Hilbert Space: perspective of Quantum Mechanics

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Abstract *:* The major goal of this work is to determine the significance of Hilbert space in Quantum Mechanics and to investigate associated postulates. Hilbert Space is an important part of Quantum Mechanics and can be defined as the complete space of the inner product. In quantum mechanics it is used for determining the interpretation of a wave function. In a mathematical domain Hilbert space is observed to be infinite-dimensional space.

IndexTerms **– Linear vector space, Inner product space, Matrix mechanics, Wave mechanics.**

I. INTRODUCTION

Hilbert spaces are important in a variety of fields of mathematics, including analysis, differential geometry, group theory, stochastics, and even number theory. In addition, the notion of a Hilbert space provides the mathematical foundation of quantum mechanics. Today, the most spectacular new application of Hilbert space theory is given by Noncommutative Geometry [1], where the motivation from pure mathematics is merged with the physical input from quantum mechanics. Consequently, this is an important area of study in both pure mathematics and mathematical physics. From the last decades of eighteen century physicist had begun to recognize classical mechanics, Newtonian gravity and electrodynamics could not describe all of nature. Max Planck gives a new concept which was ended in 1925-1927 with the discovery of quantum mechanics. Heisenberg discovered a form of quantum mechanics known as 'matrix mechanics'. In 1926, Schrodinger was led to a formulation of quantum theory called 'wave mechanics'. von Neumann alone who, at the age of 23, recognized the mathematical structure of quantum mechanics. In this process, he defined the abstract concept of a Hilbert space. Von Neumann proposed the mathematical formulation of quantum mechanics. The observables of a given physical system are the self-adjoint (possibly unbounded) linear operators on Hilbert space (H)[2-4]. The expectation value of an observable a in a state ψ is given by $(\psi, a\psi)$. The transition probability between two states ψ and ϕ is $|\langle\psi,\phi\rangle|^2$. From the definition of inner product this number is equal to $(cos\theta)^2$, where θ is the angle between ψ and ϕ. Thus the geometry of Hilbert space has a direct physical interpretation in quantum mechanics. Given the importance of Hilbert space theory to quantum mechanics, a thorough mathematical understanding of the Hilbert space theory that underpins much of quantum mechanics will likely aid in the future development of quantum theory.

II. PREREQUISITES

Before going to discussions on Hilbert space I put some background on in linear algebra and real analysis. Nonetheless, for the sake of clarity, we start with a discussion of three notions that are fundamental to the sector of functional analysis, namely metric spaces, normed linear spaces, and inner product spaces.

IIA. Metric spaces

A metric space (X, d) is a set (X) together with an assigned metric function $d\colon X \times X \to \mathbb{R}$ that has the following properties

(i) $d(x,y) \geq 0$ for all $x, y, z \in X$,
(ii) $d(x,y) = 0$ if and only if $x = y$,
(iii) $d(x,y) = d(y,x)$ for all $x, y, z \in X$,
(iv) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

IIB. Normed Linear space

A (complex) normed linear space $(L, \|\cdot\|)$ is a linear vector space with a function $\|\cdot\| : L \to \mathbb{R}$ called a norm that satisfies the properties:

(i) $\|v\| \geq 0$ for all $v \in L$,
(ii) $\|v\| = 0$ if and only if $v = 0$,
(iii) $\|\lambda v\| = |\lambda|\|v\|$ for all $v \in L$ and $\lambda \in \mathbb{C}$,
(iv) $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in L$.

IIC. Inner product space (IPS)

If V is a linear space, then a function $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ is said to be an inner product provided that

(i) $\langle v, v\rangle \geq 0$ for all $v \in V$,
(ii) $\langle v, v\rangle = 0$ if and only if $v = 0$,
(iii) $\langle \lambda u, v\rangle = \lambda \langle u, v\rangle$ for all $u, v \in V$ and $\lambda \in \mathbb{C}$,
(iv) $\langle u, v\rangle = \overline{\langle v, u\rangle}$ whenever $u, v \in V$,
(v) $\langle u + w, v\rangle = \langle u, v\rangle + \langle w, v\rangle$ for all $u, v, w \in V$.

Basic Facts about Inner Product space:

Let X be a IPS with the inner product $\langle\cdot,\cdot\rangle$ and the associated norm $\|\cdot\|$.

(i) Cauchy-Schwarz inequality

$$|\langle x, y\rangle| \leq \|x\|\,\|y\|$$

with the equality if and only if x and y are linearly dependent.
(ii) $\langle\cdot,\cdot\rangle$ is continuous function from $X \times X$ to the corresponding scalar field.
(iii) Parallelogram Law

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \qquad \forall\, x, y \in X$$

Parallelogram Law is a sufficient and necessary condition for a norm $\|x\|$ to be the associated norm for an inner product on X.

III. PROPERTIES OF HILBERT SPACE

These are the main properties of Hilbert space

(i) Hilbert space is a linear vector space.

(ii) It has an inner product operation that satisfies certain condition. Inner product of vectors ψ_1 and ψ_2:

$$\langle\psi_1, \psi_2\rangle \in \mathbb{C}$$

$\mathbb{C}$ is a set of complex numbers.

(a) conjugate symmetry: $\langle\psi_1, \psi_2\rangle = \langle\psi_2, \psi_1\rangle^*$
(b) linear with respect to second vector $\langle\psi_1, a\psi_2 + b\psi_3\rangle = a\langle\psi_1, \psi_2\rangle + b\langle\psi_1, \psi_3\rangle$
(c) anti-linear with respect to first vector $\langle a\psi_1 + b\psi_2, \psi_3\rangle = a^*\langle\psi_1, \psi_2\rangle^* + b^*\langle\psi_2, \psi_3\rangle^*$
(d) $\langle\psi, \psi\rangle = |\psi|^2$, positive definite.
(e) $|\psi_2 - \psi_1| = \sqrt{\langle\psi_2 - \psi_1, \psi_2 - \psi_1\rangle} = d$, d is the distance between two vectors in Hilbert space.

(iii) Hilbert spaces are separable, so they contain a countable, dense, subset.

(iv) Hilbert spaces are complete (no gaps)

Cauchy sequence $\{\psi_i\}$:

$$\lim_{m,n\to\infty} |\psi_m - \psi_n| = 0$$

$$\lim_{n\to\infty} |\phi - \psi_n| = 0$$

ϕ is the element of Hilbert space. That is Hilbert space is a separable real or complex inner product space (IPS) such that it is a Banach space with the norm induced by the inner product.

IV. TYPES OF HILBERT SPACE

(a) Finite dimensional Hilbert space, e.g., $\mathbb{R}^n, \mathbb{C}^n$ are real and complex numbers with n basis vectors. Inner product of $\mathbb{R}^n$: typical dot product.

$$X_1 = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad X_2 = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad \text{then} \quad X_1^T \cdot X_2 = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \tag{1}$$

Inner product of $\mathbb{C}^n$: complex inner product.

$$Z_1 = \begin{bmatrix} a_1 + ib_1 \\ a_2 + ib_2 \\ \vdots \\ a_n + ib_n \end{bmatrix}, \quad Z_2 = \begin{bmatrix} c_1 + id_1 \\ c_2 + id_2 \\ \vdots \\ c_n + id_n \end{bmatrix}, i^2 = -1$$

Inner product $(Z_1^*)^T \cdot Z_2 = [a_1 - ib_1, \cdots] \begin{bmatrix} c_1 + id_1 \\ \vdots \end{bmatrix}$ (2)

(b) Infinite dimensional Hilbert spaces: Vector space of complex valued functions with inner product:

$$\langle\psi, \phi\rangle = \int_{-\infty}^{\infty} \psi^* \phi \, dx,$$

need square integrable function because $\langle\psi, \phi\rangle$ will be finite for ψ, ϕ, square integrable. As $\langle e^x, e^{2x}\rangle = \int_{-\infty}^{\infty} (e^x)^* \, e^{2x} \, dx \to \infty$ (inner product does not exist!)

Square –integrable functions:

$$\int_{-\infty}^{\infty} |\psi|^2 \, dx \to \text{finite}$$

$$\int_{-\infty}^{\infty} |\psi|^2 \, dx = 1 \to \text{normalized.}$$

V. HILBERT SPACE IN QUANTUM MECHANICS

In case of quantum mechanics, all operators related to observables are matrix (linear transformation) in Hilbert space and all states are just vectors in that space. Functions can also satisfy the vector conditions. That's why in some places, function space is used in place of vector space. The simplest example of Hilbert space in C^n ($n \to \infty$) with the inner product:

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i^* y_i \tag{3}$$

where $x = (x_1, x_2, x_3, \ldots\ldots, x_n)$ and $y = (y_1, y_2, y_3, \ldots\ldots, y_n)$. Any vector in a linear vector space (LVS) is represented by a Ket vector $|\psi\rangle$. If the space is n dimensional, then in simply $|\psi\rangle$ as a column vector,

$$|\psi\rangle = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{bmatrix}$$

Here $\psi_1, \psi_2, \ldots$ are the strength of the basis. Every linear vector space has a dual space (L) that is also a linear vector space. The notation used for the element of dual space is $\langle\psi|$ which is called bra vector. In case of n dimension $\langle\psi|$ can be represented by a row vector.

$$\langle\psi| = [\psi_1' \quad \psi_2' \quad \ldots \quad \psi_n']$$

The dual of the dual space $\bar{L}$ is again a dual space (L) i.e., $\bar{\bar{L}} = L$.

VA. RESULTS AND DISCUSSION

The modern quantum mechanics start with the famous Schrodinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi = i\hbar\frac{\partial^2\Psi}{\partial t^2} \tag{4}$$

We start by solving the 1-dimentional square-well potential problem. The solution of Schrodinger equation in this case has an orthonormal sequence $\left\{sin\left(\frac{n\pi}{a}x\right)\right\}$ and the general state is given by

$$\psi(x) = \sum_{n=1}^{\infty} A_n sin\left(\frac{n\pi}{a}x\right) \tag{5}$$

Using the property of complete orthonormal sequence if we consider $sin\left(\frac{n\pi}{a}x\right)$ as basis then it represents a Hilbert space as it follows the all properties of Hilbert space. Here, for normalized $\psi(x)$

$$\int_{-\infty}^{\infty}|\psi|^2\,dx = \int_{-\infty}^{\infty}\psi^*\psi dx \tag{6}$$

To represent a particle by $\psi(x,t)$ one condition is $\psi(x) \to 0$ at $\to \pm\infty$. As $|\psi|^2$represents probability density so the value of $|\psi|^2$is maximum near the particle which is represented by the wave function. The sum over all the states with appropriate constant gives the complete states and in 1-dimensional case

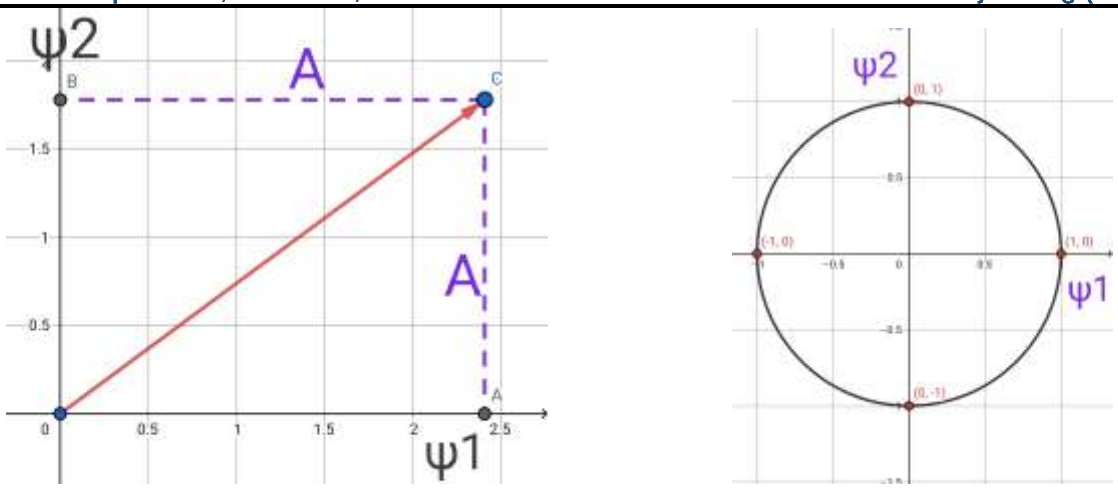
$$\int_{-\infty}^{\infty}\psi^*\psi\,dx = 1.$$

There are two main formulations of quantum mechanics: (1) wave mechanics and (2) matrix mechanics. In wave mechanics functions are represented as states of particle. They can be treated as the member of linear vector space. In matrix mechanics states are represented as row or ket vector. The elements or components of the vectors can be function or some constants. There are other formulations of quantum mechanics like path integral formulation of Feynman or phase space formulation of Wigner. But here we will just focus on wave mechanics and matrix mechanics formulation.

Considering an example [5] of wave function

$$\psi(x, 0) = A[\psi_1(x) + \psi_2(x)] \tag{7}$$

Now we have to find A.



(a) **(b)**

Figure 1: Coordinate plane whose axis are represented by two wave vectors (eigen state). (a) Corresponding to the discuss problem and (b) normalized condition curve.

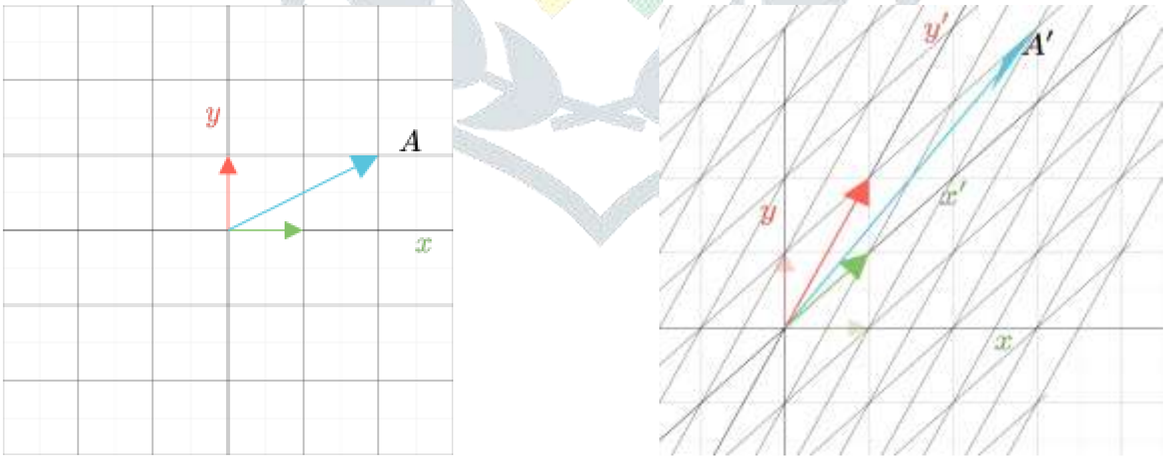
If we draw $\psi_1$and $\psi_2$as axis, then the state $\psi(x,0)$ can be represented by the red vector drawn from the origin in Fig-1a. There as shown A is represented the coefficient of basis, *i.e.*, value of the coordinates of the state point (C). Now, in quantum mechanics, if a state is $\psi(x)$, then $|\psi|^2$ represent probability density and

$$\sum_{n=1}^{\infty} A_n^2 = 1 \qquad (8)$$

This means, if we draw ψ_n's as axis (n-dimension, normally $n \to \infty$) then for any initial condition each possible state will be on the surface of a n-dimensional unit hyper-sphere and each A_n will be always < 1. If we draw $\psi_1$and $\psi_2$as axis then the space which they create is our vector space H and each point of this space represents a state of the system. From eq.(2) we know that each points which are only on the circle (Fig.1b) of radius 1 (one) are only physically valid. As the coefficient of $\psi_1$and $\psi_2$are same so the state point will be on the line with slope equal to 1 and which is also on the unit circle where $A = \frac{1}{\sqrt{2}}$.

OPERATORS

In simple terms, the operators [6] are treated as matrices. Let's take an example, suppose, we have a vector $\vec{A} = \begin{bmatrix}2\\1\end{bmatrix}$. In the plane, It is described as the arrow (Fig.2)



(a) Before transformation **(b) After transformation**

Figure 2: Linear transformation of a vector using matrix.

Now if I apply the matrix $B = \begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}$ on A, it rotates the vector as shown in Fig.2b. In vector notation $\vec{A} = 2i + j$ and after the transformation it will becomes

$$\overrightarrow{A_{new}} = 2i_{new} + j_{new}$$

That is

$$\overrightarrow{A_{new}} = 2\binom{1}{1} + 1\binom{1}{2} = \binom{3}{4} \qquad (9)$$

This can also be seen using matrix multiplication $B \cdot A$. In general, when we apply matrix on a vector it rotates the vector with appropriate stretching and squeezing in Hilbert space as we have seen as similar in quantum mechanics.

VI. CONCLUSIONS

The concept of a Hilbert space appears to be technical and special. By definition vector space with inner product that is complete in the associated metric is known as Hilbert space. As we have seen in this article in view of their special nature, it may therefore come as a surprise that Hilbert spaces play a central role in many areas of mathematics and quantum mechanics. As Hilbert space is linear vector space, complete and no gaps the matrix representation and as well as wave mechanics are greatly span in this space. Matrix is actually a special example of a particular type of operator that lives in the seedy underground of the mathematical representation of quantum mechanics. The operator and spectral theory are both outside the realm of this paper, Sterling Berberian [4], David Cohen [7] and J.R. Retherford [8] all provide approachable discourses on both spectral and operator theory. This topic has a wide range of potential of future works. In future we will extend our work on this topic.

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