On The Application Of Fuzzy Probability

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Abstract : Probability is the measure of uncertainty about the happening of an event. In probability theory an event A is a subset of a sample space S. The object of this paper is to study the basic theory of Fuzzy probability. In this paper we have study about Fuzzy event, probability of Fuzzy event and conditional properties of Fuzzy event. Also some results of probability theory are proved with respect to the Fuzzy mathematics.

Keywords: Fuzzy Linear Programming, Fuzzy Objective Function, Fuzziness in constraints and optimal value.

I. INTRODUCTION

With the founding of Fuzzy set theory by L.A. Zadeh in 1965 , a new branch of mathematics is being created which we call as Fuzzy mathematics. Several new branches of mathematics such as- Fuzzy probability, Fuzzy algebra, Fuzzy vector space etc. are being developed. L.A. Zudch in 1968 defined the probability of Fuzzy event, on the basic of this definition the probability of Fuzzy events and their application was introduced in different field by yager in 1984, Michacle and J.W.seaman in 1995 , R.Inton in 2015, J.M.Mendal in 2017. Now let us discuss the probability of Fuzzy events and some new approach for obtaining the solution of related examples.

II. PROBABILITY MEASURE: -

A probability measure P is a real valued function defined on the collection of events of a random experiment that satisfies the following axioms.

I. $\mathrm{p(A)}$ is non-negative for any event A.
II. $\mathrm{P(s) = 1}$.
III. If $\mathrm{A_i}$ for i in I is a countable collection of pair wise disjoint events then-

$$P\bigcup_{i\in 1} A = \sum_{i=1}^{\infty} P(\mathrm{A_i})$$

Where III is known as countable additive identity.
In everyday life we frequently face the situation in which an event is not the part of well defined collection of objects, for example:-
It is a cold day, she is a beautiful girl , x is approximately equal to 4. We can say that the above event is not well defined but they are ill defined events. In another words we say the above event is an element of Fuzzy set.

1. PROBABILITY SPACE: -

A probability space is a triplet $(\mathrm{R^n, S, P})$ where $\mathrm{R^n}$ is n-dimensional Euclidean space , S is the σ field of Borel set in $\mathrm{R^n}$ and P is a probability measure over $\mathrm{R^n}$. A point in R will be denoted by X. Let $\mathrm{A} \in S$. Then , the probability of A can be expressed as-

$$\mathrm{P(A)} = \int_{\mathrm{A}} dp \text{ ....(I)}$$

If $A(X)$ denotes the characteristic function of crisp set A [i.e. $\mathrm{A(X) = 0}$ or 1] and $\mathrm{E(A)}$ is the expectation of A. then the above Equation can be expressed as

$$\mathrm{P(A)} = \int_{\mathrm{R_n}} \mathrm{A(x)dp = E(A)} \text{ ....(II)}$$

Equation (II) is equal to the probability of an event A with the expectation of the characteristic function of A.

1. FUZZY EVENT:-

Let $(\mathrm{R^n, S, P})$ be a probability space in which S is the σ field of Borel sets in $\mathrm{R^n}$ and P is a probability measure over $\mathrm{R^n}$. Then a fuzzy event in$\mathrm{R^n}$ is a fuzzy set A in $\mathrm{R^n}$ whose membership function A(X) is Borel measurable defined as

$$\mathrm{A: R^n \to [0,1]}$$

2. PROBABILITY OF A FUZZY EVENT: -

The probability of a fuzzy event A in $\mathrm{R^n}$ with membership function A(X) is defined by the Lebesgue-stieltjes integral as-

$$\mathrm{P(A)} = \int_{\mathrm{R^n}} \mathrm{A(x)dp = E(A)} \text{...........(III)}$$

Where

$$\mathrm{E(A)} = \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{A(x_i)P(x_i)}$$

i.e. the probability of a Fuzzy event is the expectation $\mathrm{E(A)}$ of its membership function.
The definition of Fuzzy event and its probability form a basic for generalising the theory of fuzzy set, its several concept and results of probability theory.
Following the above definition, we have the following results-
Theorem 1 :- if A & B be two Fuzzy events in R^n , then

(a) $\mathrm{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$
(b) $\mathrm{P(A \oplus B) = P(A) + P(B) - P(AB)}$

Proof (a):-

Let us suppose that A and B are any two fuzzy sets in n-dimensional Euclidean space R^n . Now from the definition of union and intersection of two fuzzy sets. We have –

$$(A \cup B)(X) = \text{Max}\,\{A(x), B(x)\} \forall\, x \in X$$

And

$$(A \cap B)(X) = \text{Min}\,\{A(x), B(x)\} \forall\, x \in X$$

Let $A(x) > B(x)$, then

$(A \cup B)(X) = A(X)$ and

$(A \cap B)(X) = B(X)$

Thus $(A \cup B)(X) + (A \cap B)(X) = A(X) + B(X)$

$$\Longrightarrow (A \cup B)(X) = A(X) + B(X) - (A \cap B)(X) \qquad \text{............(IV)}$$

Now from the definition of probability of fuzzy event, we have-

$$P(A \cup B) = \int_{R^n} (AUB)(X)dp$$

$$= \int_{R^n}[A(X) + B(X) - (A \cap B)(X)]\,dp \qquad \text{Using (IV)}$$

$$= \int_{R^n} A(X)dp + \int_{R^n} B(X)dp - \int_{R^n} (A \cap B)(X)dp$$

$$= P(A) + P(B) - P(A \cap B) \qquad \text{(using (III))}$$

$$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Hence the result (a) .

Proof (b) :-

By the definition of direct sum of two fuzzy sets A and B , we have-

$(A \oplus B)(X) = A(X) + B(X) - [A(X)B(X)]$

$\Longrightarrow (A \oplus B)(X) = A(X) + B(X) - AB(X)$

By definition

$$P(A \oplus B) = \int_{R^n} (A \oplus B)(X)dp$$

$$= \int_{R^n}[A(X) + B(X) - \{A(X).B(X)\}]\,dp$$

$$\Longrightarrow P(A \oplus B) = \int_{R^n} [A(X) + B(X) - (AB)(X)]\,dp$$

$$= \int_{R^n} A(X)dp + \int_{R^n} B(X)dp - \int_{R^n} (AB)(X)dp$$

$$= P(A) + P(B) - P(AB) \qquad \text{(using (III))}$$

$$\therefore P(A \oplus B) = P(A) + P(B) - P(AB)$$

Hence the result (b) .

Now we will discuss the definition and meaning of statistical independence in probability theory of non-fuzzy events. We however know that the probability theory of non-fuzzy events is based on ordinary set theory in which an element either “belongs” to a set or “does not belong to” the set. Thus $A \cap B = \phi$ means that A and B are disjoint. If A & B are two events in a sample space, then the meaning of disjoint becomes independent. On other hand the probability theory of fuzzy events depend on fuzzy set theory in which an element need not either “belong to” or “not belong to” a set.

Consequently , if A and B are two fuzzy events then $A \cap B = \phi$ implies that either$A(X) = 0$ or $B(X) = 0$ or both $A(X) = 0 = B(X) \forall x \in X$. Then $P(A \cap B) = P(A) \cdot P(B)$ which implies both sides of the equation are identically equal to zero, which does not express the meaning of independence that we want.

3. STATISTICAL INDEPENDENCE :-

Two events *A* & *B* are said to be statistically independent if the occurrence of an event *A* does not depend on the occurrence of *B* and vice-versa.

In fuzzy set theory there is no such concept corresponding to “belonging to” and “not belonging to” in a sharp way , we therefore consider it differently.

4. INDEPENDENCE OF FUZZY EVENTS: -

Let us consider a probability measure P defined on a sample space S. Let P be the Cartesian product of two measure P_1& P_2 which are respectively denoted on S_1& $S_2$and we have $S = S_1 \times S_2$. Let A_1 and A_2 are two fuzzy events defined on S_1& S_2 with membership function given by –

$$A_1(X_1, X_2) = A_1(X_1)$$
$$A_2(X_1, X_2) = A_2(X_2)$$

Then we observe that the occurrence of event A and B are independent. Moreover A & B satisfy the following relation

$$P(AB) = \int_{S} A_1(X_1, X_2)A_2(X_1, X_2)d(P_1 \times P_2)$$

$$= \int_{S_1 \times S_2} A_1(X_1)\,A_2(X_2)(dP_1 \times dP_2)$$

$$= \int_{S_1} A_1(X_1)\,dP_1 \cdot \int_{S_2} A_2(X_2)\,dP_2$$

$$= P(A) \cdot P(B)$$

$$\Rightarrow P(AB) = P(A)\cdot P(B)$$

Two fuzzy events A&B are said to be "Statistically independence" if $(AB) = P(A)\cdot P(B)$.

5. CONDITIONAL PROBABILITY: -

Let A&B two fuzzy events in a sample spaceR^n, then the conditional probability of A given B is defined by –

$$P(A/B) = \frac{P(AB)}{P(B)} \quad \text{proved} \qquad P(B) > 0.$$

Let us discuss some examples based on the above.

Example 1:- An ordinary die is thrown. What is the probability that the number which turns up is close to 4 ?

Solution:- consider the experiment of rolling an ordinary six-faced die. Then the sample space S for this experiment be given by-

$$S = \{1,2,3,4,5,6\}$$

The probability $P(X_i)$ for each of the outcomes is obviously $1/6$ where $X_i = i, i = 1,2,3,4,5,6.$

Let A be a fuzzy event and $A(X)$ is the number that the die turns up which is close to 4.

Let assume that A is described by a membership function A(X) such that

$A(1) = 0.2, A(2) = 0.4, A(3) = 0.6,\ A(4) = 1, A(5) = 0.6\ , A(6) = 0.4$

1	2	3	4	5	6
A(1)=0.2	A(2)=0.4	A(3)=0.6	A(4)=1	A(5)=0.6	A(6)=0.4

Membership function may be shown on real line,

Then the probability that the number which turn up is close to 4 is given by-

$$P(A) = \sum_{i=1}^{6} A(X_i)P(X_i)$$

$$= A(X_1)P(X_1) + A(X_2)P(X_2) + A(X_3)P(X_3) + A(X_4)P(X_4) + A(X_5)P(X_5) + A(X_6)P(X_6)$$
$$= (0.2\times 1/6) + (0.4\times 1/6) + (0.6\times 1/6) + (1\times 1/6) + (0.6\times 1/6) + (1/6\times 0.4)$$

$$= 1/6(0.2 + 0.4 + 0.6 + 1 + 0.6 + 0.4)$$
$$= 1/6\ (3.2) = \frac{3.2}{6} = 0.533$$

$$\therefore P(A) = 0.533$$

Thus the probability that the number which turn up close to 4 is 0.533.

Example 2:- weather of the Mumbai city is given in the probability matrix -

$$P = (P_{ij}) = \begin{array}{c|cccc} \text{Day}\downarrow \ \text{Weather}\rightarrow & S & R & C & SN \\ \hline S & 5/8 & 1/8 & 1/8 & 1/8 \\ R & 1/4 & 1/2 & 1/8 & 1/8 \\ C & 3/16 & 3/16 & 7/16 & 3/16 \\ SN & 1/8 & 1/8 & 1/4 & 1/2 \end{array}$$

Where S, R, C & SN stands for Sunny day, Rain day, Cloud day and snowing day. The entries in the first row represent the probabilities for various kind of weather in Sunny day and similarly in 2nd ,3rd, and 4th row represent Rainy, Cloud and Snowy day respectively.

Suppose today's weather of Mumbai city x is Sunny. Then what would be the probability of having good weather after two days from today?

Solution :- Since the weather of the Mumbai city given in probability matrix

Day↓ Weather→	S	R	C	SN
S	5/8	1/8	1/8	1/8
R	1/4	1/2	1/8	1/8
C	3/16	3/16	7/16	3/16
SN	1/8	1/8	1/4	1/2

Let S be the sample space. Then –

$$S = \{S, R, C, SN\}$$

Let A be a fuzzy event = good weather is defined in R whose membership function is subjectively defined by-

$$A(S) = 1,\ A(R) = 0.2,\ A(C) = 0.5,\ A(SN) = 0.1.$$

In given probability matrix the 1st row is the probabilities for various kind of weather following a Sunny day. Then the probabilities for the various kind of weather two days after a Sunny day may be calculated by-

$$S[P_{ij}] = [5/8, 1/8, 1/8, 1/8] \begin{bmatrix} 5/8 & 1/8 & 1/8 & 1/8 \\ 1/4 & 1/2 & 1/8 & 1/8 \\ 3/16 & 3/16 & 7/16 & 3/16 \\ 1/8 & 1/8 & 1/4 & 1/2 \end{bmatrix}$$

$$= \left[\left(\frac{25}{64}+\frac{1}{32}+\frac{3}{128}+\frac{1}{64}\right),\left(\frac{5}{64}+\frac{1}{16}+\frac{3}{128}+\frac{1}{64}\right),\left(\frac{5}{64}+\frac{1}{64}+\frac{7}{128}+\frac{1}{32}\right),\left(\frac{5}{64}+1/64+3/128+\frac{1}{16}\right)\right]$$

$$= \left[\frac{50+4+3+2}{128},\frac{10+8+3+2}{128},\frac{10+2+7+4}{128},\frac{10+2+3+8}{128}\right]$$

$$= \left[\frac{59}{128},\frac{23}{128},\frac{23}{128},\frac{23}{128}\right]$$

Thus the probability of having good weather two days after a sunny day is given by-

$$\mathrm{P(A)} = \sum_{i=1}^{4} \mathrm{A(X_i)P(X_i)}$$

$$= 1 \times \frac{59}{128} + (0.2) \times \frac{23}{128} + (0.5) \times \frac{23}{128} + (0.1) \times \frac{23}{128}$$

$$= \frac{59}{128} + \frac{4.6}{128} + \frac{11.5}{128} + \frac{2.3}{128}$$

$$= \frac{59+4.6+11.5+2.3}{128} = \frac{77.4}{128}$$

$$\therefore \mathrm{P(A)} = 0.605$$

Thus the probability of having good weather two days after a Sunny day is (0.605).

6. CONCLUSION:-

In this paper we have proved some probability rule by fuzzy setting and defined fuzzy event, Probability of fuzzy event, conditional probability etc. in lucid manner. Also we have provided an unique rule , for solving the examples related to fuzzy event whose solution can't be obtain by probability theory. After all we have tried to extend the range of probability theory with respect to the fuzzy Mathematics.

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