log mean temperature difference (LMTD) method, and the number of transfer units (NTU) which is based on a formal definition of effectiveness [2]. In Heat Exchangers text books, the derivation of ε-NTU relations are based on figures which are difficult to understand and the physical meaning of the steps in the derivation is unclear.

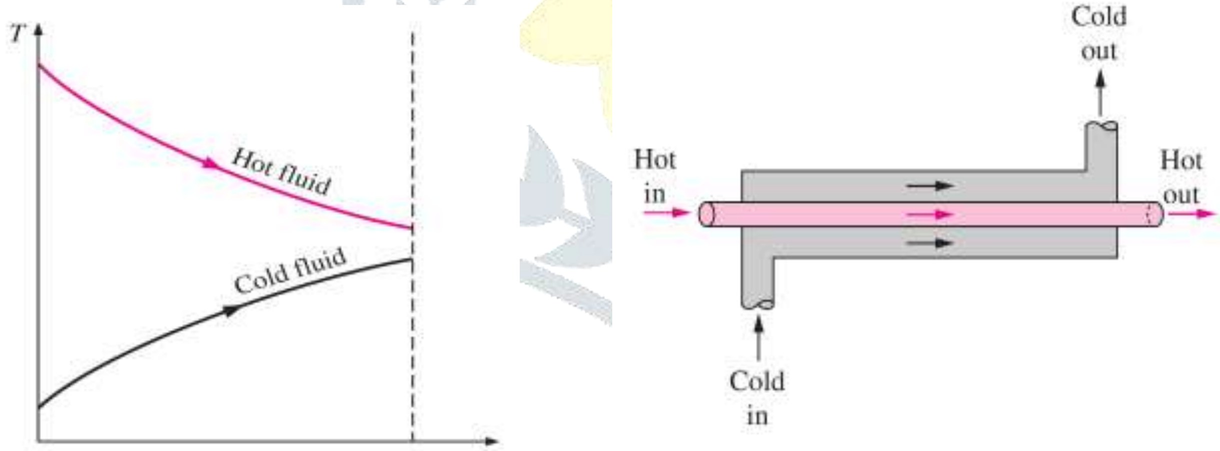

Fig (1-1): The double-pipe heat exchanger in parallel flow arrangement

1.1 Actual rate of heat transfer ($\dot{Q}$)

The actual heat transfer rate can be defined from an energy balance of the cold fluid or the hot fluid which means the heat transfer rate of the hot fluid is equal to the heat transfer rate of the cold fluid.

$$\dot{Q} = C_h \left(T_{h,in} - T_{h,out}\right) \qquad (1.1)$$

$$\dot{Q} = C_c \left(T_{c,out} - T_{c,in}\right) \qquad (1.2)$$

where

$$C_h = \dot{m}_h \, C_{p_h} \qquad (1.3)$$

$$C_c = \dot{m}_c \, C_{p_c} \qquad (1.4)$$

1.

2 The maximum possible heat transfer rate ($\dot{Q}_{max}$)

The maximum heat transfer is the maximum heat that could be transferred between the fluids. The outlet temperature of the colder fluid becomes equal to the inlet temperature of the hotter fluid when $\dot{m}_c\, C_{p_c} < \dot{m}_h\, C_{p_h}$ of infinite length as shown in figure (1.3) [3].

The heat transfer of the cold fluid equals

$$\dot{Q} = C_c \left(T_{c,out} - T_{c,in}\right)$$

Since $T_{c,out} = T_{h,in}$ and $C_c < C_h = C_{\min}$

$$\dot{Q} = C_{\min} \left(T_{h,in} - T_{c,in}\right) = \dot{Q}_{\max}$$

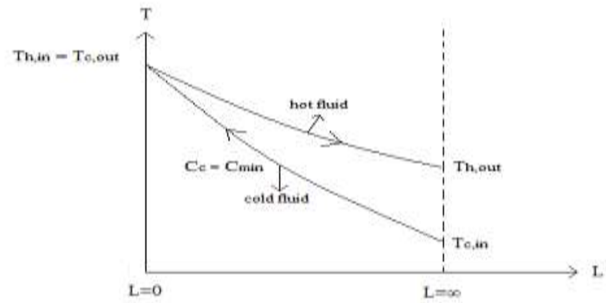

Fig (1-3): The relationship between the outlet temperature of the colder fluid and the inlet temperature of the hotter fluid of infinite length.

The outlet temperature of the hotter fluid becomes equal to the inlet temperature of the colder fluid $\dot{m}_h\, C_{p_h} < \dot{m}_c\, C_{p_c}$ of infinite length as shown in figure (1.2).

The heat transfer of the hot fluid equals

$$\dot{Q} = C_h \left(T_{h,in} - T_{h,out}\right)$$

Since $T_{h,out} = T_{c,in}$ and $C_h < C_c = C_{\min}$

$$\dot{Q} = C_{\min} \left(T_{h,in} - T_{c,in}\right) = \dot{Q}_{\max}$$

$$\dot{Q}_{\max} = C_{\min} \left(T_{h,in} - T_{c,in}\right) \qquad (1\text{-}5)$$

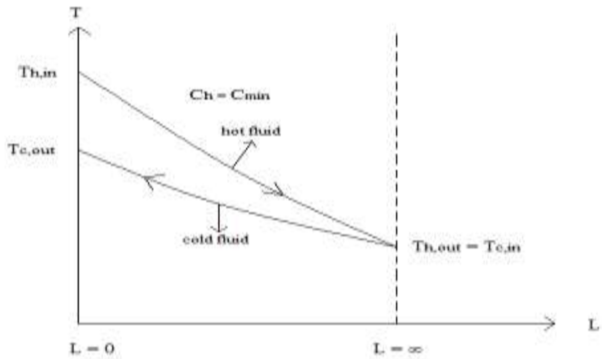

Fig (1-4): The relationship between the outlet temperature of the hotter fluid and the inlet temperature of the colder fluid of infinite length.

1.3 The heat capacity ratio (Cr)

The heat capacity ratio is defined as the ratio of the minimum heat capacity rate to the maximum heat capacity rate [3].

$$C_r = \frac{C_{\min}}{C_{\max}} \qquad (1\text{-}6)$$

1.4 The number heat transfer units (NTU)

It is defined as the ratio of the overall conductance to the smaller heat capacity rate

$$NTU = \frac{UA}{C_{\min}} \qquad (1\text{-}7)$$

1.5 The effectiveness of the heat-exchanger (ε)

The effectiveness of the heat-exchanger is defined as the ratio of the actual heat transfer rate to the maximum possible heat transfer rate.

$$\varepsilon = \frac{\dot{Q}}{\dot{Q}_{\max}} \qquad (1\text{-}8)$$

The effectiveness is a dimensionless quantity between (0 and 1).

Also $\varepsilon = f\left(NTU, C_r\right)$

II. The derivation

2.1 Energy equations of parallel flow

The first law for parallel flow per unit length of (x) direction for:

2.1.1 The hot fluid

The energy transfer of the hot fluid of length (dx) is equal to the temperature difference between the hot and the cold fluids multiply in the overall conductance which is equal to the energy transfer of the cold fluid of the same length (dx). From the control volume as shown

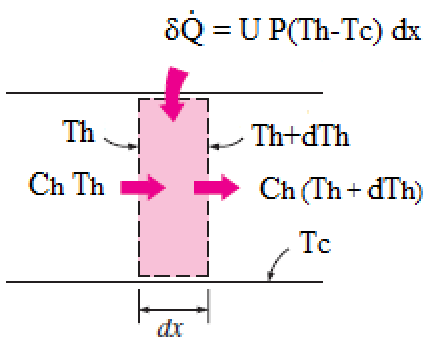

fig (2-1)

in figure (2-1), the energy balance of the total heat transfer between two fluids inside the heat exchanger is:

$$C_h \frac{dT_h(x)}{dx} = -PU\,(T_h(x) - T_c(x)) \qquad (2\text{-}1)$$

2.1.2 The cold fluid

$$C_c \frac{dT_c(x)}{dx} = PU\,(T_h(x) - T_c(x)) \qquad (2\text{-}2)$$

Putting equation (2-1) and (2-2) in matrix form gives

$$\begin{bmatrix} C_h & 0 \\ 0 & C_c \end{bmatrix} \begin{bmatrix} \frac{dT_h}{dx} \\ \frac{dT_c}{dx} \end{bmatrix} = \begin{bmatrix} -PU & PU \\ PU & -PU \end{bmatrix} \begin{bmatrix} T_h \\ T_c \end{bmatrix} \qquad (2\text{-}3)$$

Assuming the solution to be

$$\begin{bmatrix} T_h \\ T_c \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} e^{SX} \qquad (2\text{-}4)$$

So

$$\begin{bmatrix} \frac{dT_h}{dx} \\ \frac{dT_c}{dx} \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} s\, e^{SX} \qquad (2\text{-}5)$$

Put equations (2-4) and (2-5) in equation (2-3) gives

$$\begin{bmatrix} C_h & 0 \\ 0 & C_c \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} s\, e^{SX} = \begin{bmatrix} -PU & PU \\ PU & -PU \end{bmatrix} . \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} e^{SX}$$

$$\begin{bmatrix} C_h S + PU & -PU \\ -PU & C_c S + PU \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which has a nonzero solution for (A_1 and A_2) only if the determinate of coefficient array is zero. This determinate condition is called the characteristic equation.

Hence.

$$C_h C_c\, S^2 + C_h PU\, S + C_c PU\, S + P^2U^2 - P^2U^2 = 0 \qquad (2\text{-}6)$$

Or

$$C_h C_c\, S^2 + PU\,(C_h + C_c)\, S = 0 \qquad (2\text{-}7)$$

Or

$$S\,((C_h C_c\, S) + PU\,(C_h + C_c)) = 0 \qquad (2\text{-}8)$$

Either

$$S = 0$$

Or

$$S = \frac{-PU\,(C_h + C_c)}{(C_h\, C_c)}$$

Let $m = \dfrac{(C_h + C_c)}{(C_h\, C_c)}$

So $S = -PU\,m$

Case I. $S = 0$

$$\begin{bmatrix} PU & -PU \\ -PU & PU \end{bmatrix} . \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$PU\, A_1 - PU\, A_2 = 0$$

So $A_1 = A_2 = A$

Case II. $S = -PU\,m$

$$\begin{bmatrix} C_h(-PU\,m) + PU & -PU \\ -PU & C_c(-PU\,m) + PU \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -C_h m+1 & -1 \\ -1 & -C_c m+1 \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -C_h \dfrac{(C_h+C_c)}{C_h\ C_c}+1 & -1 \\ -1 & -C_c \dfrac{(C_h+C_c)}{C_h\ C_c}+1 \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{-C_h-C_c+C_c}{C_c} & -1 \\ -1 & \dfrac{-C_h-C_c+C_h}{C_h} \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} C_h & C_c \\ C_h & C_c \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$C_h\ B_1+C_c\ B_2=0$$

$$B_2=-\frac{C_h}{C_c}B_1$$

Assume that $B_1=C_c\ B$ & $B_2=-C_h\ B$

The solution becomes

$$\begin{bmatrix} T_h(x) \\ T_c(x) \end{bmatrix}=A\begin{bmatrix} 1 \\ 1 \end{bmatrix}+B\begin{bmatrix} C_c \\ -C_h \end{bmatrix}e^{-PUmX}$$

The boundary conditions for this type are:

At $X=0 \Rightarrow T_h(0)=T_{hin}$

At $X=0 \Rightarrow T_c(0)=T_{cin}$

$$T_{hin}=A+C_c\ B \qquad (2\text{-}9)$$

$$T_{cin}=A-C_h\ B \qquad (2\text{-}10)$$

From (2-9) & (2-10)

$$B=\frac{T_{hin}-T_{cin}}{C_h+C_c}$$

$$A=\frac{C_h\ T_{hin}-C_c\ T_{cin}}{C_h+C_c}$$

2.2 The actual rate of heat transfer for parallel flow

The heat transfer between the two fluids over the small length (dx) is:

$$\delta\dot{Q}=U\ P\left(T_h(x)-T_c(x)\right)dx$$

To obtain the total heat transfer between the two fluids inside the heat exchanger, the above expression is integrated from (0 to L) (the length of the heat exchanger).

$$\dot{Q}=\int_0^L U\ P\left(T_h(x)-T_c(x)\right)dx \qquad (2\text{-}11)$$

Applying $T_h(x)$ *and* $T_c(x)$ in equation (2-11)

$$\dot{Q}=UP\int_0^L B\left(C_h+C_c\right)e^{-UPmX}\,dx$$

$$\dot{Q}=UP\ B\left(C_h+C_c\right)\int_0^L e^{-UPmX}\,dx$$

$$\dot{Q}=\frac{-UP\ B\left(C_h+C_c\right)}{UPm}\left[e^{-UPmX}\right]_0^L$$

Applying

$$m = \frac{(C_h + C_c)}{(C_h\ C_c\ )}\ \&\ B = \frac{T_{h\,in} - T_{c\,in}}{C_h + C_c}$$

$$\dot{Q} = \frac{\left(T_{h\,in} - T_{c\,in}\right)\left(C_h\ C_c\right)}{C_h + C_c}\left[1 - e^{\frac{-UA(C_h + C_c)}{C_h\ C_c}}\right] \qquad (2\text{-}12)$$

2.3 The heat-exchanger effectiveness

$$\varepsilon = \frac{\dot{Q}}{\dot{Q}_{\max}}$$

from equation (2-12) and (1-5)

If $(C_h = C_{\min}\ \ and\ C_c = C_{\max})$

$$\varepsilon = \frac{(T_{h\,in} - T_{c\,in})(C_{\min}\ C_{\max})}{\left(T_{h\,in} - T_{c\,in}\right)\left(C_{\min} + C_{\max}\right)C_{\min}}\left[1 - e^{-\frac{UA}{C_{\min}}\left(\frac{C_{\min} + C_{\max}}{C_{\max}}\right)}\right]$$

$$\varepsilon = \frac{1 - \exp\left[-NTU\left(1 + C_r\right)\right]}{1 + C_r}$$

Where

$$C_r = \frac{C_{\min}}{C_{\max}}$$

$$NTU = \frac{UA}{C_{\min}}$$

If $C_c = C_{\min}\ and\ C_h = C_{\max}$

$$\varepsilon = \frac{(T_{h\,in} - T_{c\,in})(C_{\max}\ C_{\min})}{\left(T_{h\,in} - T_{c\,in}\right)\left(C_{\max} + C_{\min}\right)C_{\min}}\left[1 - e^{-\frac{UA}{C_{\min}}\left(\frac{C_{\min} + C_{\max}}{C_{\max}}\right)}\right]$$

$$\varepsilon = \frac{1 - \exp\left[-NTU\left(1 + C_r\right)\right]}{1 + C_r}$$

The last formula expression is the relationship between the effectiveness(ε)and the number of transfer units(NTU) .The plot below shows the relationship between the effectiveness(ε)and the number of transfer units(NTU) for multiple values of C_r

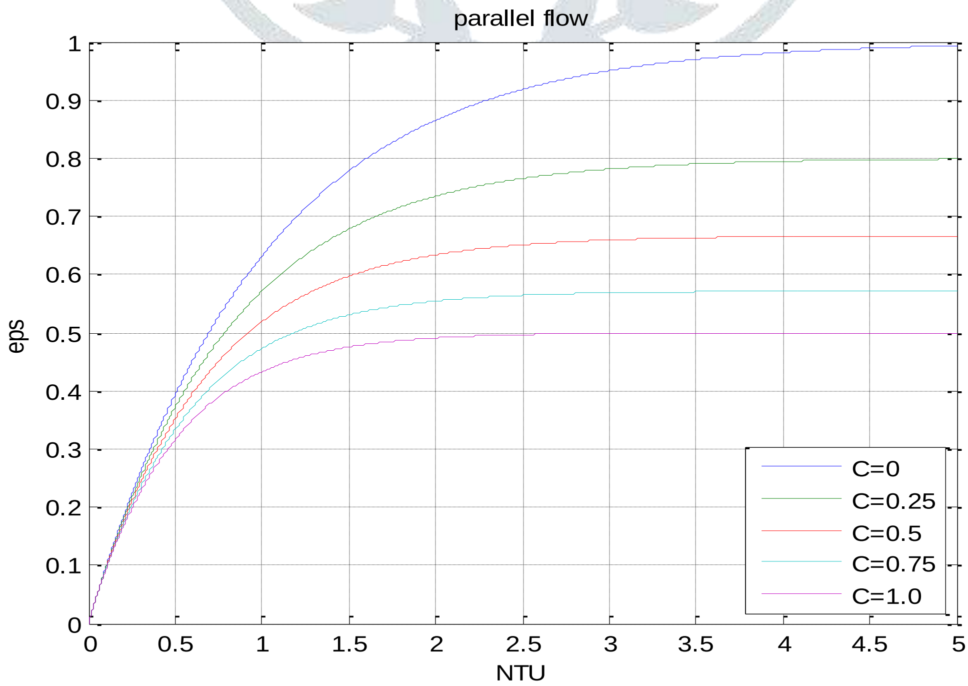

Fig (2-2) the effectiveness for parallel flow heat exchanger for for multiple values of C.

III. Conclusions

The method used in this paper to derive the NTU-Effectiveness formula for the double pipe heat exchanger in parallel flow arrangement is based on the basic energy balance equations of heat transfer with boundary conditions which is easy to understand and the physical meanings are clear.. From the plot between the effectiveness (ε) and the NTU, it was noticed for the effectiveness of parallel flow arrangement, is parallel to each other as the area increase for different Cr. However; the method used in text books based on figures which is difficult to understand.

References

1. Sadik Kakc, Hongtan Liu, and Anchasa Pramuanjaroenkij, Heat Exchangers Selection, Rating, and Thermal Design, Third Edition, CRC Press Taylor & Francis Group, USA, 2012.
2. Yunus A Cengel, Heat Transfer a Practical Approach, Second Edition, McGraw-Hill Book Company, New York, 1998.
3. B.K. HODGE, Analysis and Design of Energy System, Third Edition, Prentice-Hall, Inc. A Viacom Company, New Jersey, 1999.
4. W. M. Kays and A. L. London, Compact Heat Exchangers, Third Edition, McGraw-Hill Book Company, New York, 1984.