

SYMMETRY REDUCTIONS OF THE (2+1) DIMENSIONAL SAWADA KOTERA EQUATION WITH DAMPING TERM

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Abstract: We study the nonlinear fifth order (2+1) dimensional Sawada- Kotera equation with damping term using lie symmetry group. For this equation Lie point symmetry operators are obtained. We determine the corresponding invariant solutions and reduced equations using obtained infinitesimal generators.

Key Words: Lie symmetry, Sawada- Kotera equation, group invariant solutions.

1.Introduction :

Nonlinear phenomena are very important in applied mathematics and physics. They appear in various scientific fields such as biology, signal processing, viscoelastic materials, fluid mechanics, optical fiber, and so on. Some years ago, researchers have provided many methods for obtaining the numerical and analytical solutions of nonlinear equation, such as tanh function method, extended tanh function method [1, 2], $\left(\frac{G'}{G}\right)$ -expansion method [3, 4], sine-cosine method [5,6], simplest equation method [7] and so on. In this paper we study the Sawada- Kotera equation, namely

$$u + u_t + 5uu_{xxx} + 5u_x u_{xx} + 5u^2 u_x + u_{xxxxx} + u_{yy} = 0.$$

Sawada and kotera proposed it more than thirty years ago [4]. Because many various properties are satisfied for this equation, much effort has been made about its exact solutions.

For example, Fuchssteiner and Oevel studied its biHamiltonian structure [9], and a Darboux transformation was obtained for this system [10, 11], Liu and Dai [12] accomplished the Hirota's bilinear method to obtain exact solution of the same equation. Feng and Zheng [13] investigated this equation to establish traveling wave solution via the $\left(\frac{G'}{G}\right)$ -expansion method. In [14], wazwaz implemented the extended tanh method for constructing analytical solution of the same equation.

The outline of the paper is as follows. In section 2 we discuss the methodology of Lie symmetry analysis of the Sawada Kotera equation. Then in section 3 the solutions are obtained of the one dimensional subalgebras of the Sawada Kotera equation. In section 4 the solutions are obtained of the two dimensional subalgebras of the Sawada Kotera equation. Finally, concluding remarks are summarized in section 5.

2. Symmetries and classifications of Lie algebra for

$$u + u_t + 5uu_{xxx} + 5u_x u_{xx} + 5u^2 u_x + u_{xxxxx} + u_{yy} = 0$$

In order to derive the symmetry generators of Eqn.(1) and obtain closed form solution. We consider one parameter Lie point transformation that leaves (1) invariant. This transformation is given by

$$\tilde{x}^i = x^i + \varepsilon \xi^i(x, y, t; u) + O(\varepsilon^2), \quad i=1, \dots, 4, \quad (2)$$

Where $\xi^i = \frac{\partial x^i}{\partial \varepsilon} \Big|_{\varepsilon=0}$ defines the symmetry generator associated with (1) given by

$$V = \xi \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial x^j} + \tau \frac{\partial}{\partial x^k} + \phi \frac{\partial}{\partial x^l} \quad (3)$$

In order to determine four components ξ^i , we prolong V to fifth order. This prolongation is given by the formula (4)

$$\begin{aligned} V^{(5)} = V &+ \phi^x \frac{\partial}{\partial u_x} + \phi^y \frac{\partial}{\partial u_y} + \phi^t \frac{\partial}{\partial u_t} + \phi^{xx} \frac{\partial}{\partial u_{xx}} + \phi^{xy} \frac{\partial}{\partial u_{xy}} + \phi^{xt} \frac{\partial}{\partial u_{xt}} + \phi^{yy} \frac{\partial}{\partial u_{yy}} + \phi^{yt} \frac{\partial}{\partial u_{yt}} + \phi^{tt} \frac{\partial}{\partial u_{tt}} + \phi^{xxx} \frac{\partial}{\partial u_{xxx}} \\ &+ \phi^{xxy} \frac{\partial}{\partial u_{xxy}} + \phi^{xyt} \frac{\partial}{\partial u_{xyt}} + \phi^{xtt} \frac{\partial}{\partial u_{xtt}} + \phi^{xxx} \frac{\partial}{\partial u_{xxx}} + \phi^{xxxy} \frac{\partial}{\partial u_{xxxy}} + \phi^{xxxt} \frac{\partial}{\partial u_{xxxt}} + \phi^{xxxxy} \frac{\partial}{\partial u_{xxxxy}} + \phi^{xxxxt} \frac{\partial}{\partial u_{xxxxt}} + \phi^{xxxxt} \frac{\partial}{\partial u_{xxxxt}} \end{aligned} \quad (4)$$

In above expression every coefficient of the prolong generator is a functions of (x, y, t; u) and can be determined by the formula

$$\phi^i = D_i(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,i} + \eta u_{y,i} + \tau u_{t,i} \quad (5)$$

$$\phi^{ij} = D_i D_j(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,ij} + \eta u_{y,ij} + \tau u_{t,ij} \quad (6)$$

$$\phi^{ijk} = D_i D_j D_k(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,ijk} + \eta u_{y,ijk} + \tau u_{t,ijk} \quad (7)$$

$$\phi^{ijkn} = D_i D_j D_k D_n (\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,ijkn} + \eta u_{y,ijkn} + \tau u_{t,ijkn} \quad (8)$$

$$\phi^{ijkmn} = D_i D_j D_k D_m D_n (\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,ijkmn} + \eta u_{y,ijkmn} + \tau u_{t,ijkmn} \quad (9)$$

Where D_i represents total derivative and subscripts of u derivative with respect to the respective coordinates. To proceed with reductions of (A) we now use symmetry criterion for partial differential equations. For heat equation this criterion is expressed by the formula

$$V^{(5)}[u + u_t + 5uu_{xxx} + 5u_x u_{xx} + 5u^2 u_x + u_{xxxx} + u_{yy}] = 0 \quad (10)$$

Whenever,

$$u + u_t + 5uu_{xxx} + 5u_x u_{xx} + 5u^2 u_x + u_{xxxx} + u_{yy} = 0$$

Using this symmetry criterion with (4) in mind immediately yields

$$\phi + \phi_t + 5\phi\phi_{xxx} + 5\phi_x \phi_{xx} + 5\phi^2 \phi_x + \phi_{xxxx} + \phi_{yy} = 0 \quad (11)$$

At this stage we calculate expression for ϕ^t , ϕ^{xx} and ϕ^{yy} using (5)-(9), substitute them in (9) and then compare coefficients of various monomials in derivatives of u . This yields the following system of over determined partial differential equations

$$\begin{aligned} \xi_u &= 0 \\ \eta_u &= 0 \\ \tau_u &= 0 \\ \phi_{uu} &= 0 \\ \xi_y &= 0 \\ \tau_y &= 0 \\ \eta_x &= 0 \\ \tau_x &= 0 \\ 10u\phi - \xi_t - 5u^2 \xi_x + 5u^2 \tau_t + 5\phi_{xx} - 5u \xi_{xxx} + 15u \phi_{xxu} - \xi_{xxxx} + 5\phi_{xxxx} &= 0 \\ \phi + u\tau_t + \phi_t - u\phi_u + 5u^2 \phi_x + \phi_{yy} - 5u \phi_{xxx} + \phi_{xxxx} &= 0 \\ \phi_x - 3u \xi_{xx} + 3u \phi_{xu} - \xi_{xxx} + 2\phi_{xxu} &= 0 \\ \phi - 3u \xi_x + u \xi_{ttt} - 2\xi_{xxx} + 2\phi_{xxu} &= 0 \\ -\xi_{xx} + 2\phi_{xu} &= 0 \\ -2\xi_{xx} + \phi_{xu} &= 0 \\ -5\xi_x + \tau_t &= 0 \\ -3\xi_x + \tau_t + \phi_u &= 0 \\ -\eta_t - \eta_{yy} + 2\phi_{yu} &= 0 \\ -2\eta_y + \tau_t &= 0 \end{aligned}$$

3. Reduction of one dimensional abelian subalgebra for

$$u + u_t + 5uu_{xxx} + 5u_x u_{xx} + 5u^2 u_x + u_{xxxx} + u_{yy} = 0$$

After some more manipulations one finds that ξ and η become

$$\begin{aligned} \xi &= k_1 \\ \eta &= k_3 \end{aligned} \quad (12)$$

The remaining equations can then be used to determine τ and ϕ as

$$\begin{aligned} \tau &= k_2 \\ \phi &= 0 \end{aligned} \quad (13)$$

As this stage we construct the symmetry generators corresponding to each of the constants involved. These are a total of eight generators given by

$$\begin{aligned} V_1 &= \partial x \\ V_2 &= \partial t \\ V_3 &= \partial y \end{aligned} \quad (14)$$

It is easy to check that the symmetry generators found in (13) form a closed Lie algebra whose commutation relations are given in table 1.

Table 1

$[V_i, V_j]$	V_1	V_2	V_3
V_1	0	0	0
V_2	0	0	0
V_3	0	0	0

Commutation relations satisfied by generators

4. Reductions of two dimensional abelian subalgebra for

$$u + u_t + 5uu_{xxx} + 5u_x u_{xx} + 5u^2 u_x + u_{xxxx} + u_{yy} = 0$$

We now briefly show steps involved in the reduction of the nonlinear heat equation to a second order differential equations. Since reduction under all the subalgebra's cannot be given in the paper, we restrict ourselves to giving reductions in one case only.

i.e., $\{V_1, V_3\}$ and $\{V_2, V_3\}$.

Reduction in the remaining cases is listed in the form of Appendices A at the end of the paper.

4.1 Reduction under V_1 and V_3

From Table 1 we find that the given generators commute $[V_1, V_3]=0$. Thus either of V_1 or V_3 can be used to start the reduction with. For our purpose we begin reduction with V_1 . The characteristic equation associated with this generator is

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{0}$$

Following standard procedure we integrate the characteristic equation to get three similarity variables

$$r = y, \quad s = t, \quad u = w(r, s) \quad (15)$$

Using these similarity variables Eqn.(1) can be recast in the form

$$W + W_s + W_{rr} = 0 \quad (16)$$

At this stage we express V_3 in terms of the similarity variables defined in (14). It is straightforward to note that V_3 in the new variables takes the form

$$\square V_3 = \frac{\partial}{\partial r} \quad (17)$$

The characteristic equation for $\square V_3$ is $\frac{dr}{1} = \frac{ds}{0} = \frac{dw}{0}$. Integrating this equation as before leads to new variable $\alpha = s$ and $\beta(\alpha) = w$, which reduce (15) to a second order differential equation

$$\beta + \beta' = 0 \quad (18)$$

4.2 Reduction under V_2 and V_3

In this case the two generators V_2 and V_3 satisfy the commutation relation $[V_2, V_3] = 0$. This suggests that reduction in this case should start with V_2 . The similarity variables are

$$r = x, \quad s = y, \quad u = w(r, s)$$

The corresponding reduced partial differential equation is

$$W + 5WW_{rr} + 5W_r W_{rr} + 5W^2 W_r + W_{rrrr} + W_{ss} = 0 \quad (19)$$

The transformed V_3 is

$$\square V_3 = \partial_s \quad (20)$$

The invariants of $\square V_3$ are $\alpha = r$ and $\beta(\alpha) = w$ which reduce (18) to the ordinary differential equation

$$\beta + 5\beta\beta''' + 5\beta'\beta'' + 5\beta^2\beta' + \beta'''' = 0 \quad (21)$$

Reductions in remaining cases using generators forming sub-algebra are given in the form of Table 2 in Appendix A

Appendix A

Table 2

Algebra	Reduction
$[V_1, V_2]=0$	$\beta + \beta'' = 0$
$[V_1, V_3]=0$	$\beta + \beta' = 0$
$[V_2, V_3]=0$	$\beta + 5\beta\beta''' + 5\beta'\beta'' + 5\beta^2\beta' + \beta'''' = 0$

5. Conclusion:

In this work, we used the Lie group method to find the Lie point symmetries of the Sawada Kotera equation, which concludes to similarity variables. We utilized the corresponding invariant solution to reduce the number of independent variables in the Sawada Kotera equation. Also, we have obtained an optimal system for the Sawada Kotera equation

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