Mathematical modeling of pollutant transport in conservative and non conservative form of advection equation

[1]R.V.Waghmare, [2]S.B.Kiwne
[1]Head and Assistant Professor, [2]Head and Associate Professor
[1]Department of Mathematics,
[1]Shivaji Arts, Commerce and Science College Kannad, Dist. Aurangabad
[2] Department of Mathematics,
[2]Deogiri College, Aurangabad, Dist. Aurangabad (M.S.), India.

Abstract: This paper deals with the one-dimensional advection-dispersion equation for a nonreactive and reactive solute. The general equations are derived by applying the principle of mass balance to a small rectangular element. This paper outlines the basis for some of the simpler mathematical models of solute transport. Also we discussed mechanical dispersion due to velocity variations and molecular diffusion. Analytical solutions are provided for the one dimensional advection equation for conservative and non- conservative forms.

1. INTRODUCTION

The transport of pollutant occurs in a large variety of environmental, agricultural and industrial processes. Accurate prediction of the transport of these pollutants is crucial to the effective management of these processes. The transport of these pollutants can be adequately described by the advection-diffusion equation.

The advection-diffusion equation arises in a number of physical problems in engineering including migration of contaminants in a stream, smoke plume in atmosphere, dispersion of chemicals in reactors, tracer dispersion in a porous medium, etc.

The mass transport is the transport of solute in a solvent. The solute is dissolve and the solvent is the dissolver. Generally, the liquids are classified as solvent because thy plays an important role in industry. In environmental applications, these solvents are solutes and water and air is usually the solvents. Also advection-diffusion arises in a number of biological transport problems in which a bulk fluid like water transports a solute or even a drug with concentration. The substance being transported can be either dissolved or particulate substances.

In this paper, we developed the one-dimensional advection-dispersion equation for a nonreactive and reactive solute. The general equations are derived by applying the principle of mass balance to a small rectangular element. This paper outlines the basis for some of the simpler mathematical models of solute transport. Also we discussed mechanical dispersion due to velocity variations and molecular diffusion. And one dimensional transport of a pollutant in a channel with steady, unpolluted lateral inflow uniformly distributed over its whole length in one such problem that is considered here. This problem can be described approximately by spatially variable coefficient advection equations. Analytical techniques for the solution of the advection equations are generally restricted to simple problems with constant coefficients. There are very few analytical solutions to the one-dimensional advection equations with variable velocity. Analytical solutions have been provided by Barry and Sposito (1989) and by Basha and EI-Habel (1993) for time-dependent coefficients and by Philip (1994) for variable diffusion coefficients. Many of these analytical solutions are complicated to implement, restricted to specific initial and boundary value problems, or have limited practical relevance.

Analytical solutions will be derived for the conservative and non conservative forms of the advection equations with a particular form of spatially variable coefficients. The fluid velocity is taken to be a linear function of distance. The particular forms of the spatially variable coefficients considered here are consistent with the problem of the transport of pollutant in an open channel where the flow in the channel is augmented by steady, unpolluted lateral inflow distributed along the whole length of the channel, such as a steady inflow of ground water. Therefore the analytical solutions are solutions to a practical problem. The simple expressions considered here for the spatial variation of the coefficients facilitate the process of obtaining analytical solutions to these equations. The spatially variable coefficient equations reduce to constant coefficient equations through a simple transformation. Consequently, many of the known analytical solutions to the constant coefficient equations can be used to obtain analytical solutions to the spatially variable coefficient equations. Analytical solutions are provided for the advection of a sudden release of pollutant into the channel. The advection of an initial quasi-gaussian concentration profile in the channel is also considered. The analytical solutions are simple to evaluate and are useful for validating numerical schemes for solving the advection equation with spatially variable coefficients written in either conservative or non-conservative form (Zoppou and Knight 1994). The conservative and non-conservative forms of the equations are valid equations describing different physical problems. The analytical solutions to the conservative and non-conservative forms of the governing equations will be used to illustrate the importance of selecting the equation relevant to the physical problem, when spatially variable coefficients are involved.

2. GOVERNING EQUATIONS FOR NONREACTIVE SOLUTES

The general equations for solute transport will be derived for one-dimensional transport of a non-sorbing, non-reactive solute. Once that equation is established, it will be extended to other, more general cases [4].

The general equations are derived by applying the principle of mass balance to a small rectangular element as shown in Figure 1.1. The flow and dispersive fluxes are limited to the x direction only. Mass balance for the cube means that the total flux of solute mass through the boundary of the cube equals the time rate of change of solute mass stored inside the cube.

First, we will examine the mass flux through the boundaries of the cube. Two types of mass flux are considered: advective flux given by $F_{ax} = q_x c$. Where F_{ax} is the advective flux of solute in the x direction and c is solute concentration and macro-dispersive flux given by $F_{mx} = -nD_{mx}\frac{\partial c}{\partial x}$. Adding these two types of fluxes gives the mass flux (mass/time) in through the left side of the cube (at x) in the positive x direction, Thus

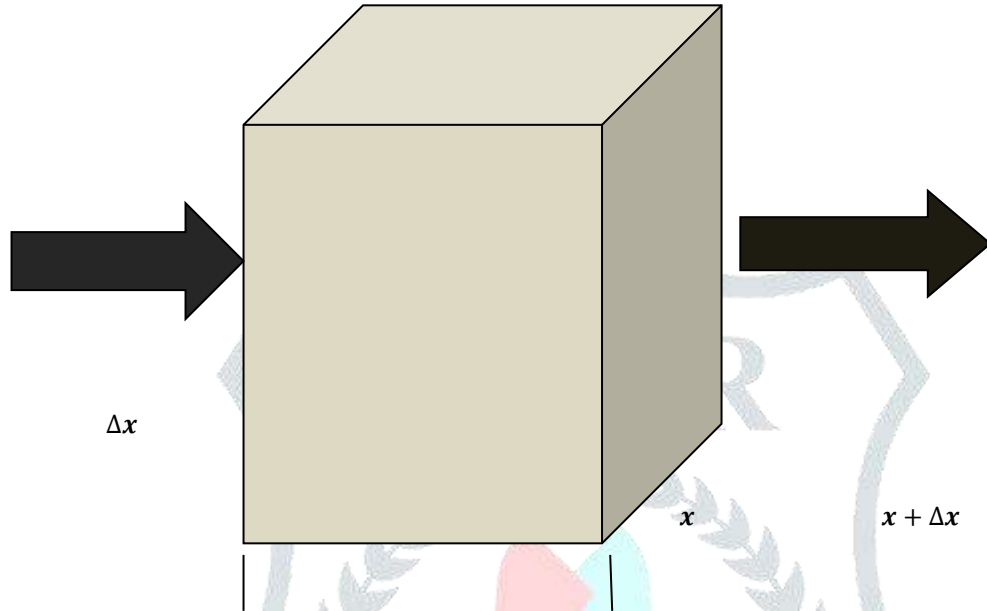Δx

x $x + \Delta x$

Figure 1.1 Element with dimensions $\Delta x \times \Delta y \times \Delta z$ for mass balance.

$$\text{The mass flux in} = \left[q_x c(x) - nD_{mx}\frac{\partial c}{\partial x}(x)\right]\Delta y \Delta z \tag{2.1}$$

And the mass flux out through the right side of the cube (at $x + \Delta x$) in the positive x direction. Thus

$$\text{The mass flux out} = \left[q_x c(x + \Delta x) - nD_{mx}\frac{\partial c}{\partial x}(x + \Delta x)\right]\Delta y \Delta z \tag{2.2}$$

The rate of change in the mass of solute stored inside the cube is

$$\frac{\partial (cn)}{\partial t}\Delta x \Delta y \Delta z \tag{2.3}$$

Where n is porosity. For mass balance, the flux in the left side minus the flux out the right side equals the rate of change in the mass stored within. Combining the previous three equations in this way and dividing through by $\Delta x \Delta y \Delta z$ gives

$$\frac{\left[nD_{mx}\frac{\partial c}{\partial x}(x+\Delta x) - nD_{mx}\frac{\partial c}{\partial x}(x) - q_x c(x+\Delta x) + q_x c(x)\right]}{\Delta x} = \frac{\partial (cn)}{\partial t} \tag{2.4}$$

In the limit as $\Delta x \to 0$, the left-hand side results in derivatives with respect to x as follows:

$$\frac{\partial}{\partial x}\left(nD_{mx}\frac{\partial c}{\partial x}\right) - \frac{\partial}{\partial x}(q_x c) = \frac{\partial (cn)}{\partial t} \tag{2.5}$$

Substituting $n\bar{u}_x = q_x$ in this equation and assuming that D_{mx} is independent of x, and that n is independent of x and t, it simplifies to

$$D_{mx}\frac{\partial^2 c}{\partial x^2} - \frac{\partial}{\partial x}(\bar{u}_x c) = \frac{\partial c}{\partial t} \tag{2.6}$$

This is the standard one-dimensional advection-dispersion equation for a nonreactive solute.

Allowing fluxes in all three dimensions would be result in a similar set of terms in each direction, giving the three-dimensional advection-dispersion equation for a nonreactive solute:

$$D_{mx}\frac{\partial^2 c}{\partial x^2} - \frac{\partial}{\partial x}(\bar{u}_x c) + D_{my}\frac{\partial^2 c}{\partial y^2} - \frac{\partial}{\partial x}(\bar{u}_y c) + D_{mz}\frac{\partial^2 c}{\partial z^2} - \frac{\partial}{\partial x}(\bar{u}_z c) = \frac{\partial c}{\partial t} \tag{2.7}$$

This can be written in a more compact form, making use of the gradient operator ∇, and writing the vector (D_{mx}, D_{my}, D_{mz}) as D_{mi}, and the vector $(\bar{u}_x, \bar{u}_y, \bar{u}_z)$ as $\bar{u}_i$:

$$D_{mi}\nabla^2 c - \nabla(\bar{u}_i c) = \frac{\partial c}{\partial t} \tag{2.8}$$

This is the general advection-dispersion equation for three dimensions, assuming that the porosity and macro-dispersivities are constant in space and time. A few variations on this equation will now be presented.

If the flow field is steady state, $\nabla \bar{u}_i = 0$ and $\bar{u}_i$ can be removed from the gradient operator in Eq. 10.23 as follows:

$$D_{mi}\nabla^2 c - \bar{u}_i \nabla c = \frac{\partial c}{\partial t} \quad \text{(steady flow)} \tag{2.9}$$

If, in addition to steady flow, the concentration field is steady state ($\partial c/\partial t = 0$), then Eq. 10.23 becomes

$$D_{mi}\nabla^2 c - \bar{u}_i \nabla c = 0 \quad \text{(steady flow and concentration)} \tag{2.10}$$

If there is no flow at all, $\bar{v} = 0$ and Eq. 10.23 becomes the diffusion equation

$$D_{mi}\nabla^2 c = \frac{\partial c}{\partial t} \quad \text{(no flow, diffusion only)} \tag{2.11}$$

Where in this case the coefficients $D_{mi} = T_i^* D$ (See in Eq. 10.14)

3. GOVERNING EQUATIONS FOR REACTIVE SOLUTES

There are many variations of the advection-dispersion equation for cases where the solutes react and/ or sorb to solid surfaces as they flow. A useful equation models a solute that decays at a constant rate. This would be appropriate for radioactive solutes, and sometimes for solutes that biodegrade. The assumption is that the decay is governed by the simple first-order rate law [4, 8].

$$\frac{dc}{dt} = -\lambda c \tag{3.1}$$

Where λ is a decay constant with dimensions of [1/T]. The half-life of the solute is linearly related to λ. When loss due to decay is added to Eq. 2.8, it becomes

$$D_{mi}\nabla^2 c - \nabla(\bar{u}_i c) - \lambda c = \frac{\partial c}{\partial t} \quad \text{(Decay)} \tag{3.2}$$

Another facet of reaction that can be easily modeled is linear sorption that is always at equilibrium. A bit of derivation is needed to get to the final governing equation. When a solute reacts with the porous media and sorbs, there is another mass flux that needs to be added to the solute transport equation. Adding $\frac{\partial c^*}{\partial t}$ (extra term) in Eq. 3.2, we get

$$D_{mi}\nabla^2 c - \nabla(\bar{u}_i c) - \lambda c - \frac{\partial c^*}{\partial t} = \frac{\partial c}{\partial t} \tag{3.3}$$

Where c^* is the sorbed mass per volume of pore water. As c^* increases, c decreases and vice versa, since the mass is being transferred from one phase to other. If the sorption is assumed to be linear and at equilibrium, the sorbed concentration and aqueous concentration are related by

$$cK_d = \frac{\text{sorbed mass}}{\text{mass aquifer solids}} \tag{3.4}$$

Where c is the aqueous solute concentration, and K_d is the sorption distribution coefficient. the right side of Eq. 3.4 can be written in terms of c^*, porosity and n, and dry bulk density ρ_b as follows:

$$cK_d = \left(\frac{\text{sorbed mass}}{\text{volume pore water}}\right)\left(\frac{\text{volume pore water}}{\text{total volume}}\right)\left(\frac{\text{total volume}}{\text{mass aquifer solids}}\right)$$

$$= (c^*)(n)\left(\frac{1}{\rho_b}\right) \tag{3.5}$$

Rearranging the previous equation gives c^* in terms of c and known constants:

$$c^* = c\left(\frac{\rho_b K_d}{n}\right) \tag{3.6}$$

Now, substituting Eq. 3.6 into 3.3 and combining the two terms with $\partial c/\partial t$, we get

$$D_{mi}\nabla^2 c - \nabla(\bar{u}_i c) - \lambda c = \frac{\partial c}{\partial t}\left(1 + \frac{\rho_b K_d}{n}\right) \tag{3.7}$$

The terms in parentheses on the right side is the retardation factor R.

$$\frac{1}{R}[D_{mi}\nabla^2 c - \nabla(\bar{u}_i c) - \lambda c] = \frac{\partial c}{\partial t} \quad \text{(sorption and decay)} \tag{3.8}$$

As discussed earlier, both advection and dispersion are slowed by $1/R$.

4. MODELING OF DISPERSION AND DIFFUSION

As solutes migrate in groundwater they tend to mix and disperse as a result of two fundamental processes one is mechanical dispersion due to velocity variations and molecular diffusion.

Solute advection is the movement of dissolved substance because the water they are in is moving. The mass flux of solute due to advection alone is

$$F_{ax} = q_x c \tag{4.1}$$

Where F_{ax} is the advective flux of solute in the x direction and c is solute concentration.

Diffusive mass flux for solutes in a saturated porous medium is governed by form of Fick's first law, which for x direction would be written as

$$F_{dx} = -nT_x^\star D \frac{\partial c}{\partial x} \tag{4.2}$$

Where F_{dx} diffusive mass is flux of solute in the x direction, n is porosity of the medium, $T_x^\star$ is the tortuosity of the liquid phase in the x direction, D is constant called the molecular diffusion coefficient, and c is solute concentration. The minus sign in this equation because mass moves toward decreasing concentrations, in the direction opposite that of concentration gradient.

Molecular diffusion is known to be governed by Fick's first law for porous media, and there is no such proven model of mass flux for the case of mechanical dispersion. It depends on the nature of spatial and transient velocity variations, which in turn depend on the nature of the medium's heterogeneity and on transient flow phenomena. Mechanical dispersion depends on too many complex processes to be described by a simple, fundamental law of mass flux. It does, however, result in spreading that is at least qualitatively similar to diffusive spreading. Therefore, mechanical dispersion is modeled as though it, too, is governed by Fick's first law.

The effects of mechanical dispersion plus molecular diffusion are typically lumped together in what is called macro-dispersion. One-dimensional macro-dispersive flux F_{mx} is governed by an equation that is a form of Fick's first law.

$$F_{mx} = -nD_{mx}\frac{\partial c}{\partial x} \tag{4.3}$$

Where n is porosity, c is concentration, and D_{mx} is a model parameter called the macro-dispersion coefficient in the x direction. The macrodispersion coefficient consists of two terms, the first for mechanical dispersion and the second for molecular diffuse on [4]:

$$D_{mx} = \propto_x |\bar{u}| + T_x^* D \qquad\qquad (4.4)$$

Where $\propto_x$ is called the Dispersivity in the x direction, $|\bar{u}|$ is the magnitude of the average linear velocity of flow, T_x^* is the tortuosity in the x direction, and D is the molecular diffusion coefficient. In all but the lowest K materials like massive clays, mechanical dispersion causes far more dispersion than molecular diffusion does, so the diffusive term can be eliminated from Eq. 4.4.

$$D_{mx} = \propto_x |\bar{u}| \qquad \text{(advection-dominated dispersion)} \qquad (4.5)$$

Mechanical dispersion is much greater in the direction of flow, so in two-or three-dimensional transport models, macro-dispersion parameters are usually defined in orthogonal directions, one parallel flow, and the other(s) transverse to flow. For transport models, the coordinate system is often a curved one, aligned with the direction of flow. For example, D_{mx} and $\propto_x$ might apply to the direction of flow, while $D_{my}, \propto_y, D_{mz}$, and $\propto_z$ apply to two orthogonal directions transverse to flow. In aquifers where the flow is predominantly horizontal, the transverse directions are usually chosen in the horizontal and vertical directions normal to flow.

If solute transport is modeled with the correct macro-dispersivity, the simulated dispersion of solute will be similar to actual dispersion of solutes on a "macro" scale. However, at a smaller scale, the actual solute plume will be more irregular or lumpy than the modeled plume. Real heterogeneous aquifers disperse contaminants into irregular, lumpy concentration distributions that follow the pattern of heterogeneities (Fitts, 1996). On the other hand, macro-dispersion models ignore the distribution of small heterogeneities, but simulate their dispersive effects as macro-dispersion. As a result, macro-dispersion models produce smoother concentration distributions than the real thing.

5. MODELING OF CONSERVATIVE AND NON-CONSERVATIVE FORM OF ADVECTION EQUATION

Advection is not only transport mass of pollutant from one place to another, it tends to spread or disperse the mass in the process. This occurs because the distribution of water velocity is not uniform in water pollution problem. The governing equation for conservative form of the advection equation can be written as [11].

$$\frac{\partial c(x,t)}{\partial t} + \frac{\partial [c(x,t)u(x)]}{\partial x} = 0 \qquad 0 < x \le \infty \quad t > 0 \qquad\qquad (5.1)$$

Where $c(x,t)$ = concentration of pollutant; $u(x)$ = one-dimensional fluid velocity field; x = distance; and t =time.

The transformation of (1) into a constant coefficient equation begins with the selection of a suitable function form for the spatial variation of $u(x)$ so that (1) becomes an equi-dimensional linear equation [Hildebrand (1962), [10]]. An equi-dimensional linear equation for (5.1) is [13].

$$\frac{\partial c(x,t)}{\partial t} + \frac{\partial [c(x,t)u_0 x]}{\partial x} = 0 \qquad 0 < x \le \infty \quad t > 0 \qquad\qquad (5.2)$$

The velocity field is simply a linear function of distance, such that $u(x,t) = u_0 x$, where u_0 is a constant.

Eq. (5.2) is further simplified by making the substitution $y = \ln(x)$ [Hildebrand (1962), [10]] so that

$$\frac{\partial [a(y,t)]}{\partial t} + u_0 \frac{\partial [a(y,t)]}{\partial y} = 0 \qquad\qquad (5.3)$$

is a constant coefficient advection equation with $a(y,t) = xc$ as the dependent variable. Known analytical solutions to the constant coefficient advection equation can be used to obtain analytical solutions to the spatially variable coefficient problem. The analytical solution for the advection of a step boundary condition and a quasi-Gaussian initial condition in a spatially variable velocity field will now be derived.

STEP PROFILE

The following initial and boundary condition is imposed on (5.2): $c(x,0) = 0$ and $c(\infty,t) = 0$. A singularity occurs at $x = 0$, therefore the boundary condition will be imposed at $x = x_0$ where $0 < x_0 < \infty, c(x_0,t) = c_0$, and the initial boundary value problem is solved in the domain $x_0 \le x \le \infty$.

Eq. (5.3) is solved using the Laplace transform with respect to time, t. The evolution in time of the step concentration profile is given by [20, 21].

$$c(x,t) = \frac{c_0}{x} H[u_0 t - \ln(x/x_0)] \qquad\qquad (5.4)$$

where H = Heaviside function.

In a variable velocity field, the concentration front is located at $x = x_0 \exp u_0 t$ and the profile decays as $1/x$. This is illustrated in Fig. 1, where $u_0 = 1, x_0 = 1, c_0 = 100$, and $t = 2$. This is what would be expected for this problem. The polluted water in the channel is diluted by the unpolluted lateral inflow entering the channel, and therefore the concentration of pollutant in the channel decreases with distance.

Since this is the conservative form of the advection equation, mass of pollutant should be conserved. The mass entering the computational domain at $x = 1$ is given by $c_0 u_0 t$ for a conservative system. Therefore

$$\int_l^\infty c(x,t) dx = c_0 u_0 t \qquad\qquad (5.5)$$

Substituting (5.4) for $c(x,t)$ into (5.5), then

$$\int_l^\infty c(x,t) dx = \int_l^\infty \frac{c_0}{x} H[u_0 t - \ln(x)] dx$$

Since $H[u_0 t - \ln(x)] = 1$ when $x \le \exp u_0 t$ and 0 otherwise, then

$$\int_l^\infty c(x,t) dx = \int_1^{\exp u_0 t} \frac{c_0}{x} dx = c_0 [\ln(x)]_1^{\exp u_0 t} = c_0 u_0 t$$

and mass is conserved.

The results for the advection of the step profile will now be compared with the evolution of the step profile predicted by the constant coefficient advection equation and the non-conservative advection equation.

For the constant coefficient advection equation given by

$$\frac{\partial c}{\partial t} + u_0 \frac{\partial c}{\partial x} = 0$$

the solution is simply

$$c(x,t) = c_0 H(u_0 t - x + x_0) \qquad (5.6)$$

The step profile is advected without changing shape at the speed u_0 and is located at $x = u_0 t + x_0$. The front for the conservative equation travels much faster, and the concentration at the front is much smaller than is predicted by the constant coefficient advection equation.

Now consider the non-conservative spatially variable coefficient advection equation

$$\frac{\partial c}{\partial t} + u_0 x \frac{\partial c}{\partial x} = 0 \qquad (5.7)$$

The analytical solution to this equation is simply

$$c(x,t) = c_0 H[u_0 t - \ln(x/x_0)] \qquad (5.8)$$

Comparing this equation with (5.4) shows that although the front is also located at $x = x_0 \exp(u_0 t)$, the profile does not decay exponentially. The profile is simply advected without deformation, as shown in Fig. 1. It is obvious that the non-conservative form of the advection equation has not conserved mass. For (5.5), (5.8) becomes

$$\int_l^\infty c(x,t)dx = \int_1^{\exp u_0 t} c_0 dx = c_0 [x]_1^{\exp u_0 t} = c_0 [\exp(u_0 t) - 1] \neq c_0 u_0 t$$

Eq. (5.7) is not conservative because mass increases exponentially with time. It is clear that the non-conservative form of the advection equation implies that the lateral inflow entering the channel is also polluted. This is in contrast to the physical problem being solved. This simple problem illustrates the importance of using the appropriate form of the governing equation for the problem being studied.

QUASI-GAUSSIAN PROFILE

Consider a pollutant with a Quasi-Guassian concentration profile. The conservative advection equation given by (5.2) can be transformed into the constant coefficient advection equation given by (5.3), which has the following analytical solution [21].

$$a(y,t) = a(y - u_0 t, 0) \qquad (5.9)$$

obtained using the method of characteristics, where the initial profile in the $y - t$ plane is translated without deformation by the amount $u_0 t$.

There is no restriction on the initial conditions that can be used. Consider the Gaussian profile

$$a(y,0) = \frac{M_0}{\sigma\sqrt{2\pi}} \exp\left[\frac{-(y-y_0)^2}{2\sigma^2}\right] \qquad (5.10)$$

in the $y - t$, plane, where the peak concentration is located at $y_0 = \ln(x_0)$ and the profile contains a mass equal to M_0.

Recalling that $a(y,t) = xc(x,t), y = \ln(x)$, and using (5.9), then the analytical solution in the $x - t$ plane of (2) for the initial conditions given by (5.10) is

$$c(x,t) = \frac{M_0}{x\sigma\sqrt{2\pi}} \exp\left\{\frac{-[\ln(x/x_0) - u_0 t]^2}{2\sigma^2}\right\} \qquad (5.11)$$

The mass under the profile is given by

$$\int_l^\infty c(x,t)dx = \int_{-\infty}^\infty xc(x,y)\, dy = \frac{M_0}{\sigma\sqrt{2\pi}} \int_{-\infty}^\infty \exp\left[\frac{-(y - u_0 t - y_0)^2}{2\sigma^2}\right] dy = M_0$$

therefore mass is conserved.

The peak concentration that occurs at $x_p = x_0 \exp(u_0 t)$ is given by

$$c(x_p, t) = \frac{M_0}{x_0 \sigma\sqrt{2\pi}} \exp(-u_0 t)$$

Therefore the peak concentration of pollutant decays exponentially with time.

The centroid of the concentration profile is given by

$$\bar{x} = \frac{\int_0^\infty xc(x,t)\, dx}{\int_0^\infty c(x,t)\, dx}$$

where the first moment is given by

$$\int_0^\infty xc(x,t)\, dx = \frac{M_0}{\sigma\sqrt{2\pi}} \int_{-\infty}^\infty e^y \exp\left[\frac{-(y - u_0 t - y_0)^2}{2\sigma^2}\right] dy = M_0 x_0 \exp[u_0 t + \sigma^2/2]$$

Therefore

$$\bar{x} = x_0 \exp\left[u_0 t + \frac{\sigma^2}{2}\right]$$

The peak concentration travels slower than the centroid of the profile and the profile is positively skewed in the $x - t$ domain.

The location of the centroid and peak concentrations coincides for a Gaussian profile. In the example shown earlier, this occurs only in the $y - t$ plane but not in the $x - t$ plane. Therefore the profile has been called a quasi-Gaussian profile.

The analytical solution given by (5.11) with $u_0 = 0.1$, $\sigma = 0.2$, $c_0 = 10$, and $x_0 = 0.2$ is for $t = 0 - 20$ in time increments of 2. As predicted, the profile decays exponentially and the mass is conserved.

For the non-conservative advection equation, (5.7), making the substitution $y = \ln(x)$ produces the following constant coefficient equation:

$$\frac{\partial a(y,t)}{\partial t} + u_0 \frac{\partial a(y,t)}{\partial x} = 0$$

where $a(y,t) = c(x,t)$. This equation has the analytical solution given by (5.9). The initial profile is translated by the amount $u_0 t$. Therefore, for the initial quasi-Gaussian profile given by (10), the analytical solution of the non-conservative advection equation becomes

$$c(x,t) = \frac{M_0}{\sigma\sqrt{2\pi}} \exp\left[\frac{-[\ln(x/x_0)-u_0 t]^2}{2\sigma^2}\right] \hspace{2cm} (5.12)$$

There is no attenuation of the peak, whose concentration is located at $x_p = x_0 \exp(u_0, t)$. This is in contrast with the analytical solution to the same problem using the conservative form of the advection equation where the peak concentration travels at the same speed, but the profile attenuates.

To establish whether the non-conservative advection equation conserves mass of pollutant, the mass under the profile is given by

$$\int_l^\infty c(x,t)dx = \frac{M_0}{\sigma\sqrt{2\pi}} \int_{-\infty}^\infty e^y \exp\left[\frac{-(y-u_0 t-y_0)^2}{2\sigma^2}\right] dy = M_0 x_0 \exp\left[u_0 t + \frac{\sigma^2}{2}\right] \neq M_0$$

Mass of pollutant is not conserved even at $t = 0$, and it increases exponentially with time t.

CONCLUSION

The analytical solution has been derived for the advection of a step boundary condition and a quasi-Gaussian initial condition in spatially variable velocity coefficients. We can simply evaluate and implement the analytical solutions and it can be used to test numerical schemes for solving the advection. It can be extended readily to multidimensional problems. Analytical techniques for the solution of the advection equations are generally restricted to simple problems with constant coefficients.

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