Analysis of Different Parameters of Finite Population Queuing Model by Using Waiting Time Distribution

[1] Damodhar F Shastrakar, [2] Sharad S Pokley,

[1] Professor, Department of Mathematics,, [2]Professor Department of Mathematics
[1] RPCE Nagpur
[2] KITS Ramtek

Abstract : The aim of this paper is to study the main parameters related to waiting time of customer and other related parameters in the system of finite population queuing model. By using binomial distribution the basic parameter waiting time of customer in the queue is studied first and using it to find the other related parameters. By using these parameters a system of finite population queuing model can be analyzed in detail.

Index Terms – Waiting time in the queue, waiting time in the system, Number in the queue, Number in the system, fluctuation of queue length

INTRODUCTION

The model we consider here is having finite capacity. So by using binomial distribution for the arrival rate of customers and service rate of server, waiting time of a customer in the queue is derived first. By using this result various others parameters can be derived and analyzed. Application of binomial distribution gives the time of finite number of arrivals and time required to serve the finite number of customers.For the system of finite population queuing model the parameters to be discuss here to analyze the system are waiting time of customer in the queue and system, number of customers in the queue and in the system, waiting time of a customer in the queue for busy system, number of customers served per busy period, probability of customer has to wait for more than a particular time and fluctuation in queue length of the customers.

Number of customers served per busy period:

Methodology:

Let capacity of system be 'M' and ' α 'be the average arrival rate of customers. ' β ' be the average service rate of customer. Model works under the queue discipline (FCFS).

Utilization factor= server is busy for service is $\rho = \dfrac{\alpha}{\beta}$. ……. .(1)

Probability of 'x' customers arrived in time't' is $P_x(t) = {}^M C_x \left(\dfrac{\alpha t}{M}\right)^x \left(1-\dfrac{\alpha t}{M}\right)^{M-x}, t < \dfrac{M}{\alpha}$ ……… (2)

From this relation it is clear that the time required for the arrival of 'M' number of customers is $o \le t < \dfrac{M}{\alpha}$.

Probability of 'x' customers served by the server in time 't' is

$$P_x(t) = {}^M C_x \left(\frac{\beta t}{M}\right)^x \left(1-\frac{\beta t}{M}\right)^{M-x}, \ t < \frac{M}{\beta} \quad \text{……. (3)}$$

From this relation it is clear that the service time required for 'M' customers is $o \le t < \dfrac{M}{\beta}$.

Probability of $'x'$ customer in the system is

$$P_x = \left(\frac{\alpha}{\beta}\right)^x \left(\frac{1-\dfrac{\alpha}{\beta}}{1-\left(\dfrac{\alpha}{\beta}\right)^{M+1}}\right) = (\rho)^x \left(\frac{1-\rho}{1-\rho^{M+1}}\right), x = 0,1,2,\ldots\ldots M \ , \rho \ne 1, \alpha \ne \beta \quad \text{….…….. (4)}$$

$$= \frac{1}{M+1}, \ \rho = 1, \alpha = \beta \quad \text{………. (5)}$$

Waiting Time Distribution of a customer in the queue:

In steady state condition the waiting time distribution of each customer is same and a continuous random variable. Let $'T'$ be the time required by the server to serve all the customers in the system.

Let $F_T(t)$ be the probability distribution function of $'T'$

Where, $F_T(0) = P_0 = \dfrac{1-\rho}{1-\rho^{M+1}}$ ………. (6)

If a customer is arriving for service and there are already $x \geq 1$ customers present in the system then the arriving customer will get service after the completion of service of all the customers in the system.

$$\therefore F_T(t) = P(T \leq t) = F_T(0) +$$

$$\int_0^t \sum_{x=1}^{M} P_x \left[\Pr ob\{(x-1) customers\, got\, service\, at\, time\, 't'\} \times \Pr ob\{One\, customer\, is\, under\, service\, during\, time\, \omega t\}\right] dt$$

$$= \frac{1-\rho}{1-\rho^{M+1}} + \int_0^t \sum_{x=1}^{M} \rho^x \frac{1-\rho}{1-\rho^{M+1}} \left[{}^M C_{x-1} \left(\frac{\beta t}{M}\right)^{x-1} \left(1-\frac{\beta t}{M}\right)^{M-(x-1)} \right] \left[{}^M C_1 \left(\frac{\beta \omega t}{M}\right) \left(1-\frac{\beta \omega t}{M}\right)^{M-1} \right] dt$$

…………. (From equations (3), (4), & (6))

$$= \frac{1-\rho}{1-\rho^{M+1}} + \int_0^t \sum_{x=1}^{M} \rho^x \frac{1-\rho}{1-\rho^{M+1}} \left[{}^M C_{x-1} \left(\frac{\beta t}{M}\right)^{x-1} \left(1-\frac{\beta t}{M}\right)^{M-(x-1)} \right] M \frac{\beta \omega t}{M} dt$$ ….. (As ωt is very small, neglecting higher powers of ωt and assuming $\omega t = 1$)

By diff w. r. to‘t’ we get probability density function of waiting time distribution

$$F_T'(t) = 0 + \sum_{x=1}^{M} \rho^x \frac{1-\rho}{1-\rho^{M+1}} \left[{}^M C_{x-1} \left(\frac{\beta t}{M}\right)^{x-1} \left(1-\frac{\beta t}{M}\right)^{M-(x-1)} \right] M \frac{\beta}{M}$$

$$= \frac{1-\rho}{1-\rho^{M+1}} \beta \sum_{x=1}^{M} \rho^x \left[{}^M C_{x-1} \left(\frac{\beta t}{M}\right)^{x-1} \left(1-\frac{\beta t}{M}\right)^{M-(x-1)} \right]$$

$$= \frac{1-\rho}{1-\rho^{M+1}} \beta \left[\sum_{x=1}^{M+1} \rho^x \, {}^M C_{x-1} \left(\frac{\beta t}{M}\right)^{x-1} \left(1-\frac{\beta t}{M}\right)^{M-(x-1)} - \rho^{M+1} \, {}^M C_{M+1-1} \left(\frac{\beta t}{M}\right)^{M} \left(1-\frac{\beta t}{M}\right)^{0} \right]$$

$$= \frac{1-\rho}{1-\rho^{M+1}} \beta \left[\rho \, {}^M C_0 \left(\frac{\beta t}{M}\right)^{0} \left(1-\frac{\beta t}{M}\right)^{M} + \rho^2 \, {}^M C_1 \left(\frac{\beta t}{M}\right)^{1} \left(1-\frac{\beta t}{M}\right)^{M-1} + \rho^3 \, {}^M C_2 \left(\frac{\beta t}{M}\right)^{2} \left(1-\frac{\beta t}{M}\right)^{M-2} \right.$$

$$\left. + \ldots\ldots + \rho^{M+1} \, {}^M C_{M+1-1} \left(\frac{\beta t}{M}\right)^{M} \left(1-\frac{\beta t}{M}\right)^{0} - \rho^{M+1} \, {}^M C_{M+1-1} \left(\frac{\beta t}{M}\right)^{M} \left(1-\frac{\beta t}{M}\right)^{0} \right]$$

$$= \frac{1-\rho}{1-\rho^{M+1}} \beta \rho$$

$$\times \left[\left(1-\frac{\beta t}{M}\right)^{M} + M \left(\rho \frac{\beta t}{M}\right) \left(1-\frac{\beta t}{M}\right)^{M-1} + \frac{M(M-1)}{2!} \left(\rho \frac{\beta t}{M}\right)^{2} \left(1-\frac{\beta t}{M}\right)^{M-2} + \ldots + \left(\rho \frac{\beta t}{M}\right)^{M} - \left(\rho \frac{\beta t}{M}\right)^{M} \right]$$

$$= \frac{1-\rho}{1-\rho^{M+1}} \beta \rho \left[\left(1-\frac{\beta t}{M} + \rho \frac{\beta t}{M}\right)^{M} - \left(\rho \frac{\beta t}{M}\right)^{M} \right]$$ ……….. (7)

Now, expected waiting time of a customer waiting in the queue is given by

$$T_q = \int_0^{M/\beta} t \times F_T'(t) dt = \frac{1-\rho}{1-\rho^{M+1}} \beta \rho \int_0^{M/\beta} t \times \left[\left(1-\frac{\beta t}{M} + \rho \frac{\beta t}{M}\right)^{M} - \left(\rho \frac{\beta t}{M}\right)^{M} \right] dt$$

On integrating by part w. r. to ‘t’

$$=\frac{1-\rho}{1-\rho^{M+1}}\beta\,\rho\left[\frac{M}{\beta}\left\{\frac{M\rho^{M+1}}{\beta\,(M+1)(\rho-1)}-\frac{\rho^{M}M}{\beta\,(M+1)}\right\}-\left\{\frac{M^{2}\rho^{M+2}}{\beta^{2}(M+1)(M+2)(\rho-1)^{2}}-\frac{\rho^{M}M^{2}}{\beta^{2}(M+1)(M+2)}\right\}\right.$$

$$\left.+\frac{M^{2}}{\beta^{2}(M+1)(M+2)(\rho-1)^{2}}\right]$$

$$\therefore T_q=\frac{\rho\,M^{2}[M\rho^{M+1}-(M+1)\rho^{M}+1]}{(M+1)(M+2)[1-\rho^{M+1}]\beta\,(1-\rho)} \qquad \ldots\ldots\ldots (8)$$

Waiting time of a customer in the system:
Expected waiting time of a customer in the system is given by

T_s = Waiting time of customer in the queue+ service time of customer

$$=T_q+\frac{1}{\beta} \qquad \ldots\ldots\ldots (9)$$

Number of customer in the system and queue:

Number of customers in the system

$$N_s=\sum_{x=1}^{M}xP_x=\sum_{x=1}^{M}x\,(\rho)^{x}\left(\frac{1-\rho}{1-\rho^{M+1}}\right)=\left(\frac{1-\rho}{1-\rho^{M+1}}\right)\sum_{x=1}^{M}x\,(\rho)^{x} \quad \text{(using (4))}$$

$$=\frac{1-\rho}{1-\rho^{M+1}}\left[\rho+2\rho^{2}+3\rho^{3}+\ldots..+M\rho^{M}\right]$$

$$=\frac{1-\rho}{1-\rho^{M+1}}\left[\frac{\rho\,(1-\rho^{M})}{(1-\rho)^{2}}-\frac{M\rho^{M+1}}{1-\rho}\right]=\frac{1}{1-\rho^{M+1}}\left[\frac{\rho\,(1-\rho^{M})}{1-\rho}-M\rho^{M+1}\right] \qquad \ldots\ldots\ldots (10)$$

And number of customers in the queue

$$N_q=\sum_{x=1}^{M}(x-1)P_x=\sum_{x=1}^{M}xP_x-\sum_{x=1}^{M}P_x=N_s-\left(\sum_{x=0}^{M}P_x-P_0\right)=N_s-(1-P_0) \quad \text{(using (4) and (6))}$$

$$=N_s-\left(1-\frac{1-\rho}{1-\rho^{M+1}}\right) \qquad \ldots\ldots\ldots (11)$$

Waiting time of a customer in the queue for busy system:
Expected waiting time of a customer in the queue for busy system is given by

$$T_b=\frac{T_q}{\Pr ob.\ of\ system\,being\,busy}=\frac{\rho\,M^{2}[M\rho^{M+1}-(M+1)\rho^{M}+1]}{(M+1)(M+2)[1-\rho^{M+1}]\beta\,(1-\rho)}\times\frac{1}{1-P_0}$$

$$=\frac{\rho\,M^{2}[M\rho^{M+1}-(M+1)\rho^{M}+1]}{(M+1)(M+2)[1-\rho^{M+1}]\beta\,(1-\rho)}\times\frac{1}{1-\frac{1-\rho}{1-\rho^{M+1}}}=\frac{M^{2}[M\rho^{M+1}-(M+1)\rho^{M}+1]}{(M+1)(M+2)\beta\,(1-\rho)\,(1-\rho^{M})} \qquad \ldots\ldots\ldots (12)$$

Number of customers served per busy period:

Expected number of a customer served per busy period is given by

$$N_b = \frac{N_s}{\text{Prob. of system being busy}} = \frac{1}{1-\rho^{M+1}}\left[\frac{\rho\left(1-\rho^M\right)}{1-\rho} - M\rho^{M+1}\right]\times\frac{1}{1-P_0}$$

$$= \frac{1}{1-\rho} - \frac{M\rho^{M+1}}{\rho\,(1-\rho^M)} \qquad \ldots\ldots\ldots\ldots (13)$$

Probability of a customer to wait in the queue for particular time:

Probability of a customer has to wait in the queue for more than time ' a ' minutes where $0 \le a < \frac{M}{\beta}$ is

$$\text{P (Waiting} \ge a) = \int_0^{M/\beta} F_T'(t)dt = \int_0^{M/\beta} \frac{1-\rho}{1-\rho^{M+1}}\beta\,\rho\left[\left(1-\frac{\beta t}{M}+\rho\frac{\beta t}{M}\right)^M - \left(\rho\frac{\beta t}{M}\right)^M\right]dt \quad \text{(using(7))}$$

On integrating w. r. to t we get

$$= \frac{M}{(M+1)(1-\rho^{M+1})}\left[\rho\left(\left(1-\frac{\beta a}{M}(1-\rho)\right)^{M+1} - \rho^M\right) + \left(\frac{\rho\,\beta\,a}{M}\right)^{M+1}(1-\rho)\right] \qquad \ldots\ldots\ldots (14)$$

Fluctuation in a queue length:

Fluctuation (Variance) of queue length for customer is

$$Var(x) = E(x^2) - [E(x)]^2 = \sum_{x=1}^{M} x^2 P_x - \left[\sum_{x=1}^{M} xP_x\right]^2 = \sum_{x=1}^{M} x^2 P_x - [N_s]^2$$

$$= \sum_{x=1}^{M} x^2(\rho)^x\left(\frac{1-\rho}{1-\rho^{M+1}}\right) - [N_s]^2$$

$$= \frac{\rho}{(1-\rho^{M+1})(1-\rho)^2}\left[1+\rho-(M+1)^2\rho^M + (2M^2+2M-1)\rho^{M+1} - M^2\rho^{M+2}\right] - [N_S]^2 \quad \ldots\ldots (15)$$

The above parameters can be analyzed for accepted and rejected customers separately.

CONCLUSION

From the above results we can analyze the different parameters of finite population queuing model. Relation (8) is a basic relation, by using this other parameters from relation (9) to (15) can be calculate and analyze it. The results of these relations analyze the entire queuing model to make some favourable improvement in the model to increase the satisfactory level of customers

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