Applications of Adomian Decomposition Method for Solving Linear Non-Homogeneous Fredholm Integral Equations of Second Kind

Sudhanshu Aggarwal[1], Anjana Rani Gupta[2], Astha Khandelwal[3]
[1]Assistant Professor, Department of Mathematics, National P.G. College Barhalganj, Gorakhpur-273402, U.P., India
[2]Professor, Department of Mathematics, Noida Institute of Engineering & Technology, Greater Noida-201306, U.P., India
[3]Assistant Professor, Department of Physics, Noida Institute of Engineering & Technology, Greater Noida-201306, U.P., India

Abstract: In this paper, we used Adomian decomposition method for solving linear non-homogeneous Fredholm integral equations of second kind and some applications are given in order to demonstrate the effectiveness of Adomian decomposition method for solving linear non-homogeneous Fredholm integral equations of second kind.

Keywords: Adomian decomposition method, Fredholm integral equation, exact solution, series solution.

I.Introduction:

Linear non-homogeneous Fredholm integral equation of second kind was given by [1-5]

$$v(x) = f(x) + \lambda \int_a^b K(x,t)v(t)dt \;\ldots\ldots\ldots(1)$$

where a and b are constants and the unknown function $v(x)$, that will be determined, occurs inside and outside the integral sign. The kernel $K(x,t)$and the function $f(x)$ are given real-valued functions, and λ is a parameter. The Fredholm integral equations appear in operator theory and it can be derived from boundary value problems with given boundary conditions. Fredholm integral equations generally appear in many physical models such as stereology, cosmic radiation, electromagnetic fields, radiography spectroscopy and image processing etc. Fredholm integral equations also arise in the theory of signal processing. Many inverse problems in science and engineering lead to the Fredholm integral equations.

In this paper, we used Adomian decomposition method (decomposition method) for solving linear non-homogeneous Fredholm integral equations of second kind because this method provides the solution in a rapidly convergent series with components that are elegantly computed. Adomian [6-7] gave non-linear stochastic systems theory and it's applications in the physics. A review of the decomposition method and some recent result for non linear equation was given by Adomian [8]. Cherruault and Saccomandi [9] gave the new results for convergence of Adomian method applied to integral equations. Adomian [10] solved the Frontier problems of Physics by using the decomposition method. After it, he solved the physical problems by decomposition method [11]. Abbaoui and Cherruault [12] discussed the convergence of Adomian's method and applied it for solving differential equations. The resolution of non linear integral equations of the first kind using the Adomian decomposition method was discussed by Cherruault and Seng [13]. Babolian et al. [14] used Adomian decomposition method for solving a system of non linear equations. Babolian and Biazar [15] also discussed the solution of a system of linear Volterra equations by Adomian decomposition method. Bakodah [16] gave the some modifications in Adomian decomposition method applied to non linear system of Fredholm integral equations of the second kind. The aim of this work is to establish exact solutions for linear non-homogeneous Fredholm integral equations of second kind using Adomian decomposition method without large computational work.

II. Solution of linear non-homogeneous Fredholm integral equations of second kind:

A solution of the linear non-homogeneous Fredholm integral equations of second kind arises in any of the following two types:

a) **Exact solution:** A solution of linear non-homogeneous Fredholm integral equations of second kind is called exact if it has a closed form such as a polynomial, trigonometric function, exponential function or the combination of two or more of these elementary functions, e.g.

$$v(x) = 2x + e^{3x},$$
$$v(x) = cosx - sinx,$$
$v(x) = 7 + sin3x + cos5x - 2e^{-x}$ and many others.

b) **Series solution:** Sometimes we can not obtain exact solution for concrete problems. In this case, we have solution in the series form that may converge to exact solution if such a solution exists. Otherwise series may not give exact solution and in this case we have approximate solution for numerical purpose. The more terms in the series give the higher accuracy in the solution.

III. Adomian decomposition method for solving linear non-homogeneous Fredholm integral equations of second kind:

The Adomian decomposition method (decomposition method) was first introduced by George Adomian [8]. This method assumes the unknown function $v(x)$ in the form of sum of an infinite number of components i.e. infinite series as

$$v(x) = \sum_{n=0}^{\infty} v_n(x) \dots \dots (2)$$

Or $v(x) = v_0(x) + v_1(x) + v_2(x) + \cdots \dots$ (3)

where all the components $v_n(x), n \geq 0$ are determined by using recursive relation.

To establish the recursive relation, we substitute (2) into (1), we get

$$\sum_{n=0}^{\infty} v_n(x) = f(x) + \lambda \int_a^b K(x,t) \sum_{n=0}^{\infty} v_n(t)\, dt \dots (4)$$

In expending form (4) can be written as

$$\begin{aligned} v_0(x) + v_1(x) + v_2(x) + v_3(x) \dots \\ = f(x) + \lambda \int_a^b K(x,t) v_0(t) dt \\ + \lambda \int_a^b K(x,t) v_1(t) dt \\ + \lambda \int_a^b K(x,t) v_2(t) dt + \cdots (5) \end{aligned}$$

Where the zeroth component $v_0(x)$ is identified by $f(x)$ and the remaining components of R.H.S. of (2) are given by the following recursive relation:

$$v_{n+1}(x) = \lambda \int_a^b K(x,t) v_n(t) dt\, , n \geq 0 \dots \dots (6)$$

After simplification(6), we have

$$\left.\begin{aligned} v_0(x) &= f(x) \\ v_1(x) &= \lambda \textstyle\int_a^b K(x,t) v_0(t) dt \\ v_2(x) &= \lambda \textstyle\int_a^b K(x,t) v_1(t) dt \\ v_3(x) &= \lambda \textstyle\int_a^b K(x,t) v_2(t) dt \end{aligned}\right\} \dots \dots (7)$$

and so on for other components.

Using (7) all the components of R.H.S. of (2) are obtained and now we get the required solution of (1) using(2), in series form which on simplification give the exact solution of (1).

III. Applications:

In this section, some applications are given in order to demonstrate the effectiveness of Adomian decomposition method for solving linear non-homogeneous Fredholm integral equations of second kind.

Application:1 Consider linear non-homogeneous Fredholm integral equations of second kind with $f(x) = e^x$, $\lambda = \frac{1}{e}, K(x,t) = 1, a = 0, b = 1$

$$v(x) = e^x + \frac{1}{e} \int_0^1 v(t)\, dt \dots (8)$$

The Adomian decomposition method assumes the unknown function $v(x)$ in the form of an infinite series as

$$v(x) = \sum_{n=0}^{\infty} v_n(x) \dots \dots (9)$$

By substituting (9) in (8), we have

$$\sum_{n=0}^{\infty} v_n(x) = e^x + \frac{1}{e} \int_0^1 \sum_{n=0}^{\infty} v_n(t)\, dt \dots \dots (10)$$

From(10), our required recursive relation is given by

$$v_0(x) = e^x \dots \dots (11)$$

$$v_{n+1}(x) = \frac{1}{e} \int_0^1 v_n(t)\, dt, n \geq 0 \dots \dots (12)$$

The first few components of R.H.S. of (9) by using (11) and (12) are given by

$$v_1(x) = \frac{1}{e} \int_0^1 v_0(t) dt$$

$$= \frac{1}{e} \int_0^1 e^t dt = \frac{(e-1)}{e} \dots (13)$$

$$v_2(x) = \frac{1}{e} \int_0^1 v_1(t) dt$$

$$= \frac{(e-1)}{e^2} \int_0^1 dt = \frac{(e-1)}{e^2} \dots (14)$$

$$v_3(x) = \frac{1}{e} \int_0^1 v_2(t) dt$$

$$= \frac{(e-1)}{e^3} \int_0^1 dt = \frac{(e-1)}{e^3} \dots (15)$$

and so on.

Now using (9), the series solution of (8) is given by

$$v(x) = v_0(x) + v_1(x) + v_2(x) + v_3(x) \dots \dots$$

$$= e^x + \frac{(e-1)}{e} + \frac{(e-1)}{e^2} + \frac{(e-1)}{e^3} + \cdots \dots \dots$$

$$= e^x + \frac{(e-1)}{e}\left[1 + \frac{1}{e} + \frac{1}{e^2} + \cdots \dots \dots\right]$$

$$= e^x + \frac{(e-1)}{e}\left[\frac{1}{1-\frac{1}{e}}\right] = e^x + 1 \dots \dots (16)$$

Which is the exact solution of (8).

Application:2 Consider linear non-homogeneous Fredholm integral equations of second kind with $f(x) = e^x - x,\ \lambda = 1, K(x,t) = xt, a = 0, b = 1$

$$v(x) = e^x - x + \int_0^1 xtv(t)\,dt \ldots..(17)$$

The Adomian decomposition method assumes the unknown function $v(x)$ in the form of an infinite series as

$$v(x) = \sum_{n=0}^{\infty} v_n(x) \ldots\ldots.(18)$$

By substituting (18) in(17), we have

$$\sum_{n=0}^{\infty} v_n(x) = e^x - x + x\int_0^1 t\sum_{n=0}^{\infty} v_n(t)\,dt \ldots\ldots(19)$$

From(19), our required recursive relation is given by

$$v_0(x) = e^x - x \ldots\ldots.(20)$$

$$v_{n+1}(x) = x\int_0^1 tv_n(t)\,dt, n \geq 0 \ldots\ldots(21)$$

The first few components of R.H.S. of (18) by using (20) and (21) are given by

$$v_1(x) = x\int_0^1 tv_0(t)dt$$

$$= x\int_0^1 t(e^t - t)dt = \frac{2x}{3} \ldots..(22)$$

$$v_2(x) = x\int_0^1 tv_1(t)dt$$

$$= x\int_0^1 \frac{2t^2}{3}dt = \frac{2x}{9} \ldots..(23)$$

$$v_3(x) = x\int_0^1 tv_2(t)dt$$

$$= x\int_0^1 \frac{2t^2}{9}dt = \frac{2x}{27} \ldots..(24)$$

and so on.

Now using (18), the series solution of (17) is given by

$$v(x) = v_0(x) + v_1(x) + v_2(x) + v_3(x) \ldots\ldots$$

$$= e^x - x + \frac{2x}{3} + \frac{2x}{9} + \frac{2x}{27} + \cdots \ldots\ldots$$

$$= e^x - x + \frac{2x}{3}\left[1 + \frac{1}{3} + \frac{1}{9} + \cdots \ldots\ldots\right]$$

$$= e^x - x + \frac{2x}{3}\left[\frac{1}{1 - \frac{1}{3}}\right] = e^x \ldots\ldots.(25)$$

Which is the exact solution of (17).

IV. Conclusion:

In this paper, we have successfully developed the Adomian decomposition method for solving linear non-homogeneous Fredholm integral equations of second kind.The given applications showed that the exact solution have been obtained using very less computational work and spending a very little time. The proposed scheme can be applied for other non-homogeneous Fredholm integral equations of second kind and their system.

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Sudhanshu Aggarwal received his M.Sc. degree from M.S. College, Saharanpur in 2007. He has also qualified CSIR NET examination (June-2010, June-2012, June-2013, June-2014 & June-2015) in Mathematical Sciences. He is working as an Assistant Professor in National P.G. College Barhalganj Gorakhpur. He is equipped with an extraordinary caliber and appreciable academic potency. He has around ten years of teaching experience at various engineering colleges affiliated to AKTU. His fields of interest include Integral Transform Methods, Differential and Partial Differential Equations, Integral Equations and Number Theory.

Dr. Anjana Rani Gupta is working as Professor and Head at NIET. She has completed her Ph.D degree from IIT Roorkee. She has more than 20 years of teaching experience at IIT Roorkee and various Engineering Colleges affiliated to UPTU. She has published many research papers in reputed journals and also a member of different Mathematical Societies.

Astha Khandelwal received her M.Sc. degree from Banasthali University Rajasthan in 2015. She is working as an Assistant Professor in Noida Institute of Engineering & Technology, Greater Noida since 2016. She has also qualified GATE exam (2016), JEST (2016). Her fields of interest include Quantum mechanics, Electromagnetic field theory.