PRIME – ANTIMAGIC LABELING OF GRAPHS

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Abstract: A Prime graph which admits antimagic labeling is called a Prime - Antimagic labeling. A Prime graph which admits odd antimagic labeling is called a Prime- odd Antimagic labeling. In the present work we investigate Prime - Antimagic labeling of Paths, Caterpillar, Spider Odd Cycles, Complete bipartite graphs, Crown and Prime- odd Antimagic labeling of Paths, Cycles, Complete bipartite graphs, Comb. Also we investigate Strongly Prime - Antimagic labeling of some special graphs of Ladder, Triangular Snake, Quadrilateral Snake, Helm and Gear graphs.

Keywords: Prime - Antimagic labeling, prime-odd antimagic labeling, Strongly prime – antimagic labeling, Path, Caterpillar, Spider, Odd Cycles, Complete bipartite graphs, Combs, Ladder, Triangular Snake, Quadrilateral Snake, Helm, Gear graphs.

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1. Introduction

We begin with simple, finite, undirected and non-trivial graph G = (V, E) with the vertex set V and the edge set E. The number of elements of V, denoted as |V| is called the *order* of the graph G while the number of elements of E, denoted as |E| is called the *size* of the graph G. In the present work C_n denotes the cycle with n vertices and P_n denotes the path of n vertices. We will give brief summary of definitions which are useful for the present investigation. We follow the notation and terminology of [10]. Different kinds of antimagic graphs were studied by T.Nicholas, S.Somasundaram and V.Vilfred [6]. According to Beineke [1] graph labeling serves as a frontier between number theory and structure of graphs. The notation of prime labeling was introduced by Roger Entringer and was discussed in a paper by Tout [7].

Definition 1.1

If the vertices of the graph are assigned values subject to certain conditions then it is known as *graph labeling*.
S.K.Vaidya and U.M.Prajapati [8] and [9] introduced the *prime labeling* of a graph which is defined as follows.

Definition 1.2

A *Prime labeling* of a graph G is an injective function $f:V(G)\rightarrow\{1,2,\ldots.|V|\}$such that every pair of adjacent vertices u and v, g.c.d(f(u),f(v)) = 1. The graph which admits prime labeling is called *Prime graph*.
N.Hartsfield and G.Ringel [1] introduced the concept of *antimagic labeling* and is defined as follows.

Definition 1.3

A connected graph G with vertex set V and edge set E is said to be *antimagic labeling* if there exists a bijection $f:V(G)\rightarrow\{1,2,\ldots|V|\}$ such that the induced mapping $f^*:E(G)\rightarrow N$ defined by $f^*(e=uv) = \sum f(u,v)$ where $(u,v) \in E(G)$ is injective and all these edge labelings are distinct.

Definition 1.4

A walk in a graph is called a *Path* in which both the vertices and edges are distinct.

Definition 1.5

Caterpillar is a tree with all vertices either on a single central path or distance one away from it. The central path may be considered to be the largest path in the caterpillar, so that both end vertices have valency one.

Definition 1.6

A *Spider* $SP(P_{n,2})$ is a caterpillar $S(X_1,X_2,\ldots..X_n)$ where $X_n = 2$ and $X_i = 0$, i = 1,2…n-1.

Definition 1.7

The graph Pn ▯ K_1 is called a *Comb.*

Definition 1.8
The graph $C_n \odot mK_1$ is a *unicyclic graph* with p = q = n (m+1) obtained from the cycle C_n by attaching m – pendent edges at each vertex of the cycle C_n.

Definition 1.9
If m = 1 the graph $C_n \odot$ m K_1 is called a *crown*.

Definition 1.10(Bertrand's Postulate)
For every positive integer n > 1 there is a prime p such that n < p < 2n.

The present work is aimed to discuss some new families of Prime – Antimagic graphs.

2. Results on Prime-Antimagic labeling
In this section we introduce the concept of Prime – Antimagic labeling and we discuss the Prime – Antimagic labeling of some special graphs.

Definition 2.1
A ***Prime-Antimagic labeling*** of a graph G is an bijective function f:V(G)→{1,2,…|V|} such that every pair of adjacent vertices u and v, g.c.d (f(u),f(v)) = 1 and the induced mapping f*:E(G)→N defined by f*(e= *uv*) $=\sum f(u,v)$ where (u,v) ∈ E(G) is injective and all these edge labelings are distinct.

Theorem 2.2
Every Path P_n, n ≥ 2, admits on Prime - Antimagic labeling.
Proof:
Define *f* on V(P_n,) by $f(v_i)$ = i, i= 1,2,…n such that every pair of adjacent vertices u and v, g.c.d $(f(v_i),f(v_{i+1}))$ = 1. The induced function f*: E(G)→N defined by $f(v_iv_{i+1})$ = 2i+1for i = 1,2,…n-1.
All these edge labelings are distinct. Thus P_n has Prime – Antimagic labeling.

Theorem 2.3
The Caterpillar $S(X_1,X_2,\ldots\ldots X_n)$ has Prime-Antimagic labeling.
Proof:
The path vertices are denoted as $v_1,v_2,\ldots v_n$ and the end vertices are denoted as $u_1,u_2,\ldots u_n$.
The assignment of vertex labels are $f(v_i)$ = 2i-1, i= 1,2,…n. and $f(u_i)$ = 2i, i=1,2,…n and the condition of prime labeling is every pair of adjacent vertices *u* and *v*, g.c.d(f(u),f(v)) = 1.
The induced edge labels are f*: E (G) → N and all are distinct.
This completes the proof.

Theorem 2.4
Every Spider SP ($P_{n,2}$) admits Prime-Antimagiclabeling.
Proof:
Define f:V(G)→{1,2,…n+2} by $f(v_i)$ = n+3-i, i=1,2,…n-2.
$f(v_{n-1})=3$
$f(v_n)=1$ and $f(a_1)=2$, $f(a_2)$= 4 and every pair of adjacent vertices *u* and *v*, g.c.d (f(u),f(v)) = 1.
The resulting edge labels are {3, 4, 5, 8, 11,13,15,…2q+1} and all are distinct.
Thus proved.

Theorem 2.5
The Cycles C_n, n – odd admits on Prime- Antimagic labeling.
Proof:
Define f:V(C_n) →{1,2,…n} by $f(v_i)$ = i, i=1,2,…n and also every pair of adjacent vertices *u* and *v*, g.c.d$(f(v_i),f(v_{i+1}))$ = 1.
The induced edge labels are defined by $f^*(v_iv_{i+1})$ = 2i+1, i=1,2,…n-1 and
$f^*(v_n)$=n+1.
All the edge labels are distinct and hence the theorem follows.

Theorem 2.6
$K_{1,n}$ is Prime – Antimagic labeling.
Proof:
Define f: V(G) →{1,2…..p} by $f(a_1)$ = 1, $f(u_i)$ = i+1, i= 1,2,….n, such that every pair of adjacent vertices *u* and *v*, g.c.d$(f(a_1),f(u_i))$ = 1.
The edge labels are f: E (G) → {3,4…p+1}. All the edge labels are distinct and hence $K_{1,n}$ is Prime – Antimagic labeling .

Theorem 2.7
$K_{m,n}$ is not Prime – Antimagic labeling if $m \geq 2$.
Proof:
Define $f: V(G) \rightarrow \{1, 2…. p\}$ such that every pair of adjacent vertices *u* and *v*, g.c.d(f(u),f(v)) = 1. This is not possible because it is an complete bipartite graph.

Theorem 2.8
The crown $C_n \odot K_1$ is Prime – Antimagic labeling.
Proof:
Let $G = C_n \odot K_1$. Here $|V(G)| = |E(G)| = 2n$.
Define $f: V(G) \rightarrow \{1,2….2n\}$ by $f(v_i) = 2i - 1$, i = 1,2,…..n and $f(u_i) = 2i$, i = 1,2,…n and also
g.c.d $(f(v_i), f(v_{i+1})) = 1$, g.c.d $(f(v_i), f(u_i)) = 1$.
Then the resulting distinct induced edge labelings are defined as follows.
$A = f^*(v_i v_{i+1}) = 4i$, i = 1, 2….n-1.
$B = f^*(v_n v_1) = 2n$
$C = f^*(v_i u_i) = 4i - 1$, i = 1, 2, .n. These sets are disjoint.
Hence $Cn \odot K_1$ is Prime – Antimagic.

Observation 2.9
The Complete graph $K_{1,n}$ is not Prime – Antimagic if $n \geq 4$.

Observation 2.10 The friendship graph F_n is not Prime – Antimagic.

3 .Prime – odd Antimagic labeling

In this section we introduce the another new concept of Prime – odd Antimagic labeling and Prime – odd Antimagic labeling of the path P_n, $n \geq 2$, the cycle C_n, $n \geq 3$, the star $K_{1,n}$ and the spider $SP(P_{n,2})$.

Definition 3.1
A connected graph G with |V| = p vertices and |E| = q edges is said to be ***Prime – odd Antimagic labeling*** if there is an bijection $f:V(G) \rightarrow \{1,3,….2|V|-1\}$ such that every pair of adjacent vertices u and v, g.c.d(f(u),f(v)) = 1 and the induced mapping $f^*:E(G) \rightarrow N$ defined by $f^*(e = uv) = \sum f(u,v)$ where $(u,v) \in E(G)$ is injective and all these edge labelings are distinct.

Theorem 3.2
Every Path P_n, $n \geq 2$, admits on Prime- odd Antimagic labeling.
Proof:
Define f on $V(P_n)$ by $f(v_i) = 2i-1$, i= 1,2,…n, such that every pair of adjacent vertices v_i and v_{i+1}, g.c.d $(f(v_i),f(v_{i+1})) = 1$.
The induced function $f^*: E(G) \rightarrow N$ defined by
$f^*(v_i v_{i+1}) = 4i$, for i=1,2,…n-1.
All these edge labelings are distinct and from this the theorem follows.

Theorem 3.3
Every cycle C_n, $n \geq 3$ admits on Prime – odd Antimagic labeling.
Proof:
Define $f:V(G) \rightarrow \{1,3,….2|V|-1\}$by $f(v_i) = 2i-1$, i = 1,2,…..n and also satisfies
g.c.d $(f(v_i), f(v_{i+1})) = 1$.
The distinct edge labels are $f^*(v_i v_{i+1}) = 4n$, i= 1,2…n-1 and
$f^*(v_n v_1) = 2n$.
Thus the Cycle C_n, $n \geq 3$ is Prime – odd Antimagic labeling.

Theorem 3.4
$K_{1,n}$ is Prime – odd Antimagic labeling.
Proof:
Define $f:V(G) \rightarrow \{1,3,….2|V|-1\}$by $f(a_1) = 1$, $f(u_i) = 2i+1$, i= 1,2,….n and
g.c.d $(f(u_i),f(a_1)) = 1$. The distinct edge labels are defined by $f^*(a_1 u_i) = 2i+2$, i= 1,2.…n and the theorem follows.

Theorem 3.5
Every Spider SP $(P_{n,2})$ admits on Prime-odd Antimagic labeling.
Proof:
Define $f:V(G) \rightarrow \{1,3,….2|V|-1\}$by $f(u_i) = 2i-1$, i=1,2,…..n+2.

The induced edge labels are distinct. Hence the condition of antimagic is satisfied.

Theorem 3.6

Any Comb Pn $\square$ K_1 is Prime – odd Antimagic labeling.

Proof:

Define a function f: V ($P_n \square K_1$) → {1,3,….2|V|-1} by

$$f(v_i) = \begin{cases} 4i-1, & 1 \le i \le n \quad \text{if i is odd} \\ 4i-3, & 1 \le i \le n \quad \text{if i is even} \end{cases}$$

$$f(u_i) = \begin{cases} 4i-3, & 1 \le i \le n \quad \text{if i is odd} \\ 4i-1, & 1 \le i \le n \quad \text{if i is even} \end{cases}$$

and also every pair of adjacent vertices v_i and v_{i+1} , g.c.d $(f(v_i),f(v_{i+1})) = 1$.

Then the induced edge labels of comb as follows.

$f^*(v_iv_{i+1}) = 8i - 1,\ 1 \le i \le n-1.$

$f^*(v_i u_i) = 8i - 4,\ 1 \le i \le n-1.$

The resulting edge labels are distinct and the theorem is satisfied.

4. Strongly Prime Antimagic Graphs

In this section we introduce the concept of strongly prime antimagic graph labeling. Prime labeling of some classes of graph were discussed by Vaidya.S.K, Prajapati.U.M in [8] and [9]. Prime labeling in the context of some graph operation was discussed in [4] and [5].Many researchers have studied prime graphs. For e.g. Fu [2] proved that P_n and $K_{1,n}$ are prime graphs. We will give brief summary of definitions which are useful for the present investigations.

Definition 4.1

A Graph G is said to be a Strongly Prime graph if for any vertex v of G there exists a Prime labeling f satisfying f(v) = 1.

Definition 4.2

A connected graph G with |V| = p vertices and |E| = q edges is said to be ***Strongly Prime – Antimagic labeling*** if there is an bijection f: V(G) →{1,3,….2|V|-1} such that every pair of adjacent vertices u and v g.c.d (f(u),f(v)) = 1 and if for any vertex v of G such that f satisfying f(v) = 1and the induced mapping f* : E(G) → N defined by f*(e=uv) $=\sum f(u,v)$ where (u,v) ∈ E(G) is injective and all these edge labelings are distinct.

Definition 4.3

A Quadrilateral Snake Q_n is obtained from a path $\{u_1, u_2 \ldots u_n\}$ by joining u_i and u_{i+1} to two vertices v_i and w_i , $1 \le i \le n-1$ respectively and then joining v_i and w_i .

Definition 4.4

The Product P_2 X P_n is called a Ladder and it is denoted by L_n .

Definition 4.5 The Corona of two graphs $G_1$and G_2 is the graph $G = G_1 \odot G_2$ formed by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2 .

Definition 4.6 The Helm H_n is a graph obtained from a wheel by attaching a pendant edge at each vertex of the n-cycle.

Definition 4.7

The gear graph G_n is obtained from the wheel by adding a vertex between each pair of adjacent vertices of the cycle. The gear graph G_n has 2n + 1 vertices and 3n edges.

Definition 4.8

Triangular Snake T_n is obtained from a path u_1 , u_2 ,…….u_n by joining u_i and u_{i+1} to a new vertex v_i for $1 \le i \le n-1$, that is every edge of a path is replaced by a triangle C_3.

Theorem 4.9

The graph $G \odot K_1$ is a Strongly Prime – Antimagic graph where G = T_n for all integer $n \ge 2$.

Proof:

Let $\{u_1,u_2 \ldots u_n\}$ be a path of length n. Let v_i, $1 \le i \le n-1$ be the new vertex joined to u_i and u_{i+1}. The resulting graph is called T_n and let x_i be the vertex which is joined to u_i , $1 \le i \le n$, let y_i be the vertex which is joined to v_i, $1 \le i \le n-1$. The resulting graph is G_1 (i.e.) $G \odot K_1$ where G = T_n graph. Now the vertex set of V (G_1) = $\{u_1,u_2 \ldots u_n, v_1,v_2 \ldots v_{n-1}, x_1,x_2, \ldots x_n, y_1 y_2, \ldots y_{n-1}\}$ and the edge set $E(G_1)= \{u_iu_{i+1}, u_iv_i /1 \le i \le n-1\} \cup \{u_ix_i / 1 \le i \le n\} \cup \{v_iu_{i+1}, v_iy_i /1 \le i \le n-1\}$. Here $|V(G_1)| = 4n-2$. Let v be the vertex for which we assign label 1 in our labeling method. Then we have the following cases:

Case i:

If v = u_j for some j ∈ {1,2…n} then the function f : V(G) → {1,2,….4n-2 } defined by

$$f(u_i) = \begin{cases} 4n + 4i - 4j - 1 & \text{if } i = 1,2,..j-1; \\ 4i - 4j + 1 & \text{if } i = j, j+1, \ldots\ldots\ldots . n. \end{cases}$$

$$f(v_i) = \begin{cases} 4n + 4i - 4j + 1 & \text{if } i = 1,2,..j-1; \\ 4i - 4j + 3 & \text{if } i = j, j+1, \ldots\ldots\ldots . n-1. \end{cases}$$

$$f(x_i) = \begin{cases} 4n + 4i - 4j & \text{if } i = 1,2,..j-1; \\ 4i - 4j + 2 & \text{if } i = j, j+1, \ldots\ldots\ldots . n. \end{cases}$$

$$f(y_i) = \begin{cases} 4n + 4i - 4j + 2 & \text{if } i = 1,2,..j-1; \\ 4i - 4j + 4 & \text{if } i = j, j+1, \ldots\ldots\ldots . n-1. \end{cases}$$

is a Prime labeling for G_1 with $f(v) = f(u_j) = 1$.

Case ii :

If $v = x_j$ for some $j \in \{1,2,\ldots..n\}$ then define a labeling f_2 using the labeling f defined in case (i) as $f_2(u_j) = f(x_j) = f(u_i)$ for $j \in \{1,2,\ldots..n\}$ and $f_2(v) = f(v)$ for all the remaining vertices.

Case iii :

If $v = v_j$ for some $j \in \{1,2,\ldots..n - 1\}$ then define a labeling f_3 using the labeling f_2 defined in case (ii) as $f_3(x_j) = f_2(v_j)$,$f_3(v_j)= f_2(x_i)$ for $j \in \{1,2,\ldots..n - 1\}$ and $f_3(v) = f_2(v)$ for all the remaining vertices.

Case iv :

If $v = y_j$ for some $j \in \{1,2,\ldots..n - 1\}$ then define a labeling f_4 using the labeling f_3 defined in case (ii) as $f_4(x_j) = f_2(y_j)$,$f_4(y_j)= f_2(x_i)$, $f_4(u_j) = f_2(v_j)$, $f_4(v_j) = f_2(u_j)$ for $j \in \{1,2,\ldots..n - 1\}$ and $f_4(v) = f_2(v)$ for all the remaining vertices.

Thus from all the cases described above G_1 is a Strongly Prime graph.

Now to find the edge labels $f^*(e = uv)= \sum f(u,v)$ where $(u, v) \in E(G)$ is injective and all these edge labelings are distinct.

Hence G_1 admits Prime – Antimagic labeling. Therefore G_1 is an Strongly Prime antimagic graph.

Theorem 4.10

The graph $G \odot K_1$ is a Strongly Prime – Antimagic graph where $G = Q_n$ for all integer $n \geq 2$.

Proof:

Let $\{u_1,u_2\ldots..u_n\}$ be a path of length n. Let v_i w_i be two vertices joined to u_i and u_{i+1} respectively and then join v_i and w_i, $1 \leq i \leq n-1$. The resulting graph is called a quadrilateral snake Q_n. Let x_i be the vertex which is joined to u_i , $1 \leq i \leq n$, let y_i be the new vertex which is joined to v_i, $1 \leq i \leq n-1$and let z_i be the new vertex which is joined to w_i, $1 \leq i \leq n-1$. The resulting graph is G_1 (i.e) $G \odot K_1$ where $G = Q_n$ graph. Now the vertex set of $V(G_1) = \{u_1,u_2, \ldots..u_n, v_1,v_2, \ldots..v_{n-1}, w_1,w_2, \ldots..w_{n-1}\ x_1,x_2, \ldots..x_n,\ y_1y_2, \ldots..y_{n-1},\ z_1,z_2, \ldots..z_{n-1}\}$ and the edge set $E(G_1) = \{u_iu_{i+1} /1 \leq i \leq n-1\} \cup \{u_ix_i /1 \leq i \leq n\} \cup \{u_iv_i, v_iy_i, v_iw_i, w_iz_i /1 \leq i \leq n-1\} \cup \{w_iu_{i+1} / 1 \leq i \leq n-1\}$. Here $|V(G_1)| = 6n - 4$. Let v be the vertex for which we assign label 1 in our labeling method. Then we have the following cases:

Case i:

If $v = u_j$ for some $j \in \{1, 2\ldots n\}$ then the function $f:V(G) \rightarrow \{1,2,\ldots.6n - 4\}$ defined by

$$f(u_i) = \begin{cases} 6n + 6i - 6j - 1 & \text{if } i = 1,2,..j-1; \\ 6i - 6j + 1 & \text{if } i = j, j+1, \ldots\ldots\ldots . n. \end{cases}$$

$$f(v_i) = \begin{cases} 6n + 6i - 6j - 3 & \text{if } i = 1,2,..j-1; \\ 6i - 6j + 3 & \text{if } i = j, j+1, \ldots\ldots\ldots . n-1. \end{cases}$$

$$f(w_i) = \begin{cases} 6n + 6i - 6j + 1 & \text{if } i = 1,2,..j-1; \\ 6i - 6j + 5 & \text{if } i = j, j+1, \ldots\ldots\ldots . n-1. \end{cases}$$

$$f(x_i) = \begin{cases} 6n + 6i - 6j & \text{if } i = 1,2,..j-1; \\ 6i - 6j + 2 & \text{if } i = j, j+1, \ldots\ldots\ldots . n. \end{cases}$$

$$f(y_i) = \begin{cases} 6n + 6i - 6j - 2 & \text{if } i = 1,2,..j-1; \\ 6i - 6j + 4 & \text{if } i = j, j+1, \ldots\ldots\ldots . n-1. \end{cases}$$

$$f(z_i) = \begin{cases} 6n + 6i - 6j + 2 & \text{if } i = 1,2,..j-1; \\ 6i - 6j + 6 & \text{if } i = j, j+1, \ldots\ldots\ldots . n-1. \end{cases}$$

is a Prime labeling for G_1 with $f(v) = f(u_j) = 1$.

Case ii :

If $v = x_j$ for some $j \in \{1,2,\ldots..n\}$ then define a labeling f_2 using the labeling f defined in case(i) as $f_2(u_j) = f(x_j) = f(u_i)$ for $j \in \{1,2,\ldots..n\}$ and $f_2(v) = f(v)$ for all the remaining vertices.

Case iii :
If $v = v_j$ for some $j \in \{1,2,\ldots..n - 1\}$ then define a labeling f_3 using the labeling f_2 defined in case (ii) as $f_3(x_j) = f_2(v_j)$, $f_3(v_j)= f_2(x_i)$ for $j \in \{1,2,\ldots..n - 1\}$ and $f_3(v) = f_2(v)$ for all the remaining vertices.
Case iv :
If $v = w_j$ for some $j \in \{1,2,\ldots..n - 1\}$ then define a labeling f_4 using the labeling f_3 defined in case (iii) as $f_4(w_j) = f_3(v_j)$, $f_4(v_j)= f_3(w_j)$ for $j \in \{1,2,\ldots..n - 1\}$ and $f_4(v) = f_3(v)$ for all the remaining vertices.
Case v: If $v = z_j$ for some $j \in \{1,2,\ldots..n - 1\}$ then define a labeling f_5 using the labeling $f_3$defined in case (iv) as follows: $f_5(z_j) = f_4(w_j)$, $f_5(w_j)= f_4(z_j)$ for $j \in \{1,2,\ldots..n - 1\}$ and $f_5(v) = f_4(v)$ for all the remaining vertices.
Case vi:
If $v = y_j$ for some $j \in \{1,2,\ldots..n - 1\}$ then define a labeling f_6 using the labeling f_2 defined in case (ii) as $f_6(u_j) = f_2(v_j)$, $f_6(v_j)= f_2(u_j)$, $f_6(x_j) = f_2(y_j)$, $f_6(y_j)= f_2(x_j)$for $j \in \{1,2,\ldots..n - 1\}$ and $f_6(v) = f_2(v)$ for all the remaining vertices.
Thus from all the cases described above G_1 is a Strongly Prime graph.
Now to find the edge labels $f^*(e = uv) = \sum f(u,v)$ where $(u,v) \in E(G)$ is injective and all these edge labelings are distinct.
Hence G_1 is an Strongly Prime – Antimagic graph.

Theorem 4.11
Thegraph $G\odot K_1$ is strongly prime antimagic graph where $G = L_n$ for all integer $n \geq 2$.
Proof:
Let G be the ladder with vertices $\{u_1, u_2.....v_1,v_2.\ldots.v_n\}$. Let u_i' be the new vertex joined to u_i $1 \leq i \leq n$ and v_i' be the new vertex joined to v_i, $1 \leq i \leq n$ in G. The resulting graph $G_1 = G\odot K_1$ where $G = L_n$ graph. Now the vertex set $V(G_1)=\{u_1,u_2,\ldots.v_1,v_2,\ldots.v_n, u_1',u_2',\ldots u_n' v_1',v_2'\ldots..v_n'\}$.
The Edge set $E(G_1) = \{ v_iv_{i+1}, u_iu_{i+1} / 1 \leq i \leq n-1 \} \cup \{ u_iv_i, u_iu_i', v_iv_i' / 1 \leq i \leq n\}$. Here
$|V(G_1)| = 4n$. Let v be the vertex for which we assign label 1 in our labeling method. Then we have the following cases.
Case (i): If $v = u_j$ for some $j \in \{1,2,\ldots n\}$ then the function $f : V(G_1) \rightarrow \{1,2,\ldots 4n\}$ defined by

$$f(u_i) = \begin{cases} 4n + 4i - 4j + 1 & \text{if } i = 1,2,\ldots\ldots.j-1 \\ 4i - 4j + 1 & \text{if } i = j, j+1,\ldots\ldots.n; \end{cases}$$

$$f(u_i') = \begin{cases} 4n + 4i - 4j + 2 & \text{if } i = 1,2,\ldots\ldots.j-1 \\ 4i - 4j + 2 & \text{if } i = j, j+1,\ldots\ldots.n \end{cases}$$

$$f(v_i) = \begin{cases} 4n + 4i - 4j + 3 & \text{if } i = 1,2,\ldots\ldots.j-2 \\ 4i - 4j + 3 & \text{if } i = j, j+1,\ldots\ldots.n \end{cases}$$

$$f(v_i') = \begin{cases} 4n + 4i - 4j + 4 & \text{if } i = 1,2,\ldots j-2 \\ 4i - 4j + 4 & \text{if } i = j, j+1,\ldots\ldots.n \end{cases}$$

$$f(v_{j-1}) = \begin{cases} 4n & \text{if } 4n-1 \text{ is multiple of } 3 \\ 4n - 1 & \text{if otherwise;} \end{cases}$$

$$f(v'_{j-1}) = \begin{cases} 4n - 1 & \text{if } 4n-1 \text{ is multiple of } 3 \\ 4n & \text{if otherwise;} \end{cases}$$

is a prime labeling for G_1 with $f(v) = f(u_j) = 1$.Thus f is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $v = u_j$ in G_1 graph.
Case (ii): If $v = u_j'$ for some $j \in \{1, 2\ldots n\}$ then define labeling f_2 using the labeling f defined in case(i) as follows:
$f_2(u_j) = f(u_j')$, $f_2(u_j') = f(u_j)$ for $j \in \{1,2,\ldots.n\}$then define a labeling f_2 using the labeling f defined in case (i) as follows:
$f_2(u_j) = f(u_j')$, $f_2(u_j') = f(u_j)$ for $j \in \{1,2,...n\}$ and $f_2(v) = f(v)$ for all the remaining vertices. Thus the labeling f_2 is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $v = u_j'$ in G_1.
Case (iii):
If $v = v_j$ for some $j \in \{1,2,...n\}$ then define a labeling f_3 using the labeling f defined in case (i) as follows:$f_3(u_i) = f(v_i)$, $f_3(v_i) = f(u_i)$, $f_3(u_i') = f(v_i')$, $f_3(v_i') = f(u_i')$ for $1 \leq i \leq n$ in G_1. Then the resulting labeling f_3 is a prime labeling and also it is possible to assign label 1 to any arbitrary vertex of $v = v_j$ in G_1.
Case (iv): If $v = v_j'$ for some $j \in \{1,2,...n\}$ then define a labeling f_4 using the labeling f_3 defined in case (iii) as follows: $f_4(v_j) = f_3(v'_j)$, $f_4(v'_j) = f_3(v_j)$ for $j \in \{1,2,...n\}$ and $f_4(v) = f_3(v)$ for all the remaining vertices. Hence f_4 is a prime labeling.
Thus in all three cases G_1 is a strongly prime graph.
And also the vertex labeling $f(u) = f(v) = \sum f(e)$ and all these labelings are distinct.
Hence G_1 is strongly prime antimagic graph.

Theorem 4.12
The Helm H_n is strongly prime antimagic graph.
Proof:
Let $v_0$be the apex vertex $v_1, v_2, \ldots.v_n$be the consecutive rim vertices of H_n and $v_1',v_2',\ldots\ldots v_n'$ be the pendent vertices of H_n. Let v be the vertex for which we assign label 1 in our labeling method. Then we have the following cases.
Case (i): If v is the apex vertex $v = v_0$ then the function $f : V(H_n) \rightarrow \{1, 2, ... 2n+1\}$ defined as $f(v_0) = 1$, $f(v_1) = 2$, $f(v_1') = 3$, $f(v_i) = 2i + 1$ if $2 \leq i \leq n$, $f(v_i') = 2i$ if $2 \leq i \leq n$. Then f is an injection function. For an arbitrary edge e = ab of H_n we claim that g.c.d (f(a), f(b)) = 1

Subcase (i): If e = v_0v_1 for some i ∈ {2, 3,.... n } then gcd(f(v_0) ,f(vi)) = gcd(1, f(v_i)) = 1.

Subcase (ii):
If e = v_iv_{i+1} for some i ∈ {1, 2, …n-1} then gcd(f(v_i), f(v_{i+1})) = gcd (2i + 1, 2i + 3) = 1 as 2i+1; 2i+3 are consecutive odd positive integers. If e = v_1v_2 then gcd(f(v_1), f(v_2)) = gcd(2,5) =1 and if e = v_nv_1 then gcd(f(v_n),f(v_1)) = gcd(2n + 1, 2) = 1 as 2n + 1 is an odd integer.
Subcase (iii): If e = $v_iv_i'$for some i ∈ {2, 3, ….n} then gcd(f(v_i), f(v_i')) = gcd(2i+1, 2i) = 1 as 2i+1, 2i are consecutive positive integers and if e = v_1v_1'then gcd(f(v_1), f(v_1')) = gcd (2, 3) = 1 as 2 and 3 are consecutive positive integers.
Case (ii): If v = $v_j$for some j ∈ {1,2,...n }, v is one of the rim vertices then we may assume that v = $v_1$then define a labeling f_2 using the labeling f defined in case(i) as follows:
$f_2(v_0)$ = f(v_1), $f_2(v_1)$ = f(v_0) and f_2(v) = f(v) for all other remaining vertices. Clearly f is an injection. For an arbitrary edge e = ab of G we claim that gcd(f(a), f(b)) = 1.
Case (iii) : If v is one of the pendent vertices then we may assume that v = $v_i'$for i = $\frac{p-1}{2}$or $\frac{p-3}{2}$, where p is the largest prime less than or equal to 2n+1. According to Bertrand's postulate such a prime p exist with $\frac{2n+1}{2}$< p < 2n+1.
Case (iv): When n = 3k+1 then define a labeling f_2 using labeling f in case (iii) as follows: $f_2(v_1)$ = f(v_1'), $f_2(v_1')$ = f(v_1) and f_2(v) = f(v) for all other remaining vertices. Then f is an injection function. For an arbitrary edge e = ab of H_n we claim that g.c.d (f(a), f(b)) = 1. Thus in all the possibilities described above f is prime labeling and also it is possible to assign label 1 to any arbitrary vertex ofH_n. Hence $H_n$is strongly prime graph.
Hence all these edge labelings are defined by the condition gcd (f(u),f(v)) = 1 .Now to find the edge labeling f*(e = uv) = ∑ f(uv). All these edge labelings are distinct. Hence $H_n$admits an strongly prime – antimagic labeling.

Theorem 4.13
The Gear graph G_n is a strongly prime - antimagic graph.
Proof:
Let $v_0$be the apex vertex $v_1, v_2, \ldots. v_n, v_1', v_2', \ldots \ldots v_n'$be the consecutive rim vertices. Let v be an arbitrary vertex of G_n that is v = v_0.Then the function f : V(G_n) →{1, 2, ...2n + 1 } defined as f(v_i) = 2i +1 for i = 0, 1, 2, ...n, f(v_i') = 2i + 2 for i = 1, 2, ...n-1, f(v_n') = 2.
Clearly f is an injection. For an arbitrary edge e = ab of G_n we claim that gcd (f(a), f(b)) = 1.
Now to find the edge labeling f*(e = uv) = ∑ f(uv) . All these edge labelings are distinct. Hence $G_n$admits astrongly prime – antimagic labeling.

5. Conclusion
In this article we have investigated Prime-Antimagic labeling of Paths, Caterpillar, Spider, Odd Cycles, Complete bipartite graphs, Crown and Prime- odd Antimagic labeling of Paths ,Cycles, Complete bipartite graphs, Spider and Comb. Also we investigated strongly prime – antimagic labeling of Triangular snake, Quadrilateral snake, Ladder, Helm, Gear graphs. Analogous work can be carried out for other families and in the context of different types of graph labeling techniques.

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