STUDY OF LANDAU'S HARMONIC OSCILLATOR AND ITS PROPERTIES

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ABSTRACT: As teaching statistical mechanics of Quantum Harmonic Oscillators (QHO) in classrooms there is a general concept conveyed to the students that these oscillators cannot exert pressure. In this research paper it has show that the harmonic oscillator whose centre of oscillation is moving cannot exert pressure. The thermodynamics of this oscillator is discussed and it is shown that it is different from ordinary QHOs. Some of the interesting properties of this oscillator are also discussed in the present research paper.

KEYWORDS: QHO, Landau oscillators, Semiconductor devices, Helmholtz energy level, Partition functions, etc.

I. INTRODUCTION

Harmonic oscillator is one of the simplest of systems that has been extensively studied both classically as well as quantum mechanically. It has served as the first approximate solution to many new physics problems and played a very important role in the development of non-relativistic quantum mechanics, and quantum field theory. Plank's radiation law theory of specific heat, molecular theory, theory of superconductivity, confinements of quarks and many other areas. Recent experiment on the Bose Einstein Condensation and nanotechnology material in the presence of magnetic field have revealed astonishing properties and brought out the importance of harmonic oscillator once again. An electron moving in a magnetic field is an excellent example of harmonic oscillator. In classical physics, an electron moving in a magnetic field follows a helical cyclotron orbit. By using quantum mechanical approach Landau had shown that the cyclotron motion of an electron in a uniform, external magnetic field is analogous to that of harmonic oscillator. Any oscillator oscillating about a fixed center of oscillation is referred to as simple harmonic oscillator. Thermodynamics of this oscillator is well known. Landau introduced the concept of harmonic oscillator whose center of oscillation in not fixed while explaining the physics of diamagnetism. The landau oscillator is a two dimensional quantum harmonic oscillator with a magnetic field in the perpendicular direction. Many harmonic oscillator models used for explaining thermodynamics of different system use the conventional oscillator where the oscillators are oscillating about an equilibrium point and this equilibrium point itself is not moving. Therefore, these oscillators will not be colliding with the walls of the container and cannot therefore, exert any pressure. In the case of Landau oscillators we can see that $P \neq 0$, because of its non-fixing of the centre of oscillation.

II. STATISTICAL THERMODYNAMICS

A canonical ensemble in statistical mechanics is an ensemble of dynamically similar systems, each of which can share its energy with a large heat reservoir, or heat bath. The distribution of the total energy amongst the possible dynamical states (i.e. the members of the ensemble) is given by partition function. In statistical mechanics, the partition function Q_N is an important quantity that encodes the statistical properties of a system in thermodynamic equilibrium. It is a function of temperature and other parameters, such as the volume enclosing a gas. Most of the thermodynamic variable of the system, such as the total energy, free energy, entropy, and pressure, can be expressed in terms of the partition function or its derivatives.

The partition function of any quantum system with energy ϵ_n is given by,

$$Q = \sum_{n=0}^{\infty} g_l \, g_n \, e^{-\beta \varepsilon_n} \quad (1)$$

and for a system of N oscillators the partition function, $Q_N = \text{Product of } Q'S$

where, $\beta = \frac{1}{kt}$, g_l = the internal degree of freedom, g_n = the degenerecy, ϵ_n = energy of the system, k = Boltzmann's constant, and T = absolute temperature.

Now the degeneracy is given by,

$$g_n = \frac{(n-d-1)!}{n!\,(d-1)!} \quad (2)$$

Here, d is the dimension. For ID- harmonic oscillator, d = 1, and hence, g_n is one except for Landau Oscillator which has an additional degeneracy. From statistical thermodynamic, Helmholtz free energy,

$$A(N,V,T) = -kT \ln Q_N \quad (3)$$

Pressure,

$$P = -\left(\frac{\partial A}{\partial V}\right)_{N,T} \quad (4)$$

Internal energy,

$$U = -\frac{\partial(\ln Q_N)}{\partial \beta} \quad (5)$$

Specific heat capacity, $C_V \quad = \quad \left(\frac{\partial U}{\partial T}\right)_V$ (6)

Entropy $\quad = \quad -\left(\frac{\partial A}{\partial T}\right)_{N,\ V}$ (7)

III. SIMPLE QUANTUM HARMONIC OSCILLATOR

Let us consider N distinguishable, no-interacting particles that have the energy level of a simple harmonic oscillator. Let ω be the frequency of the oscillators. Partition Function for an oscillator,

$$Q = \sum_{n=0}^{\infty} g_n\, e^{-\beta\epsilon_n}$$

for one dimensional oscillator $g_n = 1$, for two dimensional oscillator, $g_n = n+1$, and three dimensional oscillator it is given by,

$$g_n = \frac{(n+1)\,(n+2)}{2}$$

Now, from equation (2), for harmonic oscillator, $\epsilon_n = nh\epsilon$, neglecting zero point energy. Hence, the partition function for one dimensional oscillator,

$$Q = \sum_{n=0}^{\infty} g_n\ e^{-\beta n\hbar\omega}$$

$$Q = \sum_{n=0}^{\infty} \left(e^{-\beta\hbar\omega}\right)^n$$

$$Q = \frac{1}{1-e^{-\beta\hbar\omega}}$$

Where this last step used the standard Taylor series expansion,

$$\frac{1}{1-x} = 1 + x + x^2 + \dots\dots + \infty = \sum_{n=0}^{\infty} x^n$$

Our purpose is to compare the properties of a harmonic oscillator with Landau oscillator. Hence we will take the case of two dimensional simple harmonic oscillators. The partition function for two dimensional oscillators is,

$$Q = \sum_{n=0}^{\infty} (n+1)\ e^{-\beta n\hbar\omega}$$

It can be shown that,

$$\sum_{n=0}^{\infty} e^{-\beta\hbar\omega} \sum_{n=0}^{\infty} e^{-\beta\hbar\omega} = \sum_{n=0}^{\infty} (n+1)\ e^{-\beta\hbar\omega}$$

$$Q = \sum_{n=0}^{\infty} \left(e^{-\beta\hbar\omega}\right)^n \sum_{n=0}^{\infty} \left(e^{-\beta\hbar\omega}\right)^n$$

$$Q = \left(\frac{1}{1-e^{-\beta\hbar\omega}}\right)\left(\frac{1}{1-e^{-\beta\hbar\omega}}\right) = \left(\frac{1}{1-e^{-\beta\hbar\omega}}\right)^2$$

By using Taylor series expansion, from, N such oscillator Helmholtz free energy from eq. (3)

$$A = 2NkT\ \ln\left(1 - e^{-\beta\hbar\omega}\right)$$

Since the above term does not have volume, pressure from eq. (4) is

$$P = 0$$

Internal energy from eq. (5) is,

$$U = 2\left(\frac{N\hbar\omega}{e^{\beta\hbar\omega}-1}\right) \quad (8)$$

Specific heat capacity $C_{V,}$ from eq. (6) is,

$$C_V = 2Nk\left(\frac{\hbar\omega}{kT}\right)^2 \frac{e^{\hbar\omega/kT}}{\left(e^{\hbar\omega/kT}-1\right)^2} \quad \dots\dots\dots \quad (9)$$

Entropy Using Eq. (7) is,

$$S = 2Nk\left[\ln\left\{\frac{e^{\hbar\omega/kT}}{e^{\hbar\omega/kT}-1}\right\}+\left\{+\frac{\hbar\omega/kT}{e^{\hbar\omega/kT}-1}\right\}\right] \quad \ldots\ldots (10)$$

IV. LANDAU' S OCLILLATOR

Landau Oscillator Neglecting the zero point energy function for any quantum harmonic oscillator neglecting zero point energy is given by,

$$Q = \sum_{n=0}^{\infty} g_n e^{-\beta_n \hbar\omega}$$

The degeneracy of such an oscillator with center not fixed has been calculated by Landau and is given by,

$$g_n = \frac{Am\omega}{h} \quad (11)$$

Where, A is the area in which oscillator is confirmed, m is the mass of electron and ω is the cyclotron frequency as from eq. (8). Hence the partition function Q_L, for a Landau oscillator is given by,

$$Q_L = \sum_{n=0}^{\infty} \frac{Am\omega}{h} e^{-\beta n \hbar\omega}$$

Using by Taylor series expansion,

$$Q_L = \frac{Am\omega}{h(1 - e^{-\beta\hbar\omega})}$$

The partition for N oscillators is therefore, given by,

$$Q_{NL} = \left(\frac{Am\omega}{h}\right)^N \frac{1}{(1-e^{-\beta\hbar\omega})^N}$$

For obtaining the various thermodynamics parameters we require the logarithm of the above partition function,

$$\ln Q_{NL} = N\ln\left(\frac{Am\omega}{h}\right) - N\ln(1 - e^{-\beta\hbar\omega}) \quad (12)$$

Helmholtz free Energy is,

$$F = -NkT\ln\left(\frac{Am\omega}{h}\right) + NkT\ln(1 - e^{-\beta\hbar\omega}) \quad (13)$$

We have the expansion for pressure as,

$$P = -\left(\frac{\partial F}{\partial V}\right)_{N,T}$$

This equation shows that for a Landau oscillator the pressure is not zero. Because of the first term in eq. (4) contains area. The pressure is non-isotropic, zero in the z-direction but non-zero in x and y- directions. This anisotropy is due to the non- fixing of the direction of the motion of the centre of oscillations. As a result this oscillator can exert force along x and y-directions according to Lorentz force. Hence, consequently the internal energy of both systems is the same, whereas the entropy of the Landau oscillators is different from a two dimensional simple harmonic quantum oscillators. The entropy S from Eq. (7) is are given as,

$$S = Nk\ln\left(\frac{Am\omega}{h}\right) + Nk\left[\ln\left(\frac{e^{\hbar\omega/kT}}{e^{\hbar\omega/kT}-1}\right)+\frac{\hbar\omega/kT}{e^{\hbar\omega/kT}-1}\right] \quad (14)$$

V. PROPERTIES OF LANDAU OSCILLATORS

Thus, we conclude that confinements of electrons to two dimensions can be achieved using the technique of Molecular Beam Epitaxial (MBE), where extremely pure semiconductor crystals are grown layer by layer. If the composition of a single crystal abruptly changes during MBE growth, for example from GsAs to AlGaAs, then the conduction electrons within the crystal will be drawn to the planar interface between the two material. At low enough temperatures, these electrons will be totally confined to the interface, with freedom to move only along the interface alone and their motion will be strictly two dimensional. The 2D- electron gas has become a reality and the statistical mechanics of elections in such system in the presence of magnetic field will be of immense interest.

The energy level of an electron in a uniform magnetic field is of the harmonic oscillator type and is usually referred to as Landau levels. If confined to two dimensions, the electrons motion will be circular. These ''Landau levels'' are usually spaced by an energy given by Plank's constant time cyclotron frequency, and they are labeled n=0, 1, 2, etc. Similar quantization of electron's closed orbit also requires its wavelength to e only certain values. However unlike atomic energy levels which can accommodate only a few electrons, a single Landau level in a sufficiently large magnetic field may accommodate hundreds of billions of electrons.

In hydrogen atom the number of states available for electrons is $2n^2$. For n=1 the number of states is 2 only. Now let us calculate the number of states available for electrons in a typical Landau level. The number of states is given by equation (2),

For, B=10T, and $\omega = \frac{eB}{m}$

Where, the value was found, $\omega = 1.75824 \times 10^{12}\ \mathrm{rads}^{-1}$, and the number of states per unit area $= 2.42424 \times 10^{15}$

Hence, for a unit millions and billions of states are available for this unique system. Therefore, every state can be imagined as a band which enables it to be used as an electronic device equivalent to a semiconductor. Now, another intersecting property is the value of interval between the energy levels, for a hydrogen atom the energy Eigen values are given by,

$$E_n = \frac{-13.6}{n^2} \text{ e V}$$

V. CONCLUSION

The energy interval between the first two levels is 10.2 eV. We estimate the energy interval between two adjacent Landau levels or the Landau gap, for a typical situation: B =10T, the gap is , $\hbar\omega = 0.00115$ eV. Thus the Landau gap will be 10000 times smaller in Landau oscillator and excitation is easily possible in Landau system. In ordinary semiconductor the band gap is of the order of 1eV. Hence the materials where Landau levels are generated which are used as semiconductor will bring new vistas in the development of semiconductor physics and nanotechnology.

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