Vertex Prime Invariance Of some Path Union Graphs

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1. Abstract: we investigate path unions of C_3^+,W_4, and W_4^+ for vertex prime labeling. All different non isomorphic structures of these path unions are shown to be vertex prime. This property of graphs is called as invariance under labeling.

2.Key words: labeling, vertex prime, wheel, path union, crown.

Subject Classification: O5C78

2.Introduction:

The graphs we consider are finite, connected, simple and un- directed. We refer F.Harary[], Dynamic survey of graph labeling [3] for definitions and terminology. Deretsky, Lee, Mitchem Proposed a labeling called as vertex prime labeling of graph[2].A function f : E(G) → {1,2,..|E|} is such that for any vertex v the gcd of all labels on edges incident with v is 1 This is true to all vertices with degree at least 2.The graph that admits vertex prime labeling is called as vertex prime graph. They have shown that all forests, connected graphs, $5C_{2m}$, graph with exactly two components one of which is not odd cycle etc are vertex prime. One should refer A Dynamic survey of graph labeling by Joe Gallian [3] to find further work done in this type of labeling.

In this paper we discuss vertex prime labeling of path union graphs .Path union Pm(G) is obtained by taking m copies of graph G, identify a fixed same vertex from one copy each of G with vertex of P_m. If we change the vertex on G to identify with vertex of Pm, we will get non isomorphic pathunion. We have studied non isomorphic path unions of crown of C_3, W_4, crown W_4 and shown that they are vertex prime graphs. This is invariance under vertex prime labeling.

4. 4.Preliminaries

4.1 Crown of (p,q)graph G denoted by G^+ is obtained by attaching a pendent edge to each vertex of G. It has q + p edges and 2p vertices. A double crown G^{++} is obtained by attaching 2 pendent edges at each vertex of G. etc.

4.2 A wheel graph W_n is obtained by taking a cycle C_n and a new vertex w outside of C_n.W is joined to each vertex of Cn by an edge each. It has 2n edges and n+1 vertices.

4.3 To obtain a path union of (p, q) graph G we fuse a copy of G at a given fixed vertex of G on every vertex of path Pm. If we change the vertex on graph G used to fix with vertex of Pm we may will get a different structure of path union.

5. Theorems proved:

5.1 Path union of C_3 crown (i.e. G = $P_m(c_3^+)$) is Vertex Prime.

Proof: There are **two** non-isomorphic structures possible. **Structure I** in which a degree 3 vertex on C_3 crown is used for toidentify with vertex of Pm. We define G in terms of vertex set and edge set .

V(G) = $\{v_1,v_2,..v_m\}$ U $\{ u_{i,1}(=v_i),u_{i,2},u_{i,3},w_{i,1},w_{i,2},w_{i,3}/i = 1,2,\ldots,m\}$ E(G) = $\{ e_i=(v_iv_{i+1})/i = 1,2,\ldots m-1\}$ U $\{c_{i,1}=(v_iu_{i,2}),c_{i,2}=(u_{i,2}u_{i,3}),c_{i,3}=(u_{i,3},v_i)\}$ U $\{b_{i,1}=(u_{i,1}w_{i,1}),$ $b_{i,2} = (u_{i,2}w_{i,2}),b_{i,3}=(u_{i,3}w_{i,3})\}$. We have obtained the path union by identifying vertex w_i of i^{th} copy of w_4 with v_i, i = 1,2,..,m

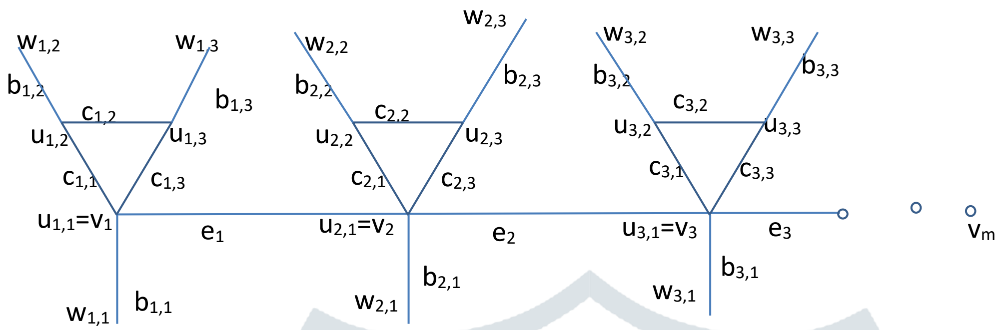

Fig 5.1 ordinary label of $P_m(c_3^+)$ Structure 1

Define a function f:E(G)→{1,2,..,|E|}as follows

f(ei) = i, i = 1,2,,,.,m-1.$f(c_{i,j})$= m-1+(i-1)6+j,j=1,2,3

$f(b_{i,j})$= m+6i-4+j, j = 1,2,3.i = 1,2,…,m.

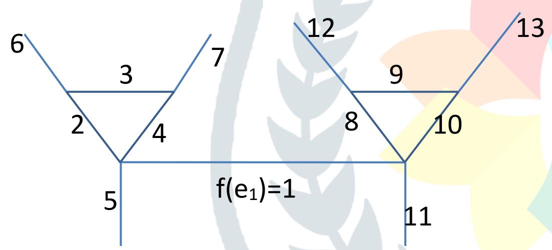

Fig 5.2 labeled copy of $P_2(c_3^+)$. **Structure 1.**

The numbers are edge labels

Define **structure 2** of G in terms of vertex set and edge set as follows:

V(G) = $\{v_1,v_2,..v_m\}$ U $\{ u_{i,1}(=v_i),u_{i,2},u_{i,3},w_{i,1},w_{i,2},w_{i,3}/i = 1,2,\ldots,m\}$

E(G) = $\{ e_i=(v_iv_{i+1})/i = 1,2,\ldots m-1\}$ U $\{c_{i,1}=(u_{i,1}u_{i,2}),c_{i,2}=(u_{i,2}u_{i,3}),c_{i,3}=(u_{i,3},u_{i,1})\}$ U $\{b_{i,1}=(u_{i,1}w_{i,1}),$ $b_{i,2} = (u_{i,2}w_{i,2}),b_{i,3}=(u_{i,3}w_{i,3})\}$.m We have obtained the path union by identifying vertex $w_{i,1}$ of i^{th} copy of w_4 with v_i,i = 1,2,..,m.

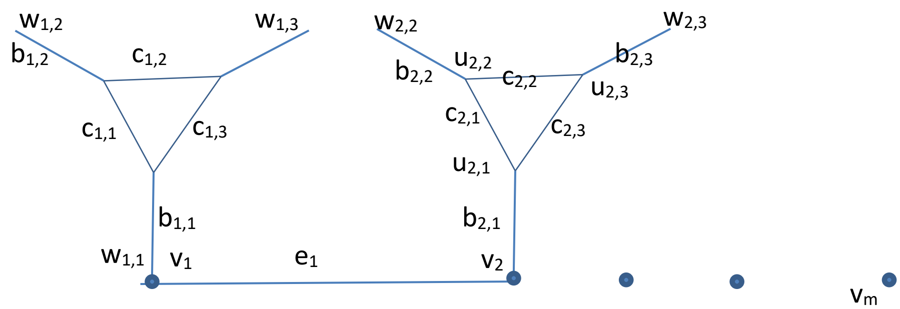

Fig 5.3 ordinary labeling of $P_m(W_4)$

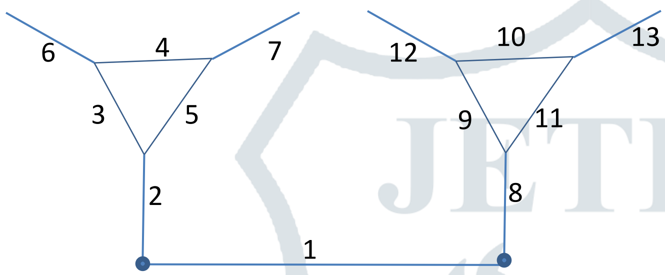

Fig 5.4 labeled copy of $P_2(W_4)$.The numbers are edge labels.

Define a function f:E(G)→{1,2,..,|E|}as follows

$f(e_i) = i$, i = 1,2,,,.,m-1.

$f(c_{i,j})$= m-1+(i-1)7+j,j=1,2,3,

$f(b_{i,j})$=m+(i-1)6 for j =1.

$f(b_{i,j})$=m+4+(i-1)6 for j =2,

$f(b_{i,j})$ = m+5+(i-1)6 for j =3.

The graph is vertex prime.

5.2 Path union of wheel graph (W_n) is vertex prime for n =4

Proof: We define the path union of wheel graph (W4) i.e G = $P_m(W_4)$ as follows.

V(G)= $\{w_i, v_1, v_2, ,..., v_m\}$ U $\{u_{i,1}, u_{i,2}, u_{i,3}, u_{i,4}\}$

E(G) = $\{e_i=(v_iv_{i+1})$/i=1,2,..,m-1} U $\{c_{i,j}=(u_{i.j}u_{i,j+1})$,j=1,2,..,n and n+1 taken (mod n), here n =4)U {bi,j=$(w_iu_{i,j})$/i = 1,2,..,m and j = 1,2,3,4}.

Define a function f:E(G)→{1,2,…q} as follows:

$f(e_i)=i$, $i=1, 2,.., m-1$ and $j = 1,2,3,4$.

$f(c_{i,j})= m-1 +(i-1)8+j$,

$i=1, 2,.., m-1$ and $j = 1,2,3,4$.

$f(b_{i,j})=f(c_{i,4})+j$,$i=1, 2,.., m-1$ and $j = 1,2,3,4$.

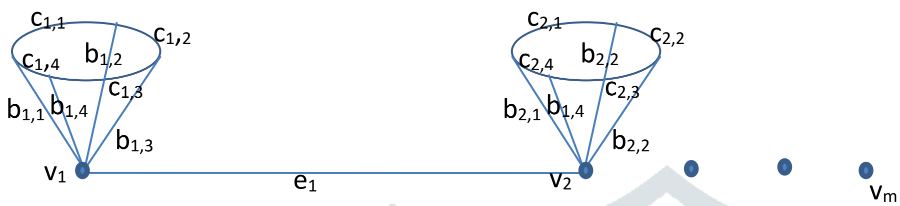

Fig 5.5 ordinary labeling of Pm(w_4)

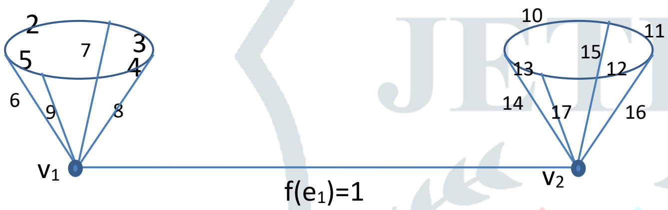

Fig 5.6 vertex prime labeling of $P_2(w_4)$

Define a function $f :E(G)\rightarrow\{1,2,\ldots,|E|\}$ as follows:

$f(e_i)= i$,$i = 1,2,\ldots,m-1$.

$f(c_{i,j})=(m-1)+8(i-1)+j$, $j = 1,2,3,4$

$f(b_{i,j})= f(c_{i,4})+j$,$j =1,2,3,4$.The graph is vertex prime.

The wheel graph (structure 2) is vertex prime.

We obtain this structure by identifying a 3-degree vertex $u_{i,1}$ on cycle C_4 of W_4 with vertex vi of path Pm.

$f(ei)=i$, $i = 1,2,..,m-1$

$f(ci,j) = m-1+8(i-1)+j$ for $i =1 ,2,\ldots,m$ and $j = 1,2,3,4$ $f(b_{i,j})=f(c_{i,4})+j$,$I = 1,2,..m$ and $j = 1,2,3,4$ .The resultant graph is vertex prime.

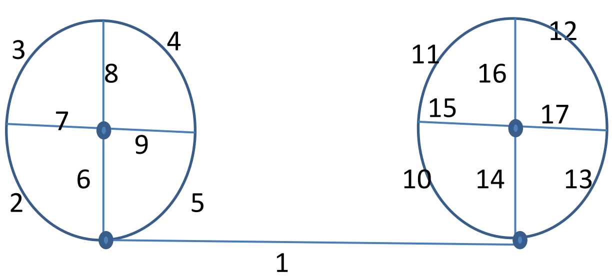

Fig 5.7 Labeled copy of $P_2(W_4)$ structure 2

The both of the structures of $P_m(W_4)$ are vertex prime graphs .

5.3 Crown of W_4 (i,e $G = W_4^+$ (all four non isomorphic structures) is vertex prime graph.

Proof: We define the graph G as: (**structure 1**)

$V(G)= \{w_i,v_1,v_2,,\ldots,v_m\} \cup \{ u_{i,1},u_{i,2},u_{i,3},u_{i,4} \} \cup \{ w_{i,j}/i= 1,2,3,\ldots,m,j=1,2,3,4\}$

$E(G) = \{e_i=(v_iv_{i+1})/i=1,2,..,m-1\} \cup \{c_{i,j}=(u_{i.j}u_{i,j+1}),j=1,2,..,n$ and $n+1$ taken (mod n), here $n =4)\} \cup \{b_{i,j}=(w_iu_{i,j})/i = 1,2,..,m$ and $j = 1,2,3,4\} \cup \{q_{i,j} =(u_{i,j}w_{i,j})/j =1,2,3,4$ and $q_{i,5} = (w_iw_{i,5}) \}$

Define a function $f:E(G)\rightarrow\{1,2,\ldots q\}$ as follows:

$f(e_i)=i, \quad i = 1,2,..m$

$f(c_{i,j})= m-1 +(i-1)8+j, \quad j = 1,2,..5, i = 1,2,..m,$

$f(c_{i,j})=m+8(i-1)+j-1,j = 1,2,..5, i = 1,2,..m,$

$f(b_{i,j})=f(c_{i,4})+j$,i=1, 2,.., m-1 and j = 1,2,3,4.

$f(q_{i,j}) = f(b_{i,4})+j/$,i=1,2,..m and j=1,2,..5;

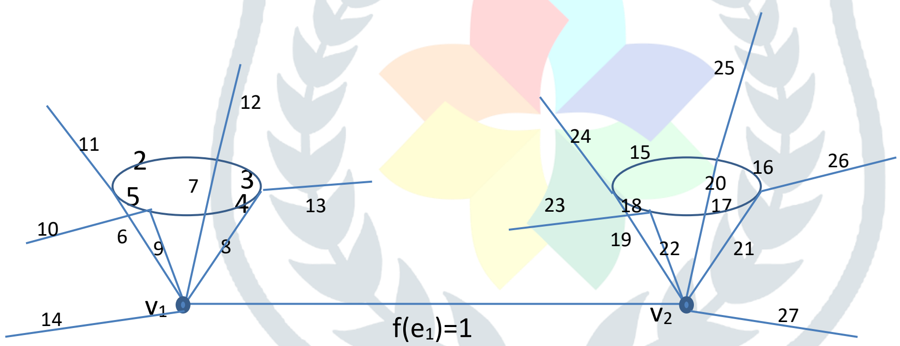

Fig 5.8 vertex prime labeling of $P_2(w_4^+)$: structure 1

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To obtain **structure 2** we take a path Pm and m copies of W_4^+. At each vertex of P_m attach a copy of W_4^+ by the pendent vertex at hub w.We define this graph as follows.

$V(G) = \{v_1,v_2,..,v_m\}\cup\{u_{i,,1},u_{i,2},u_{i,3},u_{i,4},w_i\}\cup\{w_{i,1},w_{i,2},.. w_{i,5}\}.$

$E(G) =\{e_i=(v_iv_{i+1})/I = 1,2,,,m-1\}\cup \{c_{i,1}=(u_{i,1}u_{i,2}), c_{i,2}=(u_{i,2}u_{i,3}), c_{i,3} =(u_{i,3}u_{i,4}), c_{i,4} = (u_{i,4}u_{i,1})\} \cup \{b_{i,j}=(w_iu_{i,j})/I = 1,2,\ldots,m$ and $j = 1,2,3,4.)\} \cup \{ q_{i,j}=(u_{i,j}w_{i,j})/i= 1,2,..,m,j = 1,2,3,4$and $q_{i,5}=(w_iu_{i,5}) \}$

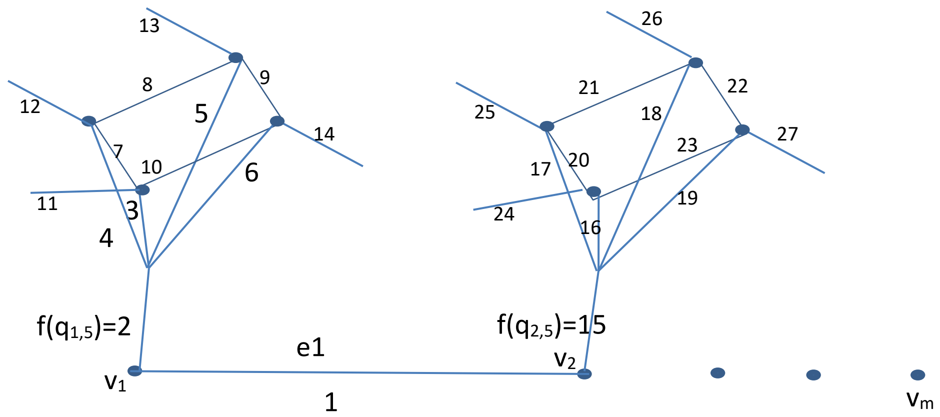

Fig 5.9 vertex prime labeling of $P_2(w_4^+)$: structure 2

Define a function as:

$f:E(G)\rightarrow\{1,2,\ldots,|E|\}$,

$f(e_i)=i$, $i = 1,2,\ldots,m-1$.

$f(q_{i,5}) = m+1+13(i-1)$, $i = 1,2,,,,m$.

$f(b_{i,j})=f(q_{i,5})+j$,

$f(c_{i,j})=f(b_{i,4})+j$, $j=2,3,4.,i=1,2,..,m$.

$f(q_{i,j})=f(c_{i,4})+j,j=1,2,3,4$.

The resultant labeling is vertex prime.

We define **structure 3** as : $V(G) = \{v_1,v_2,..,v_m\}U\{u_{i,,1},u_{i,2},u_{i,3},u_{i,4},w_i\}U\{w_{i,1},w_{i,2},w_{i,3},w_{i,4},w_{i,5}\}$.

$E(G) =\{e_i=(v_iv_{i+1})/i= 1,2,,,m-1\}U$ $\{c_{i,1}=(u_{i,1}u_{i,2}),\ c_{i,2}=(u_{i,2}u_{i,3}),\ c_{i,3} =(u_{i,3}u_{i,4}),\ c_{i,4} = (u_{i,4}u_{i,1})\}$ U { $b_{i,j}= (w_iu_{i,j})/i = 1,2,\ldots,m$, and $j = 1,2,3,4$. } U{ $q_{i,j}=(u_{i,j}w_{i,j})/i= 1,2,..,m,j = 1,2,3,4$ and $q_{i,5}=(w_iu_{i,5})$ }.

The $P_m(W_4^+)$ is obtained by taking m copies of $P_m(W_4^+)$ and identifying $q_{i,1}$ (it is the pendent vertex at point $u_{i,1}$) with v_i, $i = 1,2\ldots,m$.

Define a function f as follows:

$f:E(G)\rightarrow\{1,2,\ldots,|E|\}$,

$f(e_i)=i$, $i = 1,2,\ldots,m-1$.

$f(q_{i,1}) = m+13(i-1)$, $i = 1,2,,,,m$.

$f(c_{i,j})=f(q_{i,1})+j,i = 1,2,..m$, $j=1,2,3,4$

$f(b_{i,j})=f(c_{i,4})+j$, $j=2,3,4.,i=1,2,..,m$.

$f(q_{i,j})=f(b_{i,4})+j$. j= 2,3,4,5 The resultant labeling is vertex prime .

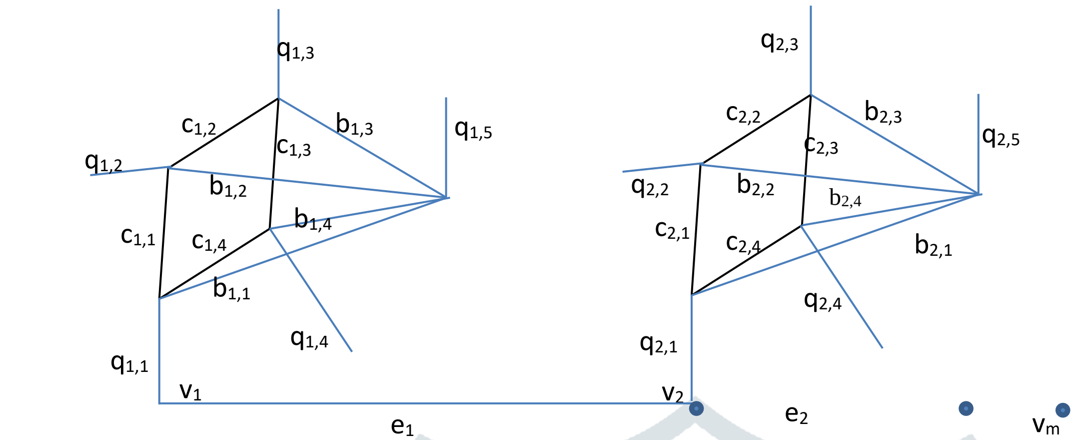

Fig 5.10 vertex prime labeling of $P_2(w_4^+)$: ordinary labeling structure 3

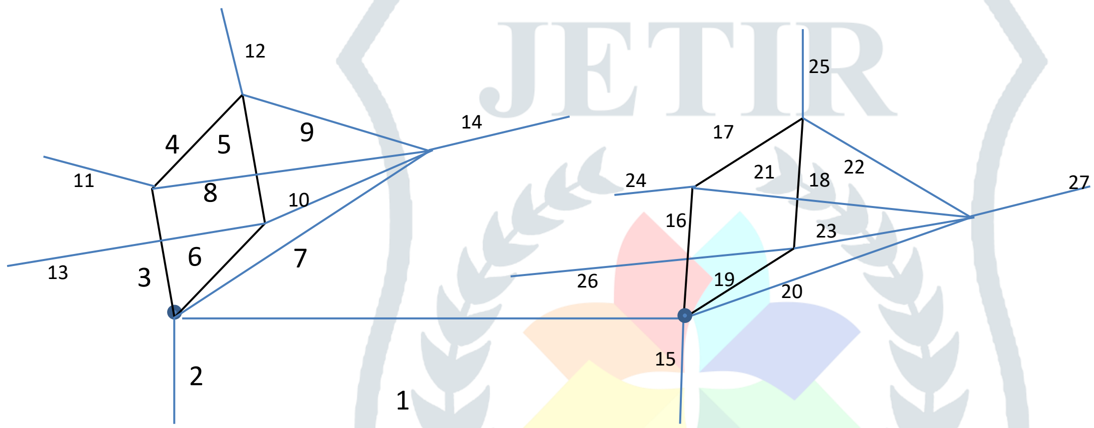

Fig 5.11 vertex prime labeling of $P_2(w_4^+)$: structure 4

Structure 4 is defined as follows. $V(G) = \{v_1,v_2,..,v_m\}U\{u_{i,1},u_{i,2},u_{i,3},u_{i,4},w_i\}U\{w_{i,1},w_{i,2},w_{i,3},w_{i,4},w_{i,5}\}$. $E(G) =\{e_i=(v_iv_{i+1})/i= 1,2,,,m-1\}U \{c_{i,1}=(u_{i,1}u_{i,2}), c_{i,2}=(u_{i,2}u_{i,3}), c_{i,3} =(u_{i,3}u_{i,4}), c_{i,4} = (u_{i,4}u_{i,1})\} U \{ b_{i,j}= (w_iu_{i,j})/i = 1,2,...,m$, and $j = 1,2,3,4. \} U\{ q_{i,j}=(u_{i,j}w_{i,j})/i= 1,2,..,m,j = 1,2,3,4$ and $q_{i,5}=(w_iu_{i,5}) \}$.

The $P_m(w_4^+)$ is obtained by attaching vertex $u_{i,1}$ to v_i

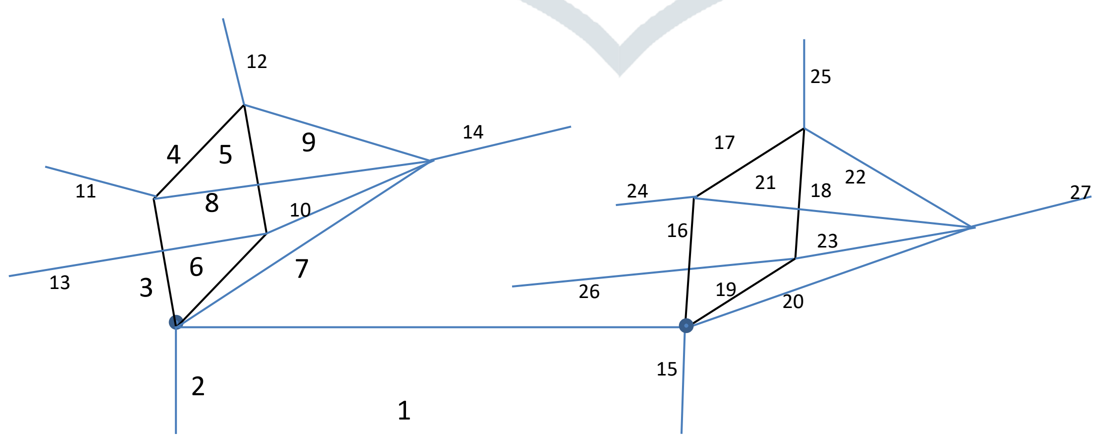

Fig 5.12 vertex prime labeling of $P_2(w_4^+)$: structure 4

$f:E(G) \rightarrow \{1,2,\dots,|E|\}$,

$f(e_i)=i$, i = 1,2,…,m-1.

$f(q_{i,1}) = m+13(i-1)$, i = 1,2,,,,m.

$f(c_{i,j})=f(q_{i,1})+j$,i = 1,2,..m , j=1,2,3,4

$f(b_{i,j})=f(c_{i,4})+j$, j=2,3,4.,i=1,2,..,m.

f(qi,j)=$f(b_{i,4})$+j.j= 2,3,4,5

The resultant labeling is vertex prime .

Conclusions : There are different structures of path union Pm(G) possible.We have shown that for G= C_3^+, W_4, C_4^+ All possible non isomorphic structures are vertex prime graphs.It is necessary to investigate this property for other graphs also.

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