A STUDY ON RELATIONS AMONG CONSECUTIVE INTEGERS AND ITS APPLICATIONS

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ABSTRACT
In this paper, we study on relations among consecutive elements of an equivalence class of congruence modulo m for any positive integer m. If x, y, z are any three consecutive elements of an equivalence class of congruence modulo m, then $y^2 = xz + m^2$. This is a very simple relation but unknown. For $m = 1$, we get the relation $y^2 = xz + 1$between any three consecutive integers x, y, z. We can extend the relation to any finite number of consecutive integers. Using these relations, we prove some new results and solve quadratic congruence of odd prime modulus. Uses of this relations are the merit of this paper.

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KEY WORDS: Consecutive elements, consecutive integers, equivalence class, consecutive triples, Fibonacci's numbers, quadratic congruence.

INTRODUCTION:
Let m be a fixed positive integer. For integers a and b, we define a relation $\Re$ on Z as $a\Re b \Leftrightarrow a \equiv b (\text{mod } m)$. Then $\Re$ is an equivalence relation on Z. This relation is called congruence modulo m. This relation has m distinct equivalence classes namely, [0], [1], [2], …….., [m-1]. If x, y, z are any three consecutive elements of an equivalence class of congruence modulo m, then $y^2 = xz + m^2$. In particular, if x, y, z are any three consecutive elements of [0], then there exists three consecutive integers a, b, c such that x=am, y=bm, z=cm and $b^2 = ac + 1$. If $a_1, a_2, a_3, ..., a_{n-1}, a_n, a_{n+1}$ $(n \geq 2)$ are consecutive elements of an equivalence class of congruence modulo m, then

$$a_2^2 + a_3^2 + ... + a_n^2 = a_1a_3 + a_2a_4 + ... + a_{n-1}a_{n+1} + (n-1)m^2$$

In particular, If $a_1, a_2, a_3, ..., a_{n-1}, a_n, a_{n+1}$ $(n \geq 2)$ are consecutive elements of [0], then there exists n+1 consecutive integers $b_1, b_2, b_3, ..., b_{n-1}, b_n, b_{n+1}$ $(n \geq 2)$ such that $a_1 = b_1 m$, $a_2 = b_2 m$, $........., a_{n+1} = b_{n+1} m$ and $b_2^2 + b_3^2 + ... + b_n^2 = b_1b_3 + b_2b_4 + ... + b_{n-1}b_{n+1} + (n-1)$.

PRELIMINARIES:
Definition 2.1: Any three consecutive integers x, y, z satisfy $y^2 = xz + 1$. The order tripled (x, y, z) is called consecutive triple. For a fixed positive integers m, $(mx + r, my + r, mz + r)$ where $r \in \{0, 1, 2, ..., m-1\}$ is called consecutive triple of congruence modulo m.

Definition 2.2:[1] A Pythagorean triple consists of three positive integers a, b, c such that $a^2 + b^2 = c^2$. Such a triple is commonly written as (a, b, c). If (a, b, c) is a Pythagorean triple, then so is (ka, kb, kc) for any positive integer k.

Definition 2.3:[2] We define a sequence of numbers as $f_1 = 1$, $f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 3$. The number f_n is called the nth Fibonacci's number.
We have the following properties of Fibonacci's numbers for each $n \in N$:

a. $f_1 + f_2 + \dots\dots + f_n = f_{n+2} - 1$

b. $f_1 + f_3 + \dots\dots + f_{2n-1} = f_{2n}$

c. $f_2 + f_4 + \dots\dots + f_{2n} = f_{2n+1} - 1$

d. $f_1^2 + f_2^2 + \dots\dots + f_n^2 = f_n f_{n+1}$

MAIN RESULTS:

Theorem 3.1: (3, 4, 5) is the only consecutive triple which is also Pythagorean triple.

Theorem 3.2: If $f_1, f_2, f_3, \dots\dots, f_n, \dots\dots$ are the Fibonacci's numbers, then

$$f_1 f_3 + f_2 f_4 + f_3 f_5 + \dots\dots + f_{n-1} f_{n+1} = \begin{cases} f_n f_{n+1} - 1, & if\ \ n\ is\ odd \\ f_n f_{n+1}, & if\ \ n\ is\ even \end{cases}$$

Proof: If f_{r-1}, f_r, f_{r+1} (r>1) are three consecutive Fibonacci's numbers, then

$$f_r^{\,2} = \begin{cases} f_{r-1} f_{r+1} + 1, & if\ \ r\ is\ odd \\ f_{r-1} f_{r+1} - 1, & if\ \ r\ is\ even \end{cases}$$

Putting r=2, 3, 4, ………, n and adding, we get

$$f_2^2 + f_3^2 + f_4^2 + \dots\dots\dots + f_n^2 = \begin{cases} f_1 f_3 + f_2 f_4 + f_3 f_5 + \dots\dots + f_{n-1} f_{n+1}, & if\ \ n\ is\ odd \\ f_1 f_3 + f_2 f_4 + f_3 f_5 + \dots\dots\dots + f_{n-1} f_{n+1}, & if\ \ n\ even \end{cases}$$

Thus, if n is odd, then

$$f_1 f_3 + f_2 f_4 + f_3 f_5 + \dots\dots + f_{n-1} f_{n+1} = f_2^2 + f_3^2 + f_4^2 + \dots\dots + f_n^2$$

$$= f_1^2 + f_2^2 + f_3^2 + f_4^2 + \dots\dots + f_n^2 - 1 \text{ since } f_1 = 1$$

$$= f_n f_{n+1} - 1, \text{ since } f_1^2 + f_2^2 + f_3^2 + f_4^2 + \dots\dots + f_n^2 = f_n f_{n+1}$$

And if n is even, then

$$f_1 f_3 + f_2 f_4 + f_3 f_5 + \dots\dots + f_{n-1} f_{n+1} - 1 = f_2^2 + f_3^2 + f_4^2 + \dots\dots + f_n^2$$

$$\Rightarrow f_1 f_3 + f_2 f_4 + f_3 f_5 + \dots\dots + f_{n-1} f_{n+1} = f_1^2 + f_2^2 + f_3^2 + f_4^2 + \dots\dots + f_n^2, \text{ since } f_1 = 1$$

$$= f_n f_{n+1}, \text{ since } f_1^2 + f_2^2 + f_3^2 + f_4^2 + \dots\dots + f_n^2 = f_n f_{n+1}$$

Hence we can conclude that

$$f_1 f_3 + f_2 f_4 + f_3 f_5 + \dots\dots + f_{n-1} f_{n+1} = \begin{cases} f_n f_{n+1} - 1, & if\ \ n\ is\ odd \\ f_n f_{n+1}, & if\ \ n\ is\ even \end{cases}$$

Theorem 3.3: If $x_1, x_2, x_3, \dots\dots, x_{n-1}, x_n, x_{n+1}$ $(n \geq 2)$ are any consecutive elements of any equivalence class ofcongruence modulo m for any positive integers m, then

$$x_1^2 + x_2^2 + x_3^2 + \dots\dots + x_n^2 \geq (x_1 x_3 + x_2 x_4 + x_3 x_5 + \dots\dots + x_{n-1} x_{n+1}) + 1$$

Proof: Let $x_1, x_2, x_3, \dots\dots, x_{n-1}, x_n, x_{n+1}$ $(n \geq 2)$ be any consecutive elements of any equivalence class ofcongruence modulo m for any positive integers m. Then

$$x_2^2 + x_3^2 + \dots\dots\dots + x_n^2 = x_1 x_3 + x_2 x_4 + \dots\dots + x_{n-1} x_{n+1} + (n-1) m^2$$

$$\Rightarrow x_1^2 + x_2^2 + x_3^2 + \dots\dots\dots + x_n^2 - (x_1 x_3 + x_2 x_4 + \dots\dots + x_{n-1} x_{n+1}) = x_1^2 + (n-1) m^2$$

$$\geq 0 + 1.1^2 \text{ as } x_1^2 \geq 0, n-1 \geq 1, m \geq 1$$

$$\Rightarrow x_1^2 + x_2^2 + x_3^2 + \dots\dots + x_n^2 \geq (x_1 x_3 + x_2 x_4 + x_3 x_5 + \dots\dots + x_{n-1} x_{n+1}) + 1$$

Remark: For $m = 1$, $n = 2$ and $x_1 = 0$ the equality holds.

Theorem 3.4: If x, y, z are three consecutive elements of any equivalence class ofcongruence modulo m for any positive integers m, then

$$y^2 = \begin{cases} m^2 (\text{mod } x), & if\ \ m^2 < x \\ r (\text{mod } x), & if\ \ m^2 > x \end{cases}$$

where $0 \leq r < x$. Similar result holds for modulo z.

Proof: Let x, y, z be three consecutive elements of any equivalence class ofcongruence modulo m for any positive integers m. Then

$y^2 = xz + m^2$

$\Rightarrow y^2 \equiv m^2 (\bmod x)$ and $y^2 \equiv m^2 (\bmod z)$

If $m^2 < x$, then $y^2 \equiv m^2 (\bmod x)$

If $m^2 > x$, then by division algorithm, there exists unique integers q and r such that $m^2 = xq + r$ where $0 \le r < x$. Therefore

$$y^2 \equiv (xq + r)(\bmod x)$$

$\Rightarrow y^2 \equiv r(\bmod x)$

Hence $y^2 = \begin{cases} m^2 (\bmod x), & if\ m^2 < x \\ r(\bmod x), & if\ m^2 > x \end{cases}$

where $0 \le r < x$. Similar, we can prove the same result for modulo z.

Lemma (Lagrange's theorem) 3.5:[1] If p is a prime and $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots\ldots + a_1 x + a_0$ where p does not divide a_n, is a polynomial of degree $n \ge 1$ with integral coefficients, then the congruence $f(x) \equiv 0(\bmod p)$ has at most n incongruent solutions modulo p.

Lemma 3.6:[1] If p is a prime number and $d | p-1$, then the congruence $x^d \equiv 1(\bmod p)$ has exactly d incongruent solutions modulo p.

Theorem 3.7: If p is an odd prime and the quadratic congruence $ax^2 + bx + c \equiv 0(\bmod p)$ can be write of the form $(2ax + b)^2 \equiv n^2 (\bmod p)$ where $n^2 = b^2 - 4ac$, then $2ax_1 + b$, p, $2ax_2 + b$ are three consecutive elements of some equivalence class ofcongruence modulo n and x_1, x_2 are incongruent solutions of the quadratic congruence.

Proof: Consider the quadratic congruence

$ax^2 + bx + c \equiv 0(\bmod p)$---------------- (i)

Since p is an odd prime, $\gcd(4a, p) = 1$. Thus given congruence equivalent to $4a(ax^2 + bx + c) \equiv 0(\bmod p)$

$\Rightarrow (2ax + b)^2 \equiv (b^2 - 4ac)(\bmod p)$

Putting $n^2 = b^2 - 4ac$ to get

$\Rightarrow (2ax + b)^2 \equiv n^2 (\bmod p)$------------------ (ii)

Let $p \in [r]$ for some $r \in \{0, 1, 2, \ldots\ldots, n-1\}$. Then $p = nl + r$ for some positive integer l.

We claim that $2ax_1 + b = n(l-1) + r$, $2ax_2 + b = n(l+1) + r$ are satisfy the congruence (ii). For that

$(2ax_1 + b)^2 - n^2 = \{n(l-1) + r\}^2 - n^2$

$= (nl + r)^2 - 2a(nl + r)$

$= p^2 - 2ap$

$= p(p - 2a)$

$\therefore \quad (2ax_1 + b)^2 \equiv n^2 (\bmod p)$

Similarly, we can show

$(2ax_2 + b)^2 \equiv n^2 (\bmod p)$

Therefore $x \equiv x_1, x_2 (\bmod p)$ are solutions of the quadratic congruence (i).

Lastly, we show x_1 and x_2 are incongruent solutions modulo p. If possible, let

$x_1 \equiv x_2 (\bmod p)$

$\Rightarrow 2ax_1 + b \equiv 2ax_2 + b(\bmod p)$

$\Rightarrow n(l-1) + r \equiv n(l+1) + r(\bmod p)$

$\Rightarrow p | 2n$

$\Rightarrow p | n$ as p is an odd prime.

Which contradicts the fact that p does not divide n. Therefore, our assumption is wrong. Thus x_1 and x_2 are incongruent solutions modulo p.

Example 3.8: Consider the congruence
$x^2+7x+10\equiv 0(\mathrm{mod}\,11)$
Here a=1, b=7, c=10 and p=11
We have $(4a,p)=(4.1,11)=1$, so the given congruence equivalent to
$4x^2+28x+40\equiv 0(\mathrm{mod}\,11)$
$\Rightarrow (2x+7)^2\equiv 3^2(\mathrm{mod}\,11)$
Here $p=3.3+2\in[2]$, when $[2]$ is an equivalence class ofcongruence modulo 3. Applying theorem (3.8), we have
$2x_1+7\equiv 3.2+2(\mathrm{mod}\,11)$ and $2x_2+7\equiv 3.4+2(\mathrm{mod}\,11)$
$\Rightarrow 2x_1\equiv 1(\mathrm{mod}\,11)$ and $2x_2\equiv 7(\mathrm{mod}\,11)$
Therefore $x\equiv 6,9(\mathrm{mod}\,11)$ are two incongruent solutions of the quadratic congruence.

Corollary 3.9: If p is an odd prime, then a and $p-a$ are incongruent solutions of the congruence $x^2\equiv a^2(\mathrm{mod}\,p)$. In this case $p-a,\ p,\ p+a$ are consecutive elements of some equivalence of congruencemodulo a.

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