ON THE NON-HOMOGENEOUS QUINTIC EQUATION WITH THREE UNKNOWNS

$$5(x^2+y^2)-9xy+2(x+y)+4=(k^2+19s^2)^n z^5$$

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Abstract : ***The non-homogeneous quintic equation with five unknowns represented by the diophantine equation*** $5(x^2+y^2)-9xy+2(x+y)+4=(k^2+19s^2)^n z^5$ ***is analyzed for its non-zero distinct integral solutions. Introducing the transformations*** $x=u+v, y=u-v$ ***and employing the method factorization,three different patterns of non-trival distinct integer solutions to the quintic equation under consideration are obtained. A few interesting properties between the solutions and special numbers namely,Polygonal numbers,,Centered Pyramidal numbers , Thabith-ibn-kurrah number, Gnomic number, Jacobsthal Lucas number , Jacobsthal number and five dimensional numbers are exhibited.***

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NOTATIONS:

$t_{m,n}$: Polygonal number of rank n with size m

P_n^m : Pyramidal number of rank n with size m

j_n : Jacobsthal Lucas number of rank n

J_n : Jacobsthal number of rank n

GNO_n : Gnomic number of rank n

Tk_n : Thabith-ibn-kurrah number of rank n

$Ct_{m,n}$: Centered Polygonal number of rank n with size m

$Cf_{3,n.30}$: Centered Tricontagonal Pyramidal number of rank n

$\mathrm{F}_{5,n,7}$: Fifth Dimensional Figurate Heptagonal number of rank n

$GF_n(k,s)$: Generalized Fibonacci Sequences of rank n

$GL_n(k,s)$: Generalized Lucas Sequences of rank n

I. INTRODUCTION

The theory of diophantine equations offers a rich variety of fascinating problems. In particular,quintic equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity[1,2,9].For illustration, one may refer [3-5] for quintic equation with three unknowns ,[6] for quintic equation with four unknowns and [7,8] for quintic equation with five unknowns. This communication concerns with yet another interesting a non-homogeneous sextic equation with 5 unknowns given by $5(x^2+y^2)-9xy+2(x+y)+4=(k^2+19s^2)^n z^5$ for determining its infinitely many non-zero integer quintuples (x,y,z,w,t) .Three different methods are illustrated. In mehod1,the solutions are obtained through the method of factorization. In mehod2,the binomial expansion is introduced to obtain the integral solutions. In method3,the integral solutions are expressed in terms of Generalized Fibonacci and Lucas sequences along with a few properties in terms of the above integer sequences.Also a few interesting properties among the values of x, y and z are presented.

2. METHOD OF ANALYSIS

The diophantine equation representing a non-homogeneous quitic equation with five unknowns is

$$5(x^2+y^2)-9xy+2(x+y)+4=(k^2+19s^2)^n z^5 \qquad (1)$$

Introducing the linear transformations

$$x = u + v, y = u - v \qquad (2)$$

in (1), it leads to

$$(u+2)^2 + 19v^2 = (k^2 + 19s^2)^{2n} z^5 \qquad (3)$$

The above equation (3) is solved through three different methods and thus, one obtains three distinct sets of solutions to (1)

2.1:Method: 1

Let $z = a^2 + 19b^2$ (4)

Substituting (4) in (3) and using the method of factorization,define

$$(u + 2 + i\sqrt{19}v) = (k + i\sqrt{19}s)^n (a + i\sqrt{19}b)^5 \qquad (5)$$

$$= r^n \exp(in\theta)(a + i\sqrt{19}b)^5$$

where $r = \sqrt{k^2 + 19s^2}, \quad \theta = \tan^{-1}\left(\frac{\sqrt{19}s}{k}\right)$ (6)

Equating real and imaginary parts in (5) we get

$$u = r^n\left(\cos n\theta(a^5 - 190a^3b^2 + 1805ab^4) - \sqrt{19}\sin n\theta(5a^4b - 190a^2b^3 + 3611b^5)\right) - 2$$

$$v = r^n\left(\cos n\theta(5a^4b - 190a^2b^3 + 361b^5) + \frac{\sin n\theta}{\sqrt{19}}(a^5 - 190a^3b^2 + 1805ab^4)\right)$$

Substituting the values of u and v in (2), the corresponding values of x, y and z are represented by

$$x(a,b,k) = r^n\left\{\begin{array}{l}\left(\cos n\theta(a^5 - 190a^3b^2 + 1805ab^4 + 5a^4b - 190a^2b^3 + 361b^5)\right) + \\ \left(\frac{\sin n\theta}{\sqrt{19}}(a^5 - 190a^3b^2 + 1805ab^4 - 95a^4b + 3610a^2b^3 - 6859b^5)\right)\end{array}\right\} - 2$$

$$y(a,b,k) = r^n\left\{\begin{array}{l}\left(\cos n\theta(a^5 - 190a^3b^2 + 1805ab^4 - 5a^4b + 190a^2b^3 - 361b^5)\right) - \\ \left(\frac{\sin n\theta}{\sqrt{19}}(a^5 - 190a^3b^2 + 1805ab^4 + 95a^4b - 3610a^2b^3 + 6859b^5)\right)\end{array}\right\} - 2$$

$$z(a,b) = a^2 + 19b^2$$

2.2:Method :2

Using the binomial expansion of $(k + i\sqrt{19}s)^n$ in (5) and equating real and imaginary parts, we have

$$u = f(\alpha)(a^5 - 190a^3b^2 + 1805ab^4) - 19g(\alpha)(5a^4b - 190a^2b^3 + 361b^5) - 2$$

$$v = f(\alpha)(5a^4b - 190a^2b^3 + 361b^5) + g(\alpha)(a^5 - 190a^3b^2 + 1805ab^4)$$

where

$$\left.\begin{array}{l} f(\alpha) = \sum_{r=0}^{\left[\frac{n}{2}\right]} (-1)^r (19)^r (s)^{2r} n_{c_{2r}} k^{n-2r} \\ g(\alpha) = \sum_{r=1}^{\left[\frac{n+1}{2}\right]} (-)^{r-1} (19)^{r-1} (s)^{2r-2} n_{C_{2r-1}} k^{n-2r+1} \end{array}\right\} \qquad (7)$$

In view of (2) and (7) the corresponding integer solution to (1) is obtained as

$$x = (f(\alpha) + g(\alpha))(a^5 - 190a^3b^2 + 1805ab^4) + (f(\alpha) - 19g(\alpha))(5a^4b - 190a^2b^3 + 361b^5) - 2$$

$$y = (f(\alpha) - g(\alpha))(a^5 - 190a^3b^2 + 1805ab^4) - (f(\alpha) + 19g(\alpha))(5a^4b - 190a^2b^3 + 361b^5) - 2$$

$$z = a^2 + 19b^2$$

2.3:Method :3

Taking $n = 0$ and $u + 2 = U$ in (3), we have,

$$U^2 + 19v^2 = z^5 \qquad (8)$$

Substituting (4) in (8), we get

$$U^2 + 19v^2 = (a^2 + 19b^2)^5 \qquad (9)$$

whose solution is given by

$$U_0=(a^5-190a^3b^2+1805ab^4)$$
$$v_0=(5a^4b-190a^2b^3+361b^5)$$

Again taking $n=1$ in (3), we have,

$$U^2+19v^2=(k^2+19s^2)(a^2+19b^2)^5 \qquad (10)$$

whose solution is represented by

$$U_1=kU_0-19sv_0$$
$$v_1=sU_0+kv_0$$

The general form of integral solutions to (1) is given by

$$\begin{pmatrix}U_n\\ v_n\end{pmatrix}=\begin{pmatrix}A_n & i\sqrt{19}B_n\\ -i\dfrac{B_n}{\sqrt{19}} & A_n\end{pmatrix}\begin{pmatrix}U_0\\ v_0\end{pmatrix}, n=1,2,3.....$$

where $$A_n=\frac{(k+i\sqrt{19}s)^n-(k-i\sqrt{19}s)^n}{2}$$

$$B_n=\frac{(k+i\sqrt{19}s)^n-(k-i\sqrt{19}s)^n}{2}$$

Thus, in view of (2), the values of x_n, y_n and z as follows:

$$x_n=\{(a^5-190a^3b^2+1805ab^4+5a^4b-190a^2b^3+361b^5)A_s+\frac{i}{\sqrt{19}}B_s$$
$$(95a^4b-3610a^2b^3+685b^5-a^5+190a^3b^2+1805ab^4)\}-2$$
$$y_n=\{(a^5-190a^3b^2+1805ab^4-5a^4b+190a^2b^3-361b^5)A_s+\frac{i}{\sqrt{19}}B_s$$
$$(95a^4b-3610a^2b^3+685b^5+a^5-190a^3b^2-1805ab^4)\}-2$$
$$z=a^2+11b^2$$

Thus, in view of (2), the following of integers x_n, y_n interms of Generalized Lucas and fiboanacci sequence satisfy (1) are as follows:

$$x_n=\begin{bmatrix}\frac{1}{2}GL_n(2k,-k^2-19s)(a^5-190a^3b^2+1805ab^4+5a^4b-190a^2b^3+361b^5)-\\ GF_n(2k,-k^2-19s)(95a^4b-3610a^2b^3+685b^5-a^5+190a^3b^2+1805ab^4)\end{bmatrix}-2$$

$$y_n=\begin{bmatrix}\frac{1}{2}GL_n(2k,-k^2-19s)(a^5-190a^3b^2+1805ab^4-5a^4b+190a^2b^3-361b^5)-\\ GF_n(2k,-k^2-19s)(95a^4b-3610a^2b^3+685b^5+a^5-190a^3b^2-1805ab^4)\end{bmatrix}-2$$

The above values of x_n, y_n satisfy the following recurrence relations respectively

$$x_{n+2}-2kx_{n+1}+(k^2+19s^2)x_n=0$$
$$y_{n+2}-2ky_{n+1}+(k^2+19s^2)y_n=0$$

2.3.1:Properties

1. $z(2^a,2^a)=20(j_{2a}+Tk_{2a}-9J_{2a})$
2. $z(a,a)+108Ct_{2,a}+10Gn_a=80t_{3,a}$
3. $x_n(1,a,k)-y_n(1,a,k)+2i\dfrac{B_n}{\sqrt{19}}(Ct_{372,a}+180t_{4,a^2}-93t_{3,a}-190t_{4,a})$

$$=2A_n(361F_{5,a,7}-792Cf_{3,a,30}-190P_a^3+190t_{8,a}-2166t_{4,a^2}-5t_{4,a}-1198t_{3,a})$$

4. $x_n(a,a,k)+y_n(a,a,k)+4=\begin{bmatrix}2A_n(38784F_{5,a,7}-1616t_{4,a^2}-32P_a^5)\\ \dfrac{i38B_n}{\sqrt{19}}(4224F_{5,a,7}-1056t_{4,a^2}-176Cf_{3,a,30}-2112t_{3,a}+176t_{12,a}-880t_{4,a})\end{bmatrix}$

3. CONCLUSION

To conclude, one may search for other pattern of solutions and their corresponding properties.

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