DIFFERENTIAL TRANSFORM METHOD FOR SOLVING LINEAR AND HOMOGENEOUS EQUATION WITH VARIABLE COEFFICIENTS

Dr. T.Shanmuga Priya[1] and N.Alima[2]
Asst. Professor[1], PG Research[2]
Department of Mathematics
[1] & [2] - Adhiyaman Arts and Science College for Women, Uthangarai, Tamil Nadu

Abstract: In this paper, we consider the differential transform method (DTM) for solving approximate and exact solutions of linear and homogeneous partial differential equation with the variable co-efficients. The efficiency of the considered method is illustrated by some problems.

Keywords: Homogeneous heat equation, Homogeneous wave equation, linear equation, differential transform method, partial differential equation with variable co-efficients.

Introduction

The differential transformation method was first introduced by Zhou (1986), who solved linear and non-linear problems in electrical circuit problems. The differential transform method is one of the numerical method for partial differential equation which uses the form of polynomials as the approximation to the exact solution. Chen and Ho (1999) developed this method for partial differential equations and Ayaz applied it to the system of differential equations. In this method has been used for differential algebraic equations, Partial differential equations, fractional differential equations, volterra integral equations and difference equations. This method has been utilized for Telegraph equation.

In this paper, the differential transform method has been utilized for solving the following, partial differential equation. The method can be used to evaluate the approximate solution by the finite taylor series and by an iterative process describing by the transformed equations obtained from the original equation using the operator of differential transformation.

There are many problems arising in science and engineering are modelled using linear or nonlinear partial differential equations (PDEs). Boundary and initial value problems in PDEs occur in fluid mechanics, mathematical physics, astrophysics, biology, materials science, electromagnetism, image processing, computer graphics, etc. PDEs are categorized into different types, including elliptic, parabolic, and hyperbolic PDE. These PDEs describe various physical phenomenon including deformation of beams, viscoelastic and inelastic flows, transverse vibrations of a homogeneous beam, plate deflection theory, engineering and applied sciences. In recent years, various methods have been proposed for solving the fourth-order parabolic PDEs, such that adomain decomposition method (ADM), variational iteration method (VIM), B-spline methods, homotopy pertubation method (HPM) and homotopy analysis method (HAM).

Definition

Consider $u(x,t)$ is analytic and differentiated continuously in the domain of interest, then let

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{t=t_0}, \qquad (1)$$

Where the spectrum$U_k(x)$ is the transformed function, which is called T-function.

Definition

The differential inverse transform of $U_k(x)$ is defined as follows:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)(t-t_0)^k, \qquad (2)$$

Substituting equation (1) in (2), we get

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{t=t_0}(t-t_0)^k, \qquad (3)$$

When (t_0) are taken as$(t_0 = 0)$ then equation (3) is expressed as

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{t=0} t^k,$$

And equation (2) is shown as

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k, \qquad (4)$$

In real application, the function $u(x,t)$ by a finite series of equation (4) can be written as

$$u(x,t) = \sum_{k=0}^{n} U_k(x)t^k, \qquad (5)$$

Usually, the values of n is decided by convergence of the series coefficients.

Example 1

Consider the one-dimensional heat equation with variable coefficients in the form

$$u_t(x,t) - \frac{x^2}{2}u_{xx}(x,t) - u_x(x,t) = 0 \qquad (6)$$

And the initial condition

$$u(x,0) = x^2, \qquad (7)$$

Where $u = u(x,t)$ is a function of the variables x and t.

The exact solution of this problem is $u(x,t) = (x^2 + 2x + 2)e^t$. Then, by using the reduced differential transformation, we can find the transformed form of equation (6) as,

$$(k+1)U_{k+1}(x) = \frac{x^2}{2}\frac{\partial^2}{\partial x^2}U_k(x) + \frac{\partial}{\partial x}U_k(x), \quad (8)$$

Using the initial condition, equation (7), we have

$$U_0(x) = x^2, \quad (9)$$

Now, substituting equation (9) into (8), we obtain the following $U_k(x)$ values successively

$U_1(x) = x^2+2x, U_2(x) = \frac{x^2+2x+2}{2}, U_3(x) = \frac{x^2+2x+2}{6}, U_4(x) = \frac{x^2+2x+2}{24}, U_5(x) = \frac{x^2+2x+2}{120},$

$U_6(x) = \frac{x^2+2x+2}{720}, \ldots,$

$U_k(x) = \frac{x^2+2x+2}{k!},$

Finally the differential inverse transform of $U_k(x)$ gives:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k = (x^2 + 2x + 2)\sum_{k=0}^{\infty}\frac{t^k}{k!}$$

Then, the exact solution is given as

$u(x,t) = (x^2 + 2x + 2)e^t.$

Example 2

Consider the two-dimensional heat equation with variable coefficients as

$$u_t(x,y,t) - \frac{y^2}{2}u_{xx}(x,y,t) - \frac{x^2}{2}u_{yy}(x,y,t) - u_x(x,y,t) - u_y(x,y,t) = 0 \quad (10)$$

Where the initial condition is

$$u(x,y,0) = y^2 \quad (11)$$

Taking differential transform of equation (10) and the initial condition equation (11) respectively,

$$(k+1)U_{k+1}(x,y) = y^2\frac{\partial^2}{\partial x^2}U_k(x,y) + x^2\frac{\partial^2}{\partial y^2}U_k(x,y) \quad (12)$$

Using the initial condition equation (11), we have

$$U_0(x,y) = y^2 \quad (13)$$

Now, substituting equation (13) into (12), we obtain the following $U_k(x,y)$ values successively,

$U_1(x,y) = x^2 + 2y, U_2(x,y) = \frac{y^2+2x+2}{2}, U_3(x,y) = \frac{x^2+2y+2}{6}, U_4(x,y) = \frac{y^2+2x+2}{24},$

$U_5(x,y) = \frac{x^2+2y+2}{120}, U_6(x,y) = \frac{y^2+2x+2}{720}, \ U_7(x,y) = \frac{x^2+2y+2}{5040}, \ldots,$

$$U_k(x,y) = \begin{cases} \frac{x^2+2y+2}{k!}, k \text{ is odd} \\ \frac{y^2+2x+2}{k!}, k \text{ is even} \end{cases}$$

Finally the differential inverse transform of $U_k(x,y)$ gives:

$$u(x,y,t) = \sum_{k=0}^{\infty} U_k(x,y)t^k = (x^2 + 2y + 2)\sum_{k=0}^{\infty}\frac{t^k}{k!} + (y^2 + 2x + 2)\sum_{k=0}^{\infty}\frac{t^k}{k!}$$

Then, the exact solution is given by,

$$u(x,y,t) = (x^2 + 2y + 2)\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right) + (y^2 + 2x + 2)\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right)$$

$$u(x,y,t) = (x^2 + 2y + 2)\sin ht + (y^2 + 2x + 2)\cos ht$$

Which is the exact solution of the given equation.

Example 3

Consider the one-dimensional wave equation with variable coefficient as

$$u_{tt}(x,t) - \frac{x^2}{2}u_{xx}(x,t) - u_x(x,t) = 0 \quad (14)$$

With an initial condition

$$u(x,0) = x, u_t(x,0) = x^2 \quad (15)$$

Taking differential transform of equation (14), we get

$$(k+1)(k+2)U_{k+2}(x) = \frac{x^2}{2}\frac{\partial^2}{\partial x^2}U_k(x) + \frac{\partial}{\partial x}U_k(x) \quad (16)$$

Using the initial condition equation (15), we have

$$U_0(x) = x, U_1(x) = x^2 \quad (17)$$

Now, substituting equation (17) into (16), we obtain the following $U_k(x)$ values successively

$U_k(x) = 0, k = 2,4,6, \ldots$

By applying the k values are k = 1,3,5,…

$U_3(x) = \frac{1}{6}(x^2 + 2x), \ U_5(x) = \frac{1}{120}(x^2 + 2x + 2),$

$U_7(x) = \frac{1}{5040}(x^2 + 2x + 2), \ldots$

$U_k(x) = \frac{1}{k!}(x^2 + 2x + 2)$

Finally the differential inverse transform of $U_k(x)$ gives:

$$u(x,t) = \sum_{k=0}^{\infty} U_k t^k = (x^2 + 2x + 2)\left(1 + t + \frac{t^2}{2} + \cdots + \frac{t^k}{k!}\right)$$

Thus, the exact solution is given by,

$u(x,t) = (x^2 + 2x + 2)\sin ht.$

Example 4

Consider the two dimensional wave equation is of the form

$$u_{tt}(x,y,t) - u_{xx}(x,y,t) - u_{yy}(x,y,t) - u(x,y,t) = 0 \quad (18)$$

With an initial condition

$$u(x,0) = 1 + \sin x\,, u_t(x,0) = 0. \quad (19)$$

Taking differential transform of equation (18) and the initial condition equation (19) respectively, we have

$$(k+1)(k+2)U_{k+2}(x) - \frac{\partial^2}{\partial x^2}U_k(x) - \frac{\partial^2}{\partial y^2}U_k(x) - U_k(x) = 0 \quad (20)$$

Using the initial condition, equation (19), we have

$$U_0(x) = 1 + \sin x\,, U_1(x) = 0. \quad (21)$$

Substituting equation (21) into (20), we obtain the following $U_k(x)$ values successively

$U_k(x) = 0 \quad k = 1,3,5,\ldots$

By applying the k values are k = 2,4,6,…

$U_2(x) = \frac{1}{2}, U_4(x) = \frac{1}{24}, U_6(x) = \frac{1}{720}, \ldots.$

$U_k(x) = \frac{1}{k!}$

Finally the differential inverse transform of $U_k(x)$ gives:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k = (1+\sin x) + \left(\frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \cdots\right)$$

Then, the exact solution is given by

$u(x,t) = \sin x + \cosh t.$

Conclusion

The differential transform method has been successfully, applied for solving linear and homogeneous partial differential equations with variable co-efficients. The solution is obtained by differential transform method is an infinite power series for appropriate initial condition, which can in turn express the exact solutions in a closed form. Thus we conclude that the proposed by this method can be extended to solve many PDEs with variable co-efficients which arise in physical and engineering applications.

References

[1] J.K. Zhou, Differential Transformation and Its Applications for ElectricalCircuits, Huazhong University Press, Wuhan, China (1986).

[2] C.K. Chen, S.H. Ho, Solving partial differential equations by two dimensionaldifferential transform method, Appl. Math. Comput., 106 (1999), 171-179.

[3] B. Soltanalizadeh, Numerical analysis of the one-dimensional Heat equationsubject to a boundary integral specification, Optics Communications,284 (2011), 2109-2112.

[4] H.R. Ghehsareh, B. Soltanalizadeh, S. Abbasbandy, A matrix formulationto the wave equation with non-local boundary condition, Inter. J. Comput.Math., 88 (2011), 1681-1696.

[5] B. Soltanalizadeh, Differential transformation method for solving one-space-dimensional telegraph equation, Comp. Appl. Math., 30, No. 3(2011), 639-653.

[6] F. Ayaz, on the two-dimensional differential transform method, Appl. Math. Comput. 143 (2003) 361-374.

[7] F. Ayaz, Solutions of the system of differential equations by differential transform method, Appl. Math.Comput. 147 (2004) 547-567.

[8] C.K. Chen, S.H. Ho, Application of differential transformation to eigenvalue problems, Appl. Math.Comput. 79 (1996) 173-188.

[9] A.Q.M. Khaliq and E.H. Twizell, A family of second-order methods for variable coefficient fourth-order parabolic partial differential equations, Int. J. Comput. Math., 23 (1987), 63–76.

[10] D.J. Gorman, Free Vibrations Analysis of Beams and Shafts, Wiley, New York ,1975.

[11] C. Andrade and S. McKee, High frequency A.D.I. methods for fourth-order parabolic equations with variable coefficients, Int. J. Comput. Appl. Math., 3 (1977), 11–14.

[12] S.D. Conte, A stable implicit difference approximation to a fourth-order parabolic equation, J. Assoc. Comp. Mach., 4 (1957), 210–212.

[13] S.D. Conte and W.C. Royster, Convergence of finite difference solution to a solution of the equation of a vibration beam, Proc. Amer. Math. Soc., 7 (1956), 742–749.

[14] A. Borhanifar, R. Abazari, Numerical study of nonlinear Schrodinger and coupled Schrodinger equations by differential transformation method, Optics Communications, 283 (2010), 2026-2031.

[15] C.K. Chen, Solving partial differantial equations by two dimensional differentialtransformation method, Appl. Math. Comput., 106 (1999), 171-179.

[16] M.J. Jang, C.K. Chen, Two-dimensional differential transformation method for partial differantial equations, Appl. Math. Comput., 121(2001), 261-270.

[17] F. Kangalgil, F. Ayaz, Solitary wave solutions for the KDV and mKDVequations by differential transformation method, Choas Solutions Fractals, 41 (2009), 464-472.

[18] J.D. Cole, On a quasilinear parabolic equation occurring in aerodynamics, Quart. Appl. Math., 9 (1951), 225-236.

[19] A. Arikoglu, I. Ozkol, Solution of difference equations by using differential transformation method,

[20] A.M. Wazwaz, On the solution of the fourth order parabolic equation by the decomposition method, Int. J. Comput. Math., 57 (1995), 213-217.

[21] Z.M. Odibat, Differential transformation method for solving Volterra integral equations with seperable kernels, Math. Comput. Modeling., 48 (2008), 1144-1149.